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Surroundings: history-dependent optimization and applications to circular and sustainable economics

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LISA MORHAIM*   AYSEGÜL YILDIZ ULUS†

*Université Paris-Panthéon-Assas, CRED, F-75005, Paris, France.
†Galatasaray University, Department of Mathematics

SURROUNDINGS: HISTORY-DEPENDENT OPTIMIZATION AND APPLICATIONS TO CIRCULAR AND SUSTAINABLE ECONOMICS

Lisa Morhaim* and Ayşegül Yıldız Ulus†

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ABSTRACT

Decisions and ideas arise within "surroundings", the set of natural and cultural conditions in which organisms live and human activities take place. Moreover, current decisions are dependent to the past. In this paper, we propose a framework which incorporates general history-dependencies and surroundings. It is designed to take into account both that history dependencies affect the surroundings and that history dependencies are affected by the surroundings. These are crucial issues in many domains, including Economics, Law, Artificial Intelligence and Reinforcement Learning. To be able to consider all kinds of phenomena and issues from different areas, history-dependencies are modeled in a very general way through the introduction of a memory function (not necessarily *consumption* history formation processes as previously in the economic literature). Our modeling is tractable, interpretable within many diverse contexts and allows several simultaneous history dependencies. We obtain optimization results and develop dynamic programming tools to deal with such models, in particular, we show that there exists a solution and the value function is the unique fixed point of the Bellman operator. Since environmental and sustainable variables are influenced by the memory of our past decisions and can be taken as surroundings, as a by-product, we furthermore introduce a very general sustainable framework which fits many existing environmental and sustainable models including circular economy models. It provides a basis for future environmental analysis.

KEYWORDS: History-dependent model, Surroundings, Circular Economy, Environment, Pollution, Sustainability, Growth, Habits, Satiation, Optimal management of natural resources, Law, Optimal growth, Intertemporal decisions with instantaneous history-dependencies, Dynamic programming, Reinforcement Learning, Artificial Intelligence.

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*Université Paris-Panthéon-Assas, CRED Paris Center for Law and Economics, lisa.morhaim@u-paris2.fr

†Galatasaray University, Department of Mathematics, aulus@gsu.edu.tr.

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Contents

1 Introduction

Our decisions and ideas arise within surroundings. Surroundings are the set of natural and cultural conditions in which organisms live and human activities take place, and play a crucial role in many areas, whether in History, Philosophy, Epistemology, Sociology, Economy, Law or Artificial intelligence. Surroundings' evolutions and human activities' histories are mutually dependent. We aim to build frameworks in which the *decisions* and *surroundings* may interlink in various ways. This is done by modeling their history evolutions and memory formations. While Goldsmith and Laks study the evolution of *ideas* in their area and refer to the way they abstract or not from human and social context, "The first is sometimes called *internal history*, the second *external history*; both are important for us." (Goldsmith and Laks[21]), we approach such issues in economics, considering that both decisions and surroundings must be taken into account, simultaneously or not.

The literature increasingly reveals the importance of history dependencies in the decisions' processes. Mainly, the instantaneous decision criteria crucially depend on the sequences of past events (which include, but are not restricted to, past decisions). It still remains to understand the history-dependencies formations, to highlight how history dependencies affect our decisions, and how they are affected by our decisions.

In Economics, history dependencies have mainly appeared in models with a representative consumer taking her own past consumptions as a reference point. Axiomatically (Rozen[44], Rustichini and Siconolfi[45, 46], He Dyer Butler[24], etc), theoretically (Ryder and Heal[25], Caroll, Overland and Weil[11], Bambi and Gozzi[6], Bambi, Ghilli, Gozzi and Leocata[5], etc) and empiricallly (Havranek, Rusnak and Sokolova[22], Fuhrer[18], Ravina[42], Dynan[15], etc .....), different viewpoints have contributed to explain some observed phenomena. In this literature, a current history level is introduced and several ways of history formation from consumptions have been considered. Morhaim and Ulus[40] proposed a unified framework allowing very general consumption history formation processes, opening a path to study various history-dependencies issues (such as habits or satiation, but also addiction and many other effects) while providing results and dynamic programming tools to deal with these models.

Yet some important issues remain to consider, among which the two following crucial ones. Firstly, history formation processes of the reference points arise, that are not necessarily *consumption* history formation processes. New modellings may be designed to incorporate different kinds of history-dependencies. Secondly, our decisions are affected by the surroundings in which they occur. By our definition of surroundings, they might be whether natural or cultural conditions, such as legal frameworks, (social or cultural) norms, environment , etc. They thus include but are not restricted to environmental and sustainable variables. It is important to understand, on one hand how do history dependencies affect the surroundings, and on the other hand how history dependencies are affected by the surroundings. These two important issues are addressed in this paper.

As a by-product, our framework in this paper fits many environmental and sustainable models, including circular economy models. Indeed, environmental and sustainable variables are influenced by (the memory of our past) decisions and can be taken as surroundings. This leads us to introduce a very general sustainable framework, which provides a basis for future environmental analysis.

Circular economy (CE) is an important issue on current policy agendas worldwide (Abad-Segura, de la Fuente, González-Zamar and Belmonte-Ureña[1], Fitch-Roy, Benson and Monciardini [17], de Melo, de Oliveira, de Sousa, Vieira and Amaral[12]), CE aiming to increase the efficiency of resources use to achieve a better balance and harmony between economy, environment and society (Ghisellini, Cialani and Ulgiati[20]). Among scholars, such discussions and ideas date at least back to the second half of the the twentieth century (Boulding[9], Pearce and Turner [41]). The conceptual and theoretical understandings as well as CE strategies and implementations are still currently discussed (Andersen[3], Kirchherr, Reike and Hekkert[32], Kalmykova Sadagopan and Rosado[27]). Nevertheless, CE is most frequently depicted as a combination of reduce, reuse and recycle activities (Kirchherr, Reike and Hekkert[32]), but also design, implying a focus on the entire life cycle of the processes the interaction between the process and the environment and the economy in which it is embedded (Ghisellini, Cialani and Ulgiati[20]). Although not systematically, a large strand of research analyze CE interlinked with sustainable development (Ghisellini, Cialani and Ulgiati[20], Ritzén and Sandström[43]. The Ellen Mac Arthur Foundation Report[37] thus aimed to demonstrate how circular economy principles and strategies significantly reduce greenhouse gas emissions and calls for integrating efforts to respond to climate change with those to accelerate the transition to a circular economy.

We now detail the way we address history-dependencies, history formation and surroundings in this paper. We provide a framework with general history formations. This allows to consider the various kinds of history-dependencies that are currently taken into account in the economic literature. We introduce a *memory function* to model the history formation process. The framework in Morhaim and Ulus[40], in which *consumption* history is considered becomes a particular case of transforming the decisions into a history sequence. Indeed, this is a case in which *consumption* is kept in memory, which is a function of previous date and current capital stock decisions. It means keeping in memory a particular function of these decisions. Our general history-dependent framework allows, not only to keep in memory *the particular function* defining consumption, but *any function* of the previous date and current decisions. Thus, Morhaim and Ulus[40] model as well as all its examples[1] are particular cases of our general history-dependent framework.

Furthermore, the framework we propose is conceived to take into account various kinds and widely interpretable surroundings. They can model habits (Ryder and Heal[25], Rozen[44], Rustichini and Siconolfi[46], Caroll, Overland and Weil[10]), satiation (He, Dyer and Butler[23]), environmental issues and pollution (Ikefuji[26], Löfgren[36], Van Der Ploeg and Withagen[53]), optimal management of natural resource (Smulders, Toman and Withagen[48], Ulus[51]), sustainability issues and circular economy (CE) models (George, Chi-ang Lin and Chen[19], Kasioumi[28, 29, 31]), and circular and causation (CCC) models (Donaghy[13, 14]), but also many other phenomena, as in Law, legal decisions and the Rule (Lewis[35], Farber[16]), in Game Theory, actions and social norms (Acemoglu and Jackson[2]), in Experimental Economics, decision making and social norms (Vostroknutov[54]), Artificial Intelligence, Reinforcement Learning (Tennenholtz, Merlis, Shani, Mladenov and Boutilier[50]). We also allow to interlink these with many kinds of effects and history-dependencies, and consider a very general instantaneous reward function, general surroundings evolutions and feasible sets.

Thus, our modeling is designed to study decisions and surroundings, simultaneously or not, and to allow the decisions and surroundings to interlink, while keeping it still tractable and interpretable within different contexts. We provide history-dependent models. We obtain optimization results and develop dynamic programming tools to deal with such models.

More precisely, we consider the following three history-dependent frameworks

- a general history-dependent framework (GHDF)
- an easily implementable history-dependent framework (EIHDF)

[1] including Ryder and Heal[25], Rozen[44], Rustichini and Siconolfi[46], Caroll, Overland and Weil[10], He, Dyer and Butler[23], Baucells and Sarin[7], Ikefuji[26], Löfgren[36], Safi and Ben Hassen[47].

- a general sustainable framework (GSF)

The first history-dependent model, i.e. the general history-dependent framework (GHDF), is a general tractable framework for intertemporal optimization framework in which the instantaneous reward function depends on the memory of (eventually all) the previous decisions. On one hand, we introduce a general function allowing to model many different memory processes and on the other hand, we deal with general decision variables, objective functions and feasible sets. The formalism we propose allows to discuss and study the memory process formation (through a function m) as well as the way the history enters the instantaneous reward function. We provide dynamic programming tools for such models. Without concavity assumptions, we show the existence of a solution and that the value function is the unique fixed point of the Bellman operator.

Further, we provide an easily implementable history-dependent framework (EIHDF), a version of our general history-dependent framework GHDF which is both easily implementable and as general as needed to be widely applicable. This is allowed by providing a history-dependent framework in which the primer of the problem, in particular the instantaneous reward function and the feasible set Γ, are defined in an adequate recursive way answering both issues simultaneously. We then show that such a model is a particular case of our general history-dependent framework and the results for this particular case are derived as corollaries of the results for the general history-dependent framework.

Finally, taking into account the fact that the environment keeps in memory our activities and decisions, we provide a general sustainable framework (GSF) which introduces a basis for future analysis in environmental and sustainable issues. The GSF model is designed in a very amenable and flexible manner so that it can be adapted to many contexts and one can easily remove or add different effects that will be needed to be addressed. The mathematical results (including existence of a solution and dynamic programming tools) are derived as an application of our general history-dependent framework GHDF and can be directly used.

The GSF encompasses many existing models in the environmental literature, whether linear or circular economy models. Particularly, a whole section of this paper is devoted to applications and examples of GSF to circular economy models. While incorporating the concept of circular economic activities, George Lin and Chen[19] considers a social planner twho maximizes an intertemporal utility function, which depends on aggregate consumption and the stock of pollution. This model was later generalized to incorporate the intensity of recycling by Stengos[30], and to further incorporate recycling habits by Kasioumi[28, 29]. The theoretical circular-economy model of economic growth with circular and cumulative causation (CCC) is presented in Donaghy[13, 14]. The George Lin and Chen[19]'s model is modified by Donaghy[13, 14] by including physical capital K, human capital HC, labor L, and other materials OM, as productive factors. We detail extensively some of these models within our framework and discuss how our framework is fitted for future research. It is amenable not only to treat various sustainable and environmental issues but also allows to connect these with many kinds of effects and history-dependencies (consumption, production, saving and investment, human capital, labor, consumption habits, recycling habits, pollution, stock of waste, etc).

The research on circular and sustainable economy is currently vivid. Our framework is fitted to consider and interlink economic, environmental, technological and social issues. The GSF can easily be adapted to already suggested paths for future research (Donaghy[14], Ghisellini[20], etc) and the extensive literature that is developing. The way the GSF may incorporate the history-dependence viewpoint and the memory formation that we introduce open perspectives towards several aspects and interpretations. In particular, the GSF allows to deal with many important features that are coming to be taken into account, such as recycling, reuse, reduction, design, habits, activities of harvesting exhaustible and renewable resources, the assimilative capacity of the natural environment for (non-recyclable) waste, transport activities, management of resources, interaction between the processes and the environment, preventative and regenerative eco-industrial development, etc. We discuss the way existing models are particular cases of our general framework as well as how its flexibility allows to use it in future research. As an example, the sustainable process or design variable r does not enter directly Donaghy's utility function in contrast with this possibility which is allowed in our framework. Our general framework allows to study simultaneously many effects and contexts: circular models without production waste (Section 4.4), linear economies with production waste, and furthermore circular economies with production waste, as well as other many effects. These may be interconnected: our general framework allows to enrich the analysis by treating at the same time several kinds of history dependencies. This is crucial for sustainability issues as they involve on one hand habits (consumption habits but also recycling habits), and on another hand pollution stocks and environmental quality (Mazar[38], Moreau[39]). Such a modelling choice also emphasizes the fact that the environment keeps in memory our activities and decisions.

The paper is organized as follows. Section 2 introduces the general history-dependent framework GHDF and the easily implementable history-dependent framework EIHDF, which is a version of our general history-dependent framework GHDF in which the primer of the problem are defined in an adequate recursive way so that it is both kept easily implementable and as general as needed to be widely applicable. We provide the results on the existence of a solution and dynamic programming tools for such models. Section 3 provides the general sustainable framework GSF which introduces a very amenable and flexible basis for future analysis in environmental and sustainable issues and encompasses many existing models in the environmental literature (including linear and circular economy models). We show that it is an application of the GHDF. Section 4 details and addresses many examples and applications of our frameworks from the related literature. The proofs are given in the Appendix. Section 5 concludes.

2 A general history-dependent framework (GHDF)

We propose a general intertemporal optimization framework in which the instantaneous reward function depends on the memory of all the previous decisions. The formalism we propose allows to discuss and study the memory process formation as well as the way the history enters the instantaneous reward function. We develop optimization results and dynamic programming tools. We further provide a general history-dependent framework which is both easily implementable and as general as needed to be widely applicable. This is allowed by providing a history-dependent framework in which the primer of the problem, in particular the instantaneous reward function and the feasible set, are defined in an adequate recursive way answering both issues simultaneously. We then show that such a model is a particular case of our general history-dependent framework and the results for this particular case are derived as corollaries of the results for the general history-dependent framework.

2.1 The general history-dependent model

We now present the general intertemporal optimization framework. The instantaneous reward function depends on the memory of all the previous decisions. We introduce a *memory* function m

and discuss the way the history enters the instantaneous reward function (through a function φ). Through these functions, our formalism allows to study many memory process formations and history-dependencies.

We describe a general framework of an infinite horizon intertemporal history dependent decision problem of a social planner which may be a problem of optimal growth related to consumption, saving, conservation, accumulation, pollution or sustainable issues. depending on the objectives and the constraints of the model.

Let X be a topological space which will be the set of state and control variables of the problem. Let us consider an intertemporal decision process. Let Y be a topological space. We define a *memory function* by $m : X \times X \to Y$ such that for time t, $m(x_t, x_{t+1})$ is the *memory* for the decision $x_{t+1} \in X$ given x_t. By this memory function m, we will obtain the history of the memories over time which is modeled in the following way:

Let us consider the set l_+^∞ defined by

$$l_+^\infty = \{\tilde{x} = (x_t)_{t=0}^\infty \in (\mathbb{R}^+)^{\mathbb{N}}, \|\tilde{x}\|_\infty := \sup_{t\in\mathbb{N}} x_t < +\infty\}$$

At time $t = 0$, she has a *time*-0 *history* denoted by $\tilde{h}^{(0)} := (h_0^{(0)}, h_1^{(0)}, ...) = (h_j^{(0)})_{j=1}^\infty$ lying in l_+^∞. Having on memory $m(x_0, x_1)$ at time 0, she has then *time*-1 *history* equal to $\tilde{h}^{(1)} := (m(x_0, x_1), \tilde{h}^{(0)})$ at time $t = 1$. That is, for any time $t \geq 1$, *time-t history* will be

$$\forall t \geq 1, \tilde{h}^{(t)} = (h_j^{(t)})_{j=1}^\infty := (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), ..., m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)})$$

The j-th coordinate of the sequence $\tilde{h}^{(t)}$ is denoted by $h_j^{(t)}$. For j such that $1 \leq j \leq t$, the term $h_j^{(t)}$ is the decision j periods prior to time t, i.e.

$$h_j^{(t)} = m(x_{t-j}, x_{t-j+1})$$

For $j \geq t+1$, the terms $h_j^{(t)}$ of the sequence $\tilde{h}^{(t)}$ are defined by the terms of $\tilde{h}^{(0)}$, i.e.

$$(h_j^{(t)})_{j=t+1}^\infty = \tilde{h}^{(0)}$$

Let us consider an agent facing a time t objective function F which is defined on a subset $\mathcal{D}_F$ of $l_+^\infty \times X \times X$

$$F : \mathcal{D}_F \subseteq (l_+^\infty \times X \times X) \to \mathbb{R}$$

The feasible correspondence $\Gamma : (l_+^\infty \times X) \to X$ is given such that for any $(\tilde{h}, x) \in l_+^\infty \times X$

$$\Gamma(\tilde{h}, x) \subseteq \{x' \in X, (\tilde{h}, x, x') \in \mathcal{D}_F\}$$

Assuming a fixed discount factor $\beta \in (0, 1)$, for initially given state stock $x_0 > 0$ and time-0 history $\tilde{h}^{(0)} \in l_+^\infty$; with feasible state and control variables at time t, that is, satisfying the process $x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t)$, for every $t \geq 0$, the general framework optimization problem can then be written as follows:

$$P_{F,\Gamma,m}(\tilde{h}^{(0)}, x_0) = \begin{cases} \text{Max} & \sum_{t=0}^{+\infty} \beta^t F\big(\tilde{h}^{(t)}, x_t, x_{t+1}\big) \\ \text{s.t.} & \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t) \\ & \forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \ldots.., m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)}) \\ & x_0 > 0 \text{ and } \tilde{h}^{(0)} \in l_+^\infty \text{ are given} \end{cases}$$

2.1.1 Notations and feasible sets

DEFINITION 2.1.— *For any given initial data $x_0 > 0$, and initial time-0 history $\tilde{h}^{(0)} \in l_+^\infty$, the feasible set $\Pi(k_0, \tilde{h}^{(0)})$ is defined by the set of sequences feasible from x_0 and $\tilde{h}^{(0)}$, i.e. for any $x_0 > 0$, for any $\tilde{h}^{(0)} \in l_+^\infty$,*

$$\Pi(\tilde{h}^{(0)}, x_0) = \{\tilde{x} = (x_t)_{t=1}^{+\infty} \in X^{I\!N}, \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t), \tilde{h}^{(t+1)} = (m(x_t, x_{t+1}), \tilde{h}^{(t)})\}$$

The process is analogous as usual. The feasible set is defined from x_0 but here also from the given (infinite) sequence $\tilde{h}^{(0)}$. Once x_1 is chosen in the feasible set $\Gamma(\tilde{h}^{(0)}, x_0)$, the sequence $\tilde{h}^{(1)}$ is updated from the current chosen variable x_1 and the given variable x_0. Then, x_2 is chosen given x_1 and $\tilde{h}^{(1)}$ in the feasible set $\Gamma(\tilde{h}^{(1)}, x_1)$. And so on, at time $t+1$, given x_t and $\tilde{h}^{(t)}$, the variable x_{t+1} must be chosen in the feasible set $\Gamma(\tilde{h}^{(t)}, x_t)$. Once x_{t+1} is chosen, at time $t+2$, x_{t+2} is chosen given x_{t+1} and $\tilde{h}^{(t+1)}$ that has been updated from the previous history sequence $\tilde{h}^{(t)}$ and the variables (x_t, x_{t+1}). Such a process allows to keep decisions in memory while chosing the variable x_t at each time t.

For a sequence $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(\tilde{h}^{(0)}, x_0)$, we denote by $\mathcal{U}(\tilde{x})$ the objective i.e.

$$\mathcal{U}(\tilde{x}) = \sum_{t=0}^{+\infty} \beta^t F(\tilde{h}^{(t)}), x_t, x_{t+1})$$

2.1.2 Assumptions

We set that (F, Γ) satisfies the set $(\mathcal{A})$ of assumptions if the following assumptions (**F**) on the feasible set, (**A**) on the instantaneous function, and (**M**) on the memory function are satisfied

(**F**)
(F1) Γ is a continuous nonempty compact-valued correspondence from $l_+^\infty \times X$ into X.
(F2) There exists $a \geq 0, a \neq 1$ and $a' \geq 0$ such that for any $x \in X$ and any history $\tilde{h} \in l_+^\infty$, $x' \in \Gamma(\tilde{h}, x) \Rightarrow \|x'\| \leq a'\|x\| + a$

(**A**)
(A1) $\forall x_0 > 0$ and $\tilde{h}^{(0)} \neq 0, \exists \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$ such that $\mathcal{U}(\tilde{x}) > -\infty$.
(A2) There exist $a \in I\!R^+$ with $a\beta < 1$, $a_2 \in I\!R^+$ and a continuous function $a_1 : X \to I\!R$ such that for any $x_0 > 0$ and $\tilde{h}^{(0)} \in l_+^\infty$, for any feasible sequence $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(\tilde{h}^{(0)}, x_0)$ and its associated history $\tilde{h}^{(t)}$, for any $t \geq 0$,

$$F^+(\tilde{h}^{(t)}, x_t, x_{t+1}) \leq a_1(x_0)a^t + a_2$$

(**M**) The function m is continuous.

2.1.3 The objective is well-defined

The next result shows that, under the set of assumptions $(\mathcal{A})$, the objective function of the optimization problem $\mathcal{P}_{F,\beta,m}(\tilde{h}^{(0)}, x_0)$ is well-defined. The proof is given in the Appendix.

Proposition 2.1.— *Assume* $(\mathcal{A})$ *and let* $x_0 \in X$ *and* $\tilde{h}^{(0)} \in l_+^\infty$ *be given. Then for any feasible sequence* $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(\tilde{h}^{(0)}, x_0)$*, the limit* $\lim_{T \to +\infty} \sum_{t=0}^{T} \beta^t F\big(\tilde{h}^{(t)}, x_t, x_{t+1}\big)$*, with* $\forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), ..., m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)})$*, is well-defined.*

The proof can be found in the Appendix.

2.1.4 Existence and uniqueness of the solution

In this section, we give a proposition which shows the existence of a solution to the optimization problem $\mathcal{P}_{F,\beta,m}(\tilde{h}^{(0)}, x_0)$. The uniqueness is obtained under additional assumptions, requiring strict concavity of the instantaneous function F. The proof is given in the Appendix.

Proposition 2.2.— *Assume* $(\mathcal{A})$ *and assume that* $\Pi(\tilde{h}^{(0)}, x_0) \neq \emptyset$*. Then there exists an optimal solution. Moreover, if* F *is jointly strictly concave in* (y, x, x') *on* $l_+^\infty \times X \times X$*, then the solution is unique.*

We give the existence and uniqueness results in a general framework, in particular without any differentiability assumptions. The strict joint concavity of F allows to guarantee uniqueness.
The proof can be found in the Appendix.

2.1.5 The value function and Bellman equation

We now define the value function of the optimization problem. We show that under the set of assumptions $(\mathcal{A})$, dynamic programming tools can be used to study $\mathcal{P}_{F,\beta,m}(\tilde{h}^{(0)}, x_0)$. We first study the properties of the value function. We then provide an appropriate set of functions on which the Bellman operator has a unique fixed point which is the value function.

Definition 2.2.— *The value function* V *is defined on* $l_+^\infty \times X$ *by for any* $(\tilde{h}^{(0)}, x_0) \in l_+^\infty \times X$

$$V(\tilde{h}^{(0)}, x_0) = \begin{cases} Max & \sum_{t=0}^{+\infty} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) \\ s.t. & \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t) \\ & \forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), ..., m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)}) \\ & \tilde{h}^{(0)} \in l_+^\infty \text{ and } x_0 \in X \text{ are given} \end{cases}$$

Proposition 2.3.— *Assume* $(\mathcal{A})$*. Then the value function* V *is upper semi-continuous.*

The proof of this proposition can be found in Appendix.

2.1.6 Properties of the value function

The next proposition states some further properties on the value function and the proof is given in Appendix. These properties, together with the upper-semi continuity, will provide an appropriate set of functions to consider for dynamic programming tools.

PROPOSITION 2.4.— *Assume* $(\mathcal{A})$. *Then the value function* V *satisfies*
(i) $\forall x_0, \tilde{h}^{(0)}, \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0), \overline{\lim}_{t\to+\infty} \beta^t V(\tilde{h}^{(t)}, x_t) \leq 0.$
(ii) $\forall x_0, \tilde{h}^{(0)}$, *and* $\forall \tilde{x} \in \Pi(\tilde{h}^{(0)}, , x_0)$ *such that* $\mathcal{U}(\tilde{x}) > -\infty$, $\lim_{t\to+\infty} \beta^t V(\tilde{h}^{(t)}, x_t) = 0$.

The proof can be found in the Appendix.

A standard proof (see Theorem 4.4 p.75 Stokey, Lucas and Prescott[49]) allows to show the following result.

PROPOSITION 2.5.— *Assume* $(\mathcal{A})$. *Then* $\tilde{x}^*$ *is an optimal solution if and only if*

$$\forall t \geq 0, V(\tilde{h}^{*(t)}, x_t^*) = F(\tilde{h}^{*(t)}, x_t^*, x_{t+1}^*) + \beta V(\tilde{h}^{*(t+1)}, x_{t+1}^*)$$

where $\tilde{h}^{*(t)} = (m(x_{t-1}^*, x_t^*), m(x_{t-2}^*, x_{t-1}^*), ..., m(x_1^*, x_2^*), m(x_0^*, x_1^*), \tilde{h}^{(0)})$

Let B be the Bellman operator, i.e. $B : \mathcal{F}(l_+^\infty \times X, \mathrm{I\!R}) \to \mathcal{F}(l_+^\infty \times X, \mathrm{I\!R})$ be defined by

$$\forall w \in \mathcal{F}(l_+^\infty \times X, \mathrm{I\!R}), Bw(\tilde{h}, x) = \max_{x' \in \Gamma(\tilde{h}, x)} \{F(\tilde{h}, x, x') + \beta w((m(x, x'), \tilde{h}), x')\}$$

DEFINITION 2.3.— *Let* $\mathcal{F}_b(l_+^\infty \times X, \mathrm{I\!R})$ *be the set of upper semi-continuous functions* $w \in \mathcal{F}(l_+^\infty \times X, \mathrm{I\!R})$ *such that*
(i) $\forall x_0 \in X, \forall \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0), \overline{\lim}_{t\to+\infty} \beta^t w(\tilde{h}^{(t)}, x_t) \leq 0,$
with $\tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), ..., m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)})$
(ii) $\forall x_0 \in X, \forall \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$ *such that* $\mathcal{U}(\tilde{x}) > -\infty$, *one has* $\lim_{t\to+\infty} \beta^t w(\tilde{h}^{(t)}, x_t) = 0$

2.1.7 The value function is the unique fixed-point of the Bellman operator

We now finally state that the value function is the unique fixed-point of the Bellman operator on this set of functions. The proof is given in Appendix.

PROPOSITION 2.6.— *Assume* $(\mathcal{A})$. *Then the value function* V *is the unique fixed-point of the Bellman operator on the set of functions* $\mathcal{F}_b(l_+^\infty \times X, \mathrm{I\!R})$.

Finally, this shows that dynamic programming tools can be used to deal with general history-dependent optimal growth models.

2.2 An easily implementable history-dependent model (EIHDF)

In this section, we provide a general history-dependent framework which is both easily implementable and as general as needed to be widely applicable. This is allowed by providing a history-dependent framework in which the primer of the problem, in particular the instantaneous reward function and the feasible set Γ, are defined in an adequate recursive way answering both issues simultaneously. We then show that such a model is a particular case of our general history-dependent framework and the results for this particular case are derived as corollaries of the results for the general history-dependent framework.

2.2.1 The problem

Let X, Y be topological spaces. Let us consider $F : \mathbb{R}^N \times X \times X \to \mathbb{R}$ the instantaneous reward function, $m : X \times X \to Y$ the memory function and the (fixed) discount factor $\beta \in (0,1)$. Let us consider an adjustement level function $\varphi : l_+^\infty \to \mathbb{R}^N$, with $N \in \mathbb{N}$. Let $x_0 \in X$ and $\tilde{h}^{(0)} \in l_+^\infty$ be given, and consider the problem

$$\mathcal{P}^{\varphi}_{F,\Gamma,m}(\tilde{h}^{(0)}, x_0) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t F\big(\varphi(\tilde{h}^{(t)}), x_t, x_{t+1}\big) \\ \text{s.t.} & \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t) \\ & \forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \ldots.., m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)}) \end{cases}$$

Recall that for any given initial data $x_0 > 0$, and initial time-0 history $\tilde{h}^{(0)} \in l_+^\infty$, the feasible set $\Pi(\tilde{h}^{(0)}, x_0)$ is defined by the set of sequences feasible from x_0 and $\tilde{h}^{(0)}$, i.e. for any $x_0 > 0$, for any $\tilde{h}^{(0)} \in l_+^\infty$,

$$\Pi(\tilde{h}^{(0)}, x_0) = \{\tilde{x} = (x_t)_{t=1}^{+\infty} \in X^{\mathbb{N}}, \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t), \tilde{h}^{(t+1)} = (m(x_t, x_{t+1}), \tilde{h}^{(t)})\}$$

2.2.2 Assumptions

Let us give a set $(\mathcal{A}')$ of assumptions: (**F**) on the feasible set, (**A'**) on the instantaneous function, and (**M**) on the memory function.

(**A'**)

(A1') $\forall x_0 > 0$ and $\tilde{h}^{(0)} \neq 0, \exists \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$ such that $\sum_{t=0}^{+\infty} \beta^t F\big(\varphi(\tilde{h}^{(t)}), x_t, x_{t+1}\big) > -\infty$.

(A2') The function φ is continuous and there exist $a \in \mathbb{R}^+$ with $a\beta < 1$, $a_2 \in \mathbb{R}^+$ and a continuous function $a_1 : X \to \mathbb{R}$ such that for any $x_0 > 0$ and $\tilde{h}^{(0)} \in l_+^\infty$, for any feasible sequence $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(\tilde{h}^{(0)}, x_0)$ and its associated history $\tilde{h}^{(t)}$, for any $t \geq 0$,

$$F^+(\varphi(\tilde{h}^{(t)}), x_t, x_{t+1}) \leq a_1(x_0)a^t + a_2$$

2.2.3 An easily implementable case

An easily implementable case is when the function $\varphi : l_+^\infty \to \mathbb{R}^N$, with $N \in \mathbb{N}$ can be defined recursively through a function $G : Y \times \mathbb{R}^N \to \mathbb{R}^N$, i.e. for any $\upsilon \in Y$ and any $\tilde{h} \in l_+^\infty$, as

$$\varphi(\upsilon, \tilde{h}) = G\big(\upsilon, \varphi(\tilde{h})\big)$$

and when the correspondence Γ can be defined (and easily calculated) by φ

$$\Gamma(\tilde{h}, x) = g(\varphi(\tilde{h}), x)$$

with $g : \mathbb{R}^N \times X \to X$ is given.

The problem is then written

$$\mathcal{P}^{\varphi}_{F,\Gamma,m}(\tilde{h}^{(0)}, x_0) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t F\big(\varphi(\tilde{h}^{(t)}), x_t, x_{t+1}\big) \\ \text{s.t.} & \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t) \\ & \forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \ldots.., m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)}) \\ & \forall t \geq 1, \varphi(\tilde{h}^{(t)}) = G\big(m(x_{t-1}, x_t), \varphi(\tilde{h}^{(t-1)})\big) \\ & x_0 > 0 \text{ and } \tilde{h}^{(0)} \in l_+^\infty \text{ are given} \end{cases}$$

2.2.4 A particular case of our general history-dependent framework

It is a particular case of our general history-dependent framework. Let us define the following instantaneous reward function $\hat{F} : l^\infty_+ \times X \times X \to \mathbb{R}$ by

$$\hat{F}(\tilde{h}^{(t)}, x_t, x_{t+1}) = F\big(\varphi(\tilde{h}^{(t)}), x_t, x_{t+1}\big)$$

It is straightforward to check that the problem $\mathcal{P}^{\varphi}_{F,\Gamma,m}(\tilde{h}^{(0)}, x_0)$ is equivalent to the general history-dependent problem $\mathcal{P}_{\hat{F},\Gamma,m}(\tilde{h}^{(0)}, x_0)$ and that $(\hat{F}, \Gamma)$ satisfies the set $(\mathcal{A})$ of assumptions (by $(\mathcal{A}')$),

2.2.5 Theorem/proposition (existence d'une solution, programmation dynamique etc)

The following results are then derived as corollaries of the ones previously shown for the general history-dependent model. Assume $(\mathcal{A}')$ in this subsection hereafter.

Corollary 2.1.— *Assume $(\mathcal{A}')$. The problem $\mathcal{P}^{\varphi}_{F,\Gamma,m}(\tilde{h}^{(0)}, x_0)$ is well-defined. Assume moreover that $\Pi(\tilde{h}^{(0)}, x_0) \neq \emptyset$. Then there exists an optimal solution. Moreover, if F is jointly strictly concave in (y, x, x') on $l^\infty_+ \times X \times X$ and φ is linear, then the solution is unique.*

Definition 2.4.— *The value function V is defined on $l^\infty_+ \times X$ by for any $(\tilde{h}^{(0)}, x_0) \in l^\infty_+ \times X$*

$$V(\tilde{h}^{(0)}, x_0) = \begin{cases} Max & \sum_{t=0}^{+\infty} \beta^t F(\varphi(\tilde{h}^{(t)}), x_t, x_{t+1}) \\ s.t. & \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t) \\ & \forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \ldots.., m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)}) \\ & \tilde{h}^{(0)} \in l^\infty_+ \text{ and } x_0 \in X \text{ are given} \end{cases}$$

Corollary 2.2.— *Assume $(\mathcal{A}')$. Then the value function V is upper semi-continuous.*

Corollary 2.3.— *Assume $(\mathcal{A}')$. Then the value function V satisfies*
(i) $\forall x_0, \tilde{h}^{(0)}, \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0), \overline{\lim}_{t\to+\infty} \beta^t V(\tilde{h}^{(t)}, x_t) \leq 0$.
(ii) $\forall x_0, \tilde{h}^{(0)}$, and $\forall \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$ such that $\mathcal{U}(\tilde{x}) > -\infty$, $\lim_{t\to+\infty} \beta^t V(\tilde{h}^{(t)}, x_t) = 0$.

Corollary 2.4.— *Assume $(\mathcal{A}')$. Then $\tilde{x}^*$ is an optimal solution if and only if*

$$\forall t \geq 0, V(\tilde{h}^{*(t)}, x^*_t) = F(\varphi(\tilde{h}^{*(t)}), x^*_t, x^*_{t+1}) + \beta V(\tilde{h}^{*(t+1)}, x^*_{t+1})$$

where $\tilde{h}^{*(t)} = (m(x^*_{t-1}, x^*_t), m(x^*_{t-2}, x^*_{t-1}), \ldots.., m(x^*_1, x^*_2), m(x^*_0, x^*_1), \tilde{h}^{(0)})$

Let B be the Bellman operator, i.e. $B : \mathcal{F}(l^\infty_+ \times X, \mathbb{R}) \to \mathcal{F}(l^\infty_+ \times X, \mathbb{R})$ be defined by

$$\begin{aligned} \forall w \in \mathcal{F}(l^\infty_+ \times X, \mathbb{R}), Bw(\tilde{h}, x) &= \max_{x' \in \Gamma(\tilde{h}, x)} \{F(\varphi(\tilde{h}), x, x') + \beta w((m(x, x'), \tilde{h}), x')\} \\ &= \max_{x' \in g(\varphi(\tilde{h}), x)} \{F(\varphi(\tilde{h}), x, x') + \beta w((m(x, x'), \tilde{h}), x')\} \end{aligned}$$

Definition 2.5.— *Let $\mathcal{F}_b(l^\infty_+ \times X, \mathbb{R})$ be the set of upper semi-continuous functions $w \in \mathcal{F}(l^\infty_+ \times X, \mathbb{R})$ such that*
(i) $\forall x_0 \in X, \forall \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$, $\overline{\lim}_{t\to+\infty} \beta^t w(\tilde{h}^{(t)}, x_t) \leq 0$,
with $\tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \ldots.., m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)})$
(ii) $\forall x_0 \in X, \forall \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$ such that $\mathcal{U}(\tilde{x}) > -\infty$, one has $\lim_{t\to+\infty} \beta^t w(\tilde{h}^{(t)}, x_t) = 0$

Corollary 2.5.— *Assume* $(\mathcal{A}')$. *Then the value function* V *is the unique fixed-point of the Bellman operator on the set of functions* $\mathcal{F}_b(l^\infty_+ \times X, I\!R)$.

We next propose a general sustainable framework that provides a basis for future research, including environmental and sustainable analysis. It is a particular case of our general history-dependent framework. This illustrates how our general history-dependent framework can be used to deal with many economic issues.

3 Application: a general sustainable framework (GSF)

In this section, we present a general sustainable framework. It provides a basis for future analysis on environmental and sustainable issues. It is designed in a very flexible manner. It can be adapted to many contexts and one can easily remove or add different effects that are needed to be addressed. The mathematical results (including existence of a solution and dynamic programming tools) can be directly used. They are derived as an application of the general history-dependent framework presented in the previous section.

The GSF encompasses many existing models in the literature. The next section presents the model and the following section is devoted to examples from the literature: we detail extensively some of these models within our framework and discuss how our framework is fitted for future research as it is amenable not only to treat various sustainable and environmental issues but also allows to interlink these with many kinds of effects and history-dependencies (consumption, production, saving and investment, human capital, labor, consumption habits, recycling habits, pollution, stock of waste, social and legal norms, etc).

3.1 The model

We consider an economy in which a unique final good is produced using two kinds of factors of production. The *sustainable inputs* obtained from the sustainable process (from waste, recycling actions, reuse, etc) are denoted by ξ and the *other inputs* (as polluting resources, capital, human capital, labor, other materials, etc) are denoted by κ. At each date t, the production level y_t is thus given by the production function $\hat{f}$ depending on[2] $\xi_t \in I\!R$ and $\kappa_t \in I\!R^{N_\kappa}$ (with $N_\kappa \in I\!N$) as

$$y_t = \hat{f}(\xi_t, \kappa_t)$$

In this economy, the representative consumer cares for the state of environment and for sustainability. She derives utility from consumption c_t, from the sustainable process or design r_t (as for example the recycling level), and from environmental and sustainability variables E_t (which can be the stock of pollution, the recycling habits, etc). These environmental and sustainability variables depend on all previous decisions-history, thus inducing instantaneous history-dependent preferences. At each date t, the instantaneous utility of the representative agent thus depends on $c_t, r_t \in I\!R^+$ and $E_t \in I\!R^{N_E}$ (with $N_E \in I\!N$) as

$$u(c_t, r_t, E_t)$$

The economy accumulates waste. Indeed, production and consumption generate discards. Discards generated from production come both from the production process itself and the use of the polluting

[2]Note that $\xi \in I\!R^N$ ($N \in I\!N$) can be considered in this framework, it is only for simplicity of exposition that $\xi \in I\!R$.

input for producing, so the level $\hat{\mathcal{D}}_p$ of discards from production is a function of the production level y_t and the other inputs level κ_t. The level $\hat{\mathcal{D}}_c$ of discards from consumption depends on the consumption level c_t. All what is produced (y_t) and all discards that are neither consumed (c_t), invested (i_t) nor used in the sustainable process (r_t), accumulate as waste:

$$s_{t+1} - s_t = \hat{f}(\xi_t, \kappa_t) + \hat{\mathcal{D}}_p(y_t, \kappa_t) + \hat{\mathcal{D}}_c(c_t) - c_t - i_t - r_t$$

with r_t depending on the instantaneous waste stock s_t and on the environmental and sustainability variables E_t. This is modelled through a function $\mathcal{R}$ so that we have $r_t = \mathcal{R}(s_t, E_t)$ such that for all s, E, one has $0 \leq \mathcal{R}(s, E) \leq s$. The investment i_t depends on the instantaneous and the next period input levels, i.e. $i_t = \hat{\mathcal{I}}(\xi_t, \kappa_t, \xi_{t+1}, \kappa_{t+1})$. The waste accumulation dynamics is thus given by

$$s_{t+1} - s_t = \hat{f}(\xi_t, \kappa_t) + \hat{\mathcal{D}}_p(y_t, \kappa_t) + \hat{\mathcal{D}}_c(c_t) - c_t - \hat{\mathcal{I}}(\xi_t, \kappa_t, \xi_{t+1}, \kappa_{t+1}) - \mathcal{R}(s_t, E_t)$$

The environmental and sustainability variables E_t evolve in function of their previous state and the current decisions. The law of motion of the environmental and sustainability variables is thus defined through a given function $G : \mathbb{R} \times \mathbb{R}^{N_\kappa} \times \mathbb{R} \times \mathbb{R}^{N_E} \to \mathbb{R}^{N_E}$ by, at each date t,

$$E_{t+1} = G((s_t, \kappa_t, c_t), E_t)$$

Given the initial stock of waste s_0, available inputs κ_0, and initial environmental and sustainability variables E_0, the representative agent solves the following optimization problem

$$\hat{\mathcal{P}} = \begin{cases} \text{Maximize} \sum_{t=0}^{+\infty} \beta^t u(c_t, r_t, E_t) \\ \text{s.t. } \forall t \geq 0, \\ \quad s_{t+1} - s_t = \hat{f}(\xi_t, \kappa_t) + \hat{\mathcal{D}}_p(y_t, \kappa_t) + \hat{\mathcal{D}}_c(c_t) - c_t - \hat{\mathcal{I}}(\xi_t, \kappa_t, \xi_{t+1}, \kappa_{t+1}) - \mathcal{R}(s_t, E_t) \\ \quad E_{t+1} = G(s_t, \kappa_t, c_t), E_t) \\ \quad r_t = \mathcal{R}(s_t, E_t) \in [0, s_t] \\ \quad s_0, \kappa_0 > 0, E_0 \geq 0 \text{ are given} \end{cases}$$

In a sustainable economy, a key assumption is that production involves an input which is obtained through sustainable actions. Here, it is assumed that

$$\xi_t = \mathcal{R}(s_t, E_t)$$

Then, by defining $f(s_t, \kappa_t, E_t) := \hat{f}(\mathcal{R}(s_t, E_t), \kappa_t)$, $\mathcal{D}_p(s_t, \kappa_t) := \hat{\mathcal{D}}_p(f(s_t, \kappa_t), \kappa_t)$ and $\mathcal{I}(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}) = \hat{\mathcal{I}}(\mathcal{R}(s_t, E_t), \kappa_t, \mathcal{R}(s_{t+1}, E_{t+1}), \kappa_{t+1})$, the problem is written

$$\mathcal{P} = \begin{cases} \text{Maximize} \sum_{t=0}^{+\infty} \beta^t u(c_t, r_t, E_t) \\ \text{s.t. } \forall t \geq 0, \\ \quad s_{t+1} - s_t = f(s_t, \kappa_t, E_t) + \mathcal{D}_p(s_t, \kappa_t) + \mathcal{D}_c(c_t) - c_t - \mathcal{I}(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}) - \mathcal{R}(s_t, E_t) \\ \quad E_{t+1} = G((s_t, \kappa_t, c_t), E_t) \\ \quad r_t = \mathcal{R}(s_t, E_t) \in [0, s_t] \\ \quad s_0, \kappa_0 > 0, E_0 \geq 0 \text{ are given} \end{cases}$$

This framework can be easily adapted to remove or add any environmental and sustainablity variables E related effects, as for example the ones from recycling habits, or pollution, etc, adding or removing them in the preferences and corresponding accumulation laws of motion. For simplicity of exposition, we assumed that ξ and s belong to $\mathbb{R}$ but the framework can easily be adapted to deal with several kinds of sustainable inputs and several kinds of waste that may play different roles in the economy. Linear economy models are encompassed by assuming there is no input coming from any circular process (i.e. all the functions involved are constant with respect to ξ and there is no $\mathcal{R}$ involved). This will be more extensively discussed afterwards.

3.2 Assumptions

Let us assume that the function $c \to c - \mathcal{D}_c(c)$ is bijective and let us define the function $C : \mathbb{R}^{N_v} \to \mathbb{R}$ by for all $(s, \kappa, s', \kappa', E) \in \mathbb{R}^{N_v}$ (with $N_v := 2 + 2N_\kappa + N_E$),

$$C(s, \kappa, s', \kappa', E) = (Id - \mathcal{D}_c)^{-1}\big(f(s, \kappa, E) + \mathcal{D}_p(s, \kappa) - s' - \mathcal{I}(s, \kappa, s', \kappa') + s - \mathcal{R}(s, E)\big)$$

This gives the consumption level as a function of the stocks of waste, the inputs, and the environmental and sustainablity variables.
The general sustainable problem is equivalent to

$$\mathcal{P} = \begin{cases} \text{Maximize} \sum_{t=0}^{+\infty} \beta^t u(C(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}, E_t), \mathcal{R}(s_t, E_t), E_t) \\ \text{s.t. } \forall t \geq 0, \\ \quad C(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}, E_t) \geq 0 \\ \quad \forall t \geq 0, E_{t+1} = G((s_t, \kappa_t, C(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}, E_t)), E_t) \\ \quad \mathcal{R}(s_t, E_t) \in [0, s_t] \\ \quad s_0, \kappa_0 > 0, E_0 \geq 0 \text{ are given} \end{cases}$$

Let us consider the following set $\mathcal{S}$ of assumptions:

(S1) The functions $u, f, \mathcal{D}_c, \mathcal{D}_p, \mathcal{I}, \mathcal{R}, G$ are continuous, discards are bounded (both below and from above), and the function $c \to c - \mathcal{D}_c(c)$ is bijective.
(S2) (i) There exists a bounded function $e : \mathbb{R}^{N_\kappa + N_Z} \to \mathbb{R}$ such that for all $(s, \kappa, s', \kappa', E) \in \mathbb{R}^{N_v}$,

$$C(s, \kappa, s', \kappa', E) \geq 0 \Rightarrow \|(s', \kappa')\| \leq e(s, \kappa, E)$$

(ii) There exists $a \geq 0, a \neq 1$ and $a' \geq 0$ such that for all $(s, \kappa, E) \in \mathbb{R}^{N_\kappa + N_E}$,

$$e(s, \kappa, E) \leq a'\|(s, \kappa)\| + a$$

(S3) There exist $a \in \mathbb{R}^+$ with $a\beta < 1$, $a_2 \in \mathbb{R}^+$ and a continuous function $a_1 : X \to \mathbb{R}$ such that for any $x_0 = (s_0, \kappa_0) > 0$ and E_0, for any feasible sequence $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(E_0, x_0)$ and its associated history $\tilde{h}^{(t)}$, for any $t \geq 0$,

$$u^+(C(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}, E_t), \mathcal{R}(s_t, E_t), E_t) \leq a_1(x_0)a^t + a_2$$

These assumptions are usual (see Le Van Dana[52]). They are not quite restrictive (see Le Van and Morhaim[33]). They allow to cover situations with unboundedness in the objective function combined with various types of the feasible set as used in the literature (with various returns to scale technology).

They ensure that the assumptions in Section 2.2.2 are satisfied. The assumption that $(Id - \mathcal{D}_c)$ is bijective allows to uniquely express the instantaneous consumption in terms of the waste stocks, the other inputs stocks, and environmental and sustainable variables, and greatly simplifies the notations. The continuity in (S1) and (S2) (i) ensure (F1) so that Γ is compact-valued. The existence of a function e in Assumption (S2) (ii) ensures Assumption (F2) and Assumptions (S3) and (S4) ensure Assumption (A2').

3.3 The general sustainable framework GSF is a particular case of the general history-dependent framework GHDF

In this section, we show that the GSF is a particular case of the GHDF. Recall that the GSF is

$$\mathcal{P} = \begin{cases} \text{Maximize} \sum_{t=0}^{+\infty} \beta^t u(C(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}, E_t, \mathcal{R}(s_t, E_t), E_t) \\ \text{s.t. } \forall t \geq 0, \\ \quad C(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}, E_t) \geq 0 \\ \quad E_{t+1} = G((s_t, \kappa_t, C(s_t, \kappa_t, s_{t+1}, \kappa_{t+1}, E_t)), E_t) \\ \quad \mathcal{R}(s_t, E_t) \in [0, s_t] \\ \quad s_0, \kappa_0 > 0, E_0 \geq 0 \text{ are given} \end{cases}$$

We explain hereafter that the problem is indeed a particular case of our GHDF. Let us define $X := \mathbb{R}^{1+N_\kappa}$ and $x := (s, \kappa)$. Let the memory function $m : X \times X \to Y$ be defined with $Y := \mathbb{R}^{2+2N_\kappa}$ by for any $(x, x') \in X \times X$,

$$m(x, x') = (x, x')$$

Let us define the function $\hat{G}$ by for any $(x, x') \in X \times X, E \in \mathbb{R}^{N_E}$,

$$\hat{G}((x, x'), E) = G((x, C(x, x', E)), E)$$

The adjustment level function $\varphi : l_+^\infty \to Z$ is defined recursively in the following way:

$$\forall \tilde{h} \in l_+^\infty, \forall (x, x') \in X \times X, \varphi(m(x, x'), \tilde{h}) = \hat{G}(m(x, x'), \varphi(\tilde{h}))$$

By defining the instantaneous function F for all $\tilde{h} \in l_+^\infty$ and $x, x' \in \mathbb{R}^2$,

$$F\left(\tilde{h}, x, x'\right) = u(C(x, x', \varphi(\tilde{h})), \mathcal{R}(\pi_1(x), \varphi(\tilde{h})), \varphi(\tilde{h}))$$

and the feasible correspondence Γ,

$$\Gamma(\tilde{h}, x) = \{x' \in (\mathbb{R}^+)^2, C(x, x', \varphi(\tilde{h})) \geq 0\}$$

that can also be written

$$\Gamma(\tilde{h}, x) = g(\varphi(\tilde{h}), x)$$

with the correspondence $g : Z \times X \to X$ defined for all $(k, x) \in Z \times X$ by

$$g(k, x) = \{x' \in \mathbb{R}^2, C(x, x', k) \geq 0\}$$

Then the general sustainable problem is written

$$P(\tilde{h}^{(0)}, x_0) = \begin{cases} \text{Max} & \sum\limits_{t=0}^{+\infty} \beta^t F\big(\tilde{h}^{(t)}, x_t, x_{t+1}\big) \\ \text{s.t.} & \forall t \geq 0, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t) \\ & \forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \ldots.., m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)}) \\ & x_0 > 0 \text{ and } \tilde{h}^{(0)} \in l_+^\infty \text{ are given} \end{cases}$$

which shows that our GSF fits GHDF. Hence, the results shown in Section 2 can be applied to the general sustainable problem. In particular, we obtain the existence of a solution and the dynamic programming tools described in Section 2 can be implemented.

This general sustainable framework can be easily adapted to various contexts and models. In the next section, we give existing models in the related literature as examples of our framework.

4 Some models and examples

In this section, we discuss how our general framework allows to study issues and models from various literature strands. As we introduced a general function allowing to model many different memory processes, general decision variables, objective functions and feasible sets, we are able to encompass many existing models. It generalizes the history-dependent intertemporal optimization models provided in Morhaim and Ulus[40]. Thus, the applications presented in Morhaim and Ulus[40] are encompassed. These include seminal models dealing with habit formation (Ryder and Heal[25], Rozen[44], Rustichini and Siconolfi[46], Caroll, Overland and Weil[10]) and satiation (He, Dyer and Butler[23], Baucells and Sarin[7]). Moreover, it encompasses environmental models (Ikefuji[26], Löfgren[36]) and Safi and Ben Hassen[47]) and optimal management of natural resources (Smulders, Toman and Withagen[48], Ulus[51]), as well as recent circular economy models (George, Chi-ang Lin and Chen[19], Kasioumi[28], Kasioumi and Stengos[31]) and circular and causation models (Donaghy[13, 14]).

We next discuss the way these existing models are particular cases of our general framework as well as how its flexibility allows to use it in future research. Our general framework indeed allows to study simultaneously many effects and contexts: circular models without production waste (Section 4.4), linear economies with production waste, and furthermore circular economies with production waste, as well as other many effects.

4.1 History-dependent optimal growth models with (consumption) habit formation or satiation

In Morhaim and Ulus[40], a representative agent consumes a single good on periods $t = 0, 1, 2, \ldots.$ and maximizes her intertemporal utility over the consumption stream $\tilde{c} = (c_0, c_1, \ldots.)$ in l^∞. At date t, the consumer's instantaneous utility depends on her current consumption c_t. But it also depends on her *time-t (consumption) history* $\tilde{h}^{(t)}$ which is defined from an initial history $\tilde{h}^{(0)} \in l_+^\infty$ and keeps in memory the consumption decisions as follows:

$$\forall t \geq 1, \tilde{h}^{(t)} = (h_j^{(t)})_{j=1}^\infty := (c_{t-1}, \tilde{h}^{(t-1)}) = (c_{t-1}, \ldots., c_0, \tilde{h}^{(0)})$$

Her time t-utility $u : \mathcal{D}_u \subseteq (\mathbb{R}^+ \times R) \to \mathbb{R} \cup \{-\infty\}$ changes endogenously from her time-t consumption history $\tilde{h}^{(t)}$ through the adjustment level function $\varphi : l_+^\infty \to R$ with $R = (\mathbb{R}^+)^n$ (where

$n \geq 1$). For initially given capital stock $k_0 > 0$ and time-0 history $\tilde{h}^{(0)} \in l_+^\infty$, the general framework and optimization problem, with $f : \mathbb{R}^+ \to \mathbb{R}^+$ the production function, k_t the capital stock at time t, and $\beta \in (0,1)$ the fixed discount factor , is given as follows:

$$\mathcal{P}_{u,\varphi,\beta}(k_0, \tilde{h}^{(0)}) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(c_t, \varphi(\tilde{h}^{(t)})) \\ \text{s.t.} & \forall t \geq 0, k_{t+1} = f(k_t) - c_t, k_t \geq 0 \text{ and } c_t \geq 0 \\ & \forall t \geq 1, \tilde{h}^{(t)} = (c_{t-1}, ..., c_1, c_0, \tilde{h}^{(0)}) \\ & \forall t \geq 0, (c_t, \varphi(\tilde{h}^{(t)})) \in \mathcal{D}_u \subseteq (\mathbb{R}^+ \times R) \end{cases}$$

The problem can be rewritten as follows:

$$\mathcal{P}_{u,\varphi,\beta}(k_0, \tilde{h}^{(0)}) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) \\ \text{s.t.} & \forall t \geq 0, k_{t+1} \in [0, f(k_t)] \\ & \forall t \geq 1, \tilde{h}^{(t)} = (f(k_{t-1}) - k_t, f(k_{t-2}) - k_{t-1}, ..., f(k_1) - k_2, f(k_0) - k_1, \tilde{h}^{(0)}) \\ & \forall t \geq 0, (f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) \in \mathcal{D}_u \subseteq (\mathbb{R}^+ \times R) \end{cases}$$

This is a case in which *consumption* is kept in memory, which is a function of previous date and current capital stock decisions. Thus, it means keeping in memory a function of these decisions. Our general history-dependent framework allows, not only to keep in memory *the particular function* defining consumption, but *any function* of the previous date and current decisions. By this way, the model provided in Morhaim and Ulus[40] as well as the models presented in Morhaim and Ulus[40][3] become particular cases of the GHDF. Indeed, let us define $X = Y = \mathbb{R}^+$, $x_t = k_t \in X$ for all $t = 0, \ldots, +\infty$, and $F : l_+^\infty \times X \times X \to \mathbb{R}$ by for any $\tilde{h} \in l_+^\infty$ and $x, x' \in X$,

$$F\big(\tilde{h}, x, x'\big) := u(f(x) - x', \varphi(\tilde{h}))$$

We define for any decision x' given x, the *memory function/process* $m : X \times X \to Y$ by

$$m(x, x') = f(x) - x'$$

and for history $\tilde{h} \in l_+^\infty$ and for any $x \in \mathbb{R}^+$, the feasible correspondence Γ is given by

$$\Gamma(\tilde{h}, x) = \{x' \in [0, f(x)], (f(x) - x', \varphi(\tilde{h})) \in \mathcal{D}_u\}$$

The introduction of the memory function m to the modelling allows to explicitly study many memory processes. Here, it is done by defining m which associates the consumption to the decisions.

4.2 History-dependent optimization models with environmental effects

History-dependence is important in environmental economics models. Morhaim and Ulus[40] already underlined that their general framework allows to deal with environmental effects. We show in this section how Löfgren[36]'s model is written through our new GHDF. Further, we describe how the discrete time version of Ikefuji[26]'s model is also encompassed by GHDF.

Löfgren[36] proposes a model with environmental quality habit formation and in which a consumption good moreover causes a negative external effect on the environment. The social planner maximizes the

[3] including Ryder and Heal[25], Rozen[44], Rustichini and Siconolfi[46], Caroll, Overland and Weil[10], He, Dyer and Butler[23], Baucells and Sarin[7], Ikefuji[26], Löfgren[36], Safi and Ben Hassen[47].

utility given the negative effect of the consumption good on the environment and taking into account that there is habit formation in environmental quality. The instantaneous utility $u(n_t, x_t, z_t, s_t)$ depends on n_t which is the environment that displays habit formation, x_t the "dirty" consumption good (the environmental bad), z_t the "clean" consumption good and s_t the habit level related to the environment. The following relations are satisfied with $\gamma \in (0,1), \beta \in (0,1), \delta \in (0,1)$, y being an exogenously given income and n is a given initial environment

$$\begin{cases} n_t = n - \gamma x_t \\ z_t = y - x_t \\ s_{t+1} = \beta n_t + (1-\delta) s_t \end{cases}$$

Let us define $X = Y = \mathbb{R}^+$ $x_t = k_t \in X$ for all $t = 0 \dots \infty$, and φ is defined through the following recurrence relation

$$\forall c \in \mathbb{R}^+, \forall \tilde{h} \in l_+^\infty, \varphi((c, \tilde{h})) = \beta n + (1-\delta)\varphi(\tilde{h}) - \beta\gamma c$$

Let us define $F : l_+^\infty \times X \times X \to \mathbb{R}$ by, for any $\tilde{h} \in l_+^\infty$ and $x, x' \in X$,

$$F\big(\tilde{h}, x, x'\big) := u(n - \gamma(f(x) - x'), f(x) - x', y - (f(x) - x'), \varphi(\tilde{h}))$$

We define for any decision x' given x, the *memory function/process* $m : X \times X \to Y$ by $m(x, x') = f(x) - x'$, and for any $\tilde{h} \in l_+^\infty$ and $x \in \mathbb{R}^+$, the feasible correspondence Γ is given by

$$\Gamma(\tilde{h}, x) = \{x' \in [0, f(x)], (n - \gamma(f(x) - x'), f(x) - x', y - (f(x) - x'), \varphi(\tilde{h})) \in \mathcal{D}_{\tilde{u}}\}$$

Thus, Löfgren[36]'s model is encompassed in our general framework.

Ikefuji[26] studies habit formation in consumption and pollution abatement activities when agents derive disutility both from the habit stock and pollution. The pollution P_t in period t is generated by the capital stock k_t used in production and reduced by abatement activities a_t in the same period. The problem is written as follows.

$$P(h_0, k_0, m_0) = \begin{cases} \text{Max} & \sum_{t=0}^{+\infty} \beta^t u(c_t, H_t, P_t) \\ \text{s.t.} & \forall t \geq 1, H_{t+1} = \rho c_t + (1-\rho) H_t \text{ and } P_t = \big(\frac{k_t}{a_t}\big)^\phi \\ & \forall t \geq 0, k_{t+1} = A k_t - c_t - a_t - k_t \\ & k_0 > 0, m_0 > 0 \text{ and } h_0 \text{ are given} \end{cases}$$

where c_t denotes the consumption in period t, H_t denotes the consumption habit, and P_t is the level of aggregated pollution in the economy.

In order to cover this problem in our general framework, let us define $X = (\mathbb{R}^+)^2, Y = \mathbb{R}^+$, and $x_t = (k_t, a_t) \in X$ for all $t = 0 \dots \infty$. The function $\varphi : l_+^\infty \to \mathbb{R}$ is defined by $\varphi(\tilde{h}^{(0)}) = h_0$ and

$$\forall \tilde{h} \in l_+^\infty, \forall c \in \mathbb{R}_+, \varphi(c, \tilde{h}) = G(c, \varphi(\tilde{h})).$$

with the function G defined by $G(c, y) = \rho c + (1-\rho) y$. Also, let us define $F : l_+^\infty \times X \times X \to \mathbb{R}$ by, for any $\tilde{h} \in l_+^\infty$ and $x, x' \in X$,

$$F\big(\tilde{h}, x, x'\big) := u\big(f(\pi_1(x)) - \pi_1(x'), \varphi(\tilde{h}), \big(\frac{\pi_1(x)}{\pi_2(x)}\big)^\phi\big)$$

We define for any decision x' given x, the *memory function/process* $m : X \times X \to Y$ by

$$m(x,x') = f(\pi_1(x)) - \pi_1(x')$$

and for history $\tilde{h} \in l_+^\infty$ and for any $x \in \mathbb{R}^+$, the feasible correspondence Γ is given by

$$\Gamma(\tilde{h},x) = \{x' \in [0, f(\pi_1(x))], \pi_2(x') > 0, \big(f(\pi_1(x)) - \pi_1(x'), \varphi(\tilde{h}), (\frac{\pi_1(x)}{\pi_2(x)})^\phi\big) \in \mathcal{D}_u\},$$

showing that our GHDF fits Ikefuji[26]'s model.

These two examples illustrate how our framework is general and easy to use, while keeping the environmental effects to be studied visible and interpretable. In the next section, we show how it is also fitted to optimal management of natural resources.

4.3 Optimal management of natural resources

Smulders, Toman and Withagen[48] and Ulus[51] present models referring to the optimal management of exhaustible natural resources like oil, coal, gas and etc. In these models, there exists a single planner who manages the decision of extraction and consumption of an exhaustible natural resource at each period t in order to maximize the intertemporal utility. At each period t, the instantaneous utility depends on its consumption c_t and also on the stock of the natural resource, which is denoted by s_t. The output $y_t \in \mathbb{R}_+$ is produced from capital (k_t) and extracted quantity of resource (r_t) (or the extraction flow of natural resource per unit of time) by a production function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ where $y_t = f(k_t, r_t)$. The output is either consumed as $c_t \geq 0$ or saved as capital to the next period as k_{t+1} satisfying:

$$c_t + k_{t+1} \leq f(k_t, r_t) + (1-\delta)k_t \text{ with } k_t \geq 0$$

where δ stands for the depreciation rate of capital. For initially given stocks of $k_0, s_0, r_0 > 0$ and $\beta \in (0,1)$ the fixed discount factor, the problem of the planner is given as follows:

$$\mathcal{P}\big((k_0,r_0),s_0)\big) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(c_t, s_t) \\ \text{s.t.} & \forall t \geq 0, s_{t+1} = s_t - r_t \\ & \forall t \geq 0, 0 \leq c_t = f(k_t, r_t) + (1-\delta)k_t - k_{t+1} \\ & s_0, k_0, r_0 > 0 \text{ are given} \end{cases}$$

which can be rewritten as

$$\mathcal{P}\big((k_0,r_0),s_0)\big) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(f(k_t, r_t) + (1-\delta)k_t - k_{t+1}, s_t) \\ \text{s.t.} & \forall t \geq 0, s_{t+1} = s_t - r_t \\ & \forall t \geq 0, 0 \leq k_{t+1} \leq f(k_t, r_t) + (1-\delta)k_t \\ & s_0, k_0, r_0 > 0 \text{ are given} \end{cases}$$

Let us define G by $G(s,r) := s - r$ and φ by the law of motion

$$\forall t \geq 0, \varphi(\tilde{h}^{(t+1)}) = G(r_t, \varphi(\tilde{h}^{(t)}))$$

so that the instantaneous utility $u(c_t, s_t)$ is $u(c_t, \varphi(\tilde{h}^{(t)}))$. By defining $\tilde{h}^{(0)}$ such that $\varphi(\tilde{h}^{(0)}) = s_0$ and for any $t \geq 0$, $\tilde{h}^{(t+1)}$ such that $\tilde{h}^{(t+1)} = (r_t, \tilde{h}^{(t)})$, the model can be written

$$\mathcal{P}_{u,\varphi,\beta}(k_0, \tilde{h}^{(0)}) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(f(k_t, r_t) + (1-\delta)k_t - k_{t+1}, \varphi(\tilde{h}^{(t)})) \\ \text{s.t.} & \forall t \geq 0, k_{t+1} \in [0, f(k_t, r_t) + (1-\delta)k_t] \\ & \forall t \geq 0, \tilde{h}^{(t+1)} = (r_t, ..., \tilde{h}^{(0)}) \end{cases}$$

The model can be written in our general framework by defining $x := (k, r)$ and the memory function by $m(x, x') = \pi_2(x)$. Let us define the objective function F by, for any $(x, x', \tilde{h})$,

$$F(x, x', \tilde{h}) := u\big(f(\pi_1(x), \pi_2(x)) + (1-\delta)\pi_1(x) - \pi_1(x'), \varphi(\tilde{h})\big)$$

and the feasible set by, for any $(\tilde{h}, x)$,

$$\Gamma(\tilde{h}, x) := \{x', 0 \leq \pi_1(x') \leq f(\pi_1(x), \pi_2(x)) + (1-\delta)\pi_1(x), 0 \leq \pi_2(x') \leq \varphi(\tilde{h})\}$$

Here G is defined by $G(s, r) := s - r$. However, our framework allows to consider more general forms of G, for example if it comes to taking into account the resources regeneration (whether naturally or otherwise).

In the next section, we show that our results are also fitted for circular economy issues.

4.4 Circular economy (CE) and circular and cumulative causation (CCC) models

Circular economy is most frequently depicted as a combination of reduce, reuse and recycle activities (Kirchherr, Reike and Hekkert[32]), but also design, implying a focus on the entire life cycle of the processes as well as the interaction between the process and the environment and the economy in which it is embedded (Ghisellini, Cialani and Ulgiati[20]).

In this section, we show that the circular economy models provided by George, Chi-ang Lin and Chen[19], Kasioumi and Stengos[30], Kasioumi[29] and Donaghy[13, 14] are particular cases of our general sustainable framework. We first consider circular economy models without recycling habits (George, Chi-ang Lin and Chen[19] and Kasioumi and Stengos[30]), then circular economy models with recycling habits (Kasioumi[28]). We also show that our framework is adapted to the circular and cumulative causation models as developed by Donaghy[13, 14] .

4.4.1 CE models without recycling habits

In this section, we consider the circular models without recycling habits proposed by George, Chi-ang Lin and Chen[19], and Kasioumi and Stengos[30].

George, Chi-ang Lin and Chen[19] and Kasioumi and Stengos[30] consider a closed economy with zero population growth. They abstract from capital accumulation and technical progress[4]. The social planner maximizes an intertemporal utility where the instantaneous utility $u(c, P)$ depends on consumption c and the stock of pollution P. The output is produced via a (concave) production function ϕ using two factors of production, one which corresponds to the rate of use of the recyclable

[4]However, our results apply to such models with capital accumulation and technical progress (see Section 4.4.3).

resource and another (z) corresponding to the rate of use of the environmentally polluting resource. Output produced in any given period but not consumed or used for the employment of the polluting resource, accumulates as (potentially recyclable) waste. Recycling turns waste into a useful factor of production: a proportion b of the waste stock s with intensity of recycling τ is supposed to be recycled each period. George, Chi-ang Lin and Chen[19]'s model is a particular case of Kasioumi and Stengos[30] in which the intensity of recycling τ is equal to one. The social planner solves the following optimization problem

$$\mathcal{P}\big((s_0,z_0),p_0\big) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(c_t,p_t) \\ \text{s.t.} & \forall t \geq 0, s_{t+1} - s_t = \phi(\tau b s_t, z_t) - c_t - \alpha z_{t+1} - \tau b s_t \\ & \forall t \geq 0, p_{t+1} - p_t = \theta z_t - \delta p_t + (1-b)s_t \\ & s_0, z_0 > 0, p_0 \geq 0 \text{ are given} \end{cases}$$

By defining $\kappa := z$ (thus $N_\kappa = 1$), $E := p$ (thus $N_E = 1$), $f(s,\kappa,E) := \phi(\tau bs,\kappa)$, $\mathcal{D}_p(s,\kappa) := 0$, $\mathcal{D}_c(c) := 0$, $\mathcal{I}(s,\kappa,s',\kappa') := \alpha\kappa'$, $\mathcal{R}(s,E) := \tau bs$ and $G((s,\kappa,c),E) := (1-\delta)E + \theta\kappa + (1-b)s$, both models (George, Chi-ang Lin and Chen[19] and Kasioumi and Stengos[30])) fit our general sustainable framework.

4.4.2 CE models with recycling habits

In this section, we consider the circular models with recycling habits proposed by Kasioumi[28, 29]. It is an extension of the theoretical work of Kasioumi and Stengos[30], combining elements of the circular economy model of George, Chi-ang Lin and Chen[19] with the habit formation framework of Ikefuji[26]. Kasioumi[28, 29] deals with a fixed intensity of recycling while Kasioumi[29] deals further with an intensity $\tau(H_t)$ depending (through an affine function $\tau(H_t) := \eta + rH_t$) on the level of recycling habits. The social planner solves the following optimization problem

$$\mathcal{P}\big((s_0,z_0),p_0\big) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(c_t,r_t,H_t,p_t) \\ \text{s.t.} & \forall t \geq 0, s_{t+1} - s_t = \phi(\tau(H_t) b s_t, z_t) - c_t - \alpha z_{t+1} - \tau(H_t) b s_t \\ & \forall t \geq 0, p_{t+1} - p_t = \theta z_t - \delta p_t + (1-b)s_t + \rho c_t \\ & H_{t+1} - H_t = \mu(\tau(H_t) b s_t - H_t) \\ & s_0, z_0 > 0, p_0 \geq 0 \text{ are given} \end{cases}$$

By defining $\kappa := z$ (thus $N_\kappa = 1$), $E := (p,H)$ (thus $N_E = 2$), $f(s,\kappa,E) := \phi(\tau(\pi_2(E))bs, e)$, $\mathcal{D}_p(s,\kappa) := 0$, $\mathcal{D}_c(c) := 0$, $\mathcal{I}(s,\kappa,s',\kappa') := \alpha\kappa'$, $\mathcal{R}(s,E) := \tau(\pi_2(E))bs$ and

$$G((s,\kappa,c),E) := \big((1-\delta)\pi_1(E) + \theta\kappa + (1-b)s + \rho c, \mu(\tau(\pi_2(E))bs - \pi_2(E)\big)$$

these models (Kasioumi[28, 29]) fit our general sustainable framework. Note that our framework fits also recycling intensity functions τ that are not necessarily affine.

4.4.3 CCC models

The theoretical circular-economy model of economic growth with circular and cumulative causation (CCC) is presented in Donaghy[13, 14] as follows. The George Lin and Chen[19]'s model is modified by modelling capital formation and technical change and including physical capital K, human capital

HC, labor L, and other materials OM, as productive factors. The optimization problem for the social planner is to choose control variables, consumption c and environmentally polluting resource z but also invesments in physical and social capital to maximize the intertemporal utility function.

Given the functions $b : \mathbb{R} \to \mathbb{R}$ accounting for increased efficiencies in recycling (thus depends on the accumulated recycling experience $R \in \mathbb{R}$), the function $\vartheta : \mathbb{R}^2 \to \mathbb{R}$ characterizing technical progress in reducing pollution from the polluting resource as a function of human capital deepening (thus depends on $(K, HC) \in \mathbb{R}^2$), the production function ϕ, and the instantaneous utility function u depending on consumption and pollution $(c, P) \in \mathbb{R}^2$, the model in Donaghy[13, 14] is as follows.

$$\mathcal{P} = \begin{cases} \text{Maximize} \sum_{t=0}^{+\infty} \beta^t u(c_t, P_t) \\ \\ \text{s.t. } \forall t \geq 0, \\ s_{t+1} - s_t = \phi(s_t, z_t, K_t, HC_t, L_t, OM_t, R_t) - c_t - \alpha z_{t+1} \\ \qquad -(K_{t+1} - K_t)(1 + \frac{a_1}{2K_t}) - (HC_{t+1} - HC_t)(1 + \frac{a_2}{2HC_t}) - b(R_t)s_t \\ P_{t+1} - P_t = \vartheta(K_t, HC_t)z_t - \delta P_t + (1 - b(R_t))s_t \\ R_{t+1} - R_t = b(R_t)s_t \\ s_0, z_0 > 0, P_0 \geq 0 \text{ are given} \\ K_0, R_0, HC_0, L_0 \geq 0 \text{ are given} \end{cases}$$

Let us define $\kappa := (z, K, L, HC, OM)$ (thus $N_\kappa = 5$), $E := (P, R)$, $\mathcal{I}(s, \kappa, s', \kappa') := \alpha\pi_1(\kappa') + (\pi_2(\kappa') - \pi_2(\kappa))(1 + \frac{a_1}{2\pi_2(\kappa)}) + (\pi_4(\kappa') - \pi_4(\kappa))(1 + \frac{a_2}{2\pi_4(\kappa)})$ with for any $i = 1, ..., 5$, π_i is the i-th projection[5], $\mathcal{D}_p(s, \kappa) = 0$, $\mathcal{D}_c(c) = 0$ and $\mathcal{R}(s, E) = b(\pi_2(E))s$ and G by

$$G((s, \kappa, c), E) = \big(\vartheta(\pi_2(\kappa), \pi_4(\kappa))\pi_1(\kappa) + (1 - \delta)\pi_1(E) + (1 - b(\pi_2(E)))s, \pi_2(E) + b(\pi_2(E))s\big).$$

The sustainable process or design variable r does not enter directly Donaghy's utility function in contrast with this possibility which is allowed in our framework. When the process does not enter directly in the utility, it suffices to define the utility function $\hat{u}(c, r, M) := u(c, M)$. This shows that circular and cumulative causation models fit our GSF. Note that Donaghy[13] considers the particular functions $b(R) = \left(\frac{\phi}{R} + \zeta\right)^{-1}$, $\vartheta(K, HC) = \left[g - \chi(\frac{HC}{K})^\epsilon\right]$, $u(c, P) = [\frac{1}{\gamma}\left(\omega c P^{-\eta}\right)^\gamma]$ and $\phi(s, z, K, HC, L, OM, R) = \psi[\theta_1(b(R)s)^{-\sigma} + \theta_2 z^{-\sigma} + \theta_3 K^{-\sigma} + \theta_4 HC^{-\sigma} + \theta_5 L^{-\sigma} + \theta_6 OM^{-\sigma}]^{-\frac{\kappa}{\sigma}}$, whereas we allow general functions.

4.4.4 On the circular and sustainable economy research agenda

The research on circular and sustainable economy is currently vivid. Our framework is fitted to consider and interlink economic, environmental, technological and social issues. The GSF can easily be adapted to already suggested paths for future research (Donaghy[13, 14], Ghisellini[20]) and the extensive literature that is developing. The way the GSF may incorporate the history-dependence viewpoint, and the memory formation that we introduce, open perspectives towards several aspects and interpretations. In particular, as a by-product, the GSF allows to deal with many important features that are coming to be taken into account, such as recycling, reuse, reduction, design, habits, activities of harvesting exhaustible and renewable resources, the assimilative capacity of the natural environment for (non-recyclable) waste, transport activities, management of resources, interaction

[5] i.e. $\pi_i : \mathbb{R}^5 \to \mathbb{R}$ is defined by, for any $a = (a_j)_{j=1}^5 \in \mathbb{R}^5, \pi_i(a) = a_i$

between the processes and the environment, preventative and regenerative eco-industrial development, etc. These may be interconnected.

Our framework is fitted for future research as it is amenable not only to treat various sustainable and environmental issues but also allows to interlink these with many kinds of effects and history-dependencies (consumption, production, saving and investment, human capital, labor, consumption habits, recycling habits, pollution, stock of waste, social and legal norms). The way the GSF may incorporate the history-dependence viewpoint and the memory formation that we introduce open perspectives towards several aspects and interpretations. In particular, the GSF allows to deal with many important features that are coming to be taken into account, such as recycling, reuse, reduction, design, habits, activities of harvesting exhaustible and renewable resources, the assimilative capacity of the natural environment for (non-recyclable) waste, transport activities, management of resources, interaction between the processes and the environment, preventative and regenerative eco-industrial development, etc. We discuss the way existing models are particular cases of our general framework as well as how its flexibility allows to use it in future research. As an example, the sustainable process or design variable r does not enter directly Donaghy's utility function in contrast with this possibility which is allowed in our framework. The GSF allows to study simultaneously many effects and contexts: circular models without production waste, linear economies with production waste, and furthermore circular economies with production waste, as well as other many effects. These may be interconnected: our general framework allows to enrich the analysis by treating at the same time several kinds of history dependencies. This is crucial for sustainability issues as they involve on one hand habits (consumption habits but also recycling habits), and on another hand pollution stocks and environmental quality (Mazar[38], Moreau[39]). Such a modelling choice also emphasizes the fact that the environment keeps in memory our activities and decisions.

Following the research program suggested by Donaghy[13, 14], our modelling is a step towards a clearer view on interlinked economic activities and surroundings, which are crucial when dealing with environmental issues and circular economies, thus helping in managing transitions in the economy, and in particular transitions to a circular economy (or system of circular economies).

5 Conclusion

In this paper, we introduced a new framework for the analysis of decisions' processes with history-dependencies and surroundings. The modeling we provide is very general and flexible while we keep the framework being interpretable and tractable. It is designed to be adapted to various surroundings and different kinds of history-dependencies. It allows to study the history-dependencies formation, and to highlight how history dependencies affect our decisions as well as how they are affected by our decisions.

We have developed dynamic programming tools (in particular, existence of a solution and that the value function is the unique fixed point of the Bellman operator) to solve such models.

Since environmental and sustainable variables are influenced by (the memory of our past) decisions and can be taken as surroundings, as a by-product, we introduced a very general sustainable framework which fits many existing environmental and sustainable models including circular economy models. It provides a basis for future environmental analysis (as we discussed in Section 4.4.4).

Our framework opens new paths towards the interdependencies between surroundings and decisions. It is amenable not only to treat various issues and phenomena from diverse areas, as in Law, legal decisions and the Rule (Lewis[35], Farber[16]), dynamic models of Law ([8]), in Game Theory, actions and social norms (Acemoglu and Jackson[2]), in Experimental Economics, decision making and social norms (Vostroknutov[54]), Artificial Intelligence, Reinforcement Learning (Tennenholtz, Merlis, Shani, Mladenov and Boutilier[50]). It also allows to interlink these with many kinds of effects and history-dependencies.

6 Appendix

6.1 Proof of Proposition 2.1

By the assumption $(\mathcal{A})$, along a feasible path $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(x_0, \tilde{h}^{(0)})$, with associated history $\tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \ldots.., m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)})$,

$$\forall t, F(\tilde{h}^{(t)}, x_t, x_{t+1}) \leq a_1(x_0)a^t + a_2$$

Then for all $T \in \mathbb{N}$,

$$\begin{aligned} \sum_{t=0}^{T} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) &\leq \sum_{t=0}^{T} \beta^t (a_1(x_0)a^t + a_2) \\ &= a_1(x_0) \sum_{t=0}^{T} (a\beta)^t + a_2 \sum_{t=0}^{T} \beta^t \end{aligned}$$

since $0 < a\beta < 1$ and $0 < \beta < 1$ the conclusion follows.

6.2 Proof of Proposition 2.2

Let us show that the objective is upper semi-continuous and the feasible sequence set is a compact set of the product topology. By (F), one can check by induction that for any given $k_0 \in \mathbb{R}^+$ and $\tilde{h}^{(0)} \in l_+^\infty$, for all feasible sequence $\tilde{x} = (x_t)_t \in \Pi(\tilde{h}^{(0)}, x_0)$, for all $t, x_{t+1} \in \Gamma(\tilde{h}^{(t)}, x_t)$, we have

$$\forall t, x_t \leq a^t \|x_0\| + \frac{1 - a^t}{1 - a} a' = (\|x_0\| - \frac{a'}{1 - a})a^t + \frac{a'}{1 - a}$$

The feasible set $\Pi(\tilde{h}^{(0)}, x_0)$ is included in a compact set for the product topology. Moreover, it is closed. So the feasible set $\Pi(\tilde{h}^{(0)}, x_0)$ is also compact.

We next show that the objective function $\mathcal{U}$ is upper semi-continuous.

Let us consider a sequence $\tilde{x}^n = \{(x_t^n)_{t=1}^{+\infty}\}_n \subset \Pi(\tilde{h}^{(0)}, x_0)$ that converges to $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(\tilde{h}^{(0)}, x_0)$. Note that when n converges to $+\infty$, by (m1), the sequence of associated histories $\forall t \geq 1, \widetilde{h^n}^{(t)} = (m(x_{t-1}^n, x_t)^n, m(x_{t-2}^n, x_{t-1}^n), \ldots.., m(x_1^n, x_2^n), m(x_0^n, x_1^n), \tilde{h}^{(0)})$ converges to the associated history $\forall t \geq 1, \tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), \ldots.., m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)})$.

Let us show that $\overline{\lim}_{n \to +\infty} \mathcal{U}(\tilde{x}^n) \leq \mathcal{U}(\tilde{x})$. The notation $\overline{\lim}$ means $\limsup$.

For any $t \geq 0$, by (A2), for any $\tilde{x} = (x_t)_{t=1}^{+\infty} \in \Pi(\tilde{h}^{(0)}, x_0)$ and with history $\tilde{h}^{(t)}$ associated to $\tilde{x}$

$$F^+(\tilde{h}^{(t)}, x_t, x_{t+1}) \leq a_1(x_0)a^t + a_2$$

and by $0 < a\beta < 1$, for any $\varepsilon > 0$, there exists T_ε such that for any $(x_t)_{t=1}^{+\infty} \in \Pi(\tilde{h}^{(0)}, x_0)$, and for any $T \geq T_\varepsilon$,

$$\sum_{t=T}^{+\infty} \beta^t F^+(\tilde{h}^{(t)}, x_t, x_{t+1}) \leq \varepsilon$$

So for any $\varepsilon > 0$, there exists T_ε such that for any $n \in \mathbb{N}$ and for any $T \geq T_\varepsilon$,

$$\sum_{t=T}^{+\infty} \beta^t F^+(\widetilde{h^n}^{(t)}, x_t^n, x_{t+1}^n) \leq \varepsilon$$

and for any $n \in \mathbb{N}$ and for any $T \geq T_\varepsilon$,

$$\begin{aligned}
\sum_{t=0}^{+\infty} \beta^t F(\widetilde{h^n}^{(t)}, x_t^n, x_{t+1}^n) &= \sum_{t=0}^{T} \beta^t F(\widetilde{h^n}^{(t)}, x_t^n, x_{t+1}^n) + \sum_{t=T}^{+\infty} \beta^t F(\widetilde{h^n}^{(t)}, x_t^n, x_{t+1}^n) \\
&\leq \sum_{t=0}^{T} \beta^t F(\widetilde{h^n}^{(t)}, x_t^n, x_{t+1}^n) + \sum_{t=T}^{+\infty} \beta^t F^+(\widetilde{h^n}^{(t)}, x_t^n, x_{t+1}^n) \\
&\leq \sum_{t=0}^{T} \beta^t F(\widetilde{h^n}^{(t)}, x_t^n, x_{t+1}^n) + \varepsilon
\end{aligned}$$

By taking $n \to +\infty$ (and using the continuity of F in the right-hand side of the above inequality),

$$\overline{\lim_{n\to+\infty}} \sum_{t=0}^{+\infty} \beta^t F(\widetilde{h^n}^{(t)}, x_t^n, x_{t+1}^n) \leq \sum_{t=0}^{T} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) + \varepsilon$$

Since this is true for any $T \geq T_\varepsilon$, by taking $T \to +\infty$,

$$\overline{\lim_{n\to+\infty}} \sum_{t=0}^{+\infty} \beta^t F(\widetilde{h^n}^{(t)}, x_t^n, x_{t+1}^n) \leq \sum_{t=0}^{+\infty} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) + \varepsilon$$

Since this is true for any $\varepsilon > 0$, by taking $\varepsilon \to 0$,

$$\overline{\lim_{n\to+\infty}} \sum_{t=0}^{+\infty} \beta^t F(\widetilde{h^n}^{(t)}, x_t^n, x_{t+1}^n) \leq \sum_{t=0}^{+\infty} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1})$$

So $\mathcal{U}$ is upper semi-continuous on $\Pi(\tilde{h}^{(0)}, x_0)$.

By Weierstrass Theorem (Aubin[4], Theorem 5.3.1), since $\mathcal{U}$ is upper semi-continuous and $\Pi(\tilde{h}^{(0)}, x_0)$ is a compact set for the product topology, there exists an optimal solution.

The assumptions that F is jointly strictly concave ensures the uniqueness of the solution.

6.3 Proof of Proposition 2.3

A direct proof using (A2) can be done. Indeed, let us consider a sequence $(\widetilde{h^n}^{(0)}, x_0^n)_n \subset l_+^\infty \times X$ that converges to $(\tilde{h}^{(0)}, x_0) \in l_+^\infty \times X$, use the fact that x_0^n converges to x_0 and let us consider a subsequence $(\widetilde{h^{n_i}}^{(0)}, x_0^{n_i})_i$ such that

$$\overline{\lim_{n\to+\infty}} V(\widetilde{h^n}^{(0)}, x_0^n) = \lim_{i\to+\infty} V(\widetilde{h^{n_i}}^{(0)}, x_0^{n_i})$$

Let $\varepsilon > 0$. By (A2), there exist i_0 and T_0 such that for any $i \geq i_0$ and for any $T \geq T_0$, and for optimal path $(\tilde{x}^{n_i})_i \in \Pi(\widetilde{h^{n_i}}^{(0)}, x_0^{n_i})$ and its associated history $\widetilde{h^{n_i}}^{(t)}$,

$$V(\widetilde{h^{n_i}}^{(0)}, x_0^{n_i}) = \sum_{t=0}^{+\infty} \beta^t F(\widetilde{h^{n_i}}^{(t)}, x_t^{n_i}, x_{t+1}^{n_i}) \leq \sum_{t=0}^{T} \beta^t F(\widetilde{h^{n_i}}^{(t)}, x_t^{n_i}, x_{t+1}^{n_i}) + \varepsilon$$

Fix $T \geq T_0$. The subsequence $(\tilde{x}^{n_i})_i$ that belongs[6] to $\Pi(\widetilde{h^{n_i}}^{(0)}, x_0^{n_i})$ can be assumed to converge to some $\tilde{x}$ in $\Pi(\tilde{h}^{(0)}, x_0)$. By the definition of the associated history and the continuity of m, this implies that $(\widetilde{h^{n_i}}^{(t)})_i$ converges to $\tilde{h}^{(t)}$ the history associated to $\tilde{x}$.
Let $i \to +\infty$, by the continuity of F,

$$\overline{\lim_{n\to+\infty}} V(\widetilde{h^n}^{(0)}, x_0^n) \leq \sum_{t=0}^{T} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) + \varepsilon$$

Let $T \to +\infty$,

$$\overline{\lim_{n\to+\infty}} V(\widetilde{h^n}^{(0)}, x_0^n) \leq \mathcal{U}(\tilde{x}) + \varepsilon \leq V(\tilde{h}^{(0)}, x_0)$$

by the arbitrariness of ε.

6.4 Proof of Proposition 2.4

One can check (see Le Van and Morhaim[34]) that (F2) and (A) imply that
(H) $\forall x_0 \in X, \exists \mathcal{V}(x_0)$ a compact neighborhood of x_0 in X, $\forall \varepsilon > 0, \exists T_0$ such that $\forall T \geq T_0, \forall x_0' \in \mathcal{V}(x_0), \forall \tilde{x}' \in \Pi(\tilde{h}^{(0)}, x_0')$, one has with $\tilde{h}'^{(t)} = (m(x_{t-1}', x_t'), m(x_{t-2}', x_{t-1}'), \ldots.., m(x_1', x_2'), m(x_0', x_1'), \tilde{h}^{(0)})$,

$$\sum_{t=T}^{+\infty} \beta^t F^+(\tilde{h}'^{(t)}, x_t', x_{t+1}') \leq \varepsilon$$

where $F^+(h, r, r') = \max\{0, F(h, r, r')\}$.

(i) By (H), $\exists T_0, \forall T > T_0, \forall x_0' \in \mathcal{V}(x_0), \varepsilon > 0, \forall \tilde{x}' \in \Pi(\tilde{h}^{(0)}, x_0)$,

$$\sum_{t=T}^{+\infty} \beta^t F(\tilde{h}'^{(t)}, x_t', x_{t+1}') \leq \sum_{t=T}^{+\infty} \beta^t F^+(\tilde{h}'^{(t)}, x_t', x_{t+1}') \leq \varepsilon$$

[6]by the compactness of $\Pi(\widetilde{h^{n_i}}^{(0)}, x_0^{n_i})$

Let $\tilde{x}' \in \Pi(\tilde{h}^{(0)}, x'_0), T \geq T_0$. For any $\tilde{x}'' = (x''_{T+1}, ...) \in \Pi(\tilde{h}'^{(T)}, x'_T)$, one has $(x'_1, ..., x'_T, x''_{T+1}, ...) \in \Pi(\tilde{h}^{(0)}, x'_0)$, and

$$\beta^T F(\tilde{h}'^{(T)}, x'_T, x''_{T+1}) + \beta^{T+1} F(\tilde{h}'^{(T+1)}, x''_{T+1}, x''_{T+2}) + ... \leq \varepsilon$$

so $\beta^T V(\tilde{h}^{(T)}, x'_T) \leq \varepsilon$ which implies (i).

(ii) $\forall \tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$,

$$-\infty < \mathcal{U}(\tilde{x}) \leq \sum_{t=0}^{T} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) + \beta^{T+1} V(\tilde{h}^{(T+1)}, x_{T+1})$$

and

$$0 = \lim_{T \to +\infty} [\mathcal{U}(\tilde{x}) - \sum_{t=0}^{T} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1})] \leq \underline{\lim}_{T \to +\infty} \beta^{T+1} V(\tilde{h}^{(T+1)}, x_{T+1})$$

The notation $\underline{\lim}$ means lim inf. From (i) then $\lim_{T \to +\infty} \beta^{T+1} V(\tilde{h}^{(T+1)}, x_{T+1}) = 0$

6.5 Proof of Proposition 2.6

The proof that V is a fixed-point of the Bellman operator is standard (see Stokey Lucas and Prescott[49]). Uniqueness of the fixed point is shown by contradiction. Indeed, suppose there exists W another fixed-point of $\mathcal{B}$ in $\mathcal{F}_b(l_+^\infty \times X, \mathbb{R})$. Let us first check that $W \leq V$. Let $(\tilde{h}^{(0)}, x_0)$ be given. There exists $x_1 \in \Gamma(\tilde{h}^{(0)}, x_0)$ such that $W(\tilde{h}^{(0)}, x_0) = F(\tilde{h}^{(0)}, x_0, k_1) + \beta W(\tilde{h}^{(1)}, x_1)$ and by induction, there exists a sequence $(x_t)_{t \geq 1}$ with associated history sequence $(\tilde{h}^{(t)})_t$ such that for any T,

$$W(\tilde{h}^{(0)}, x_0) = \sum_{t=0}^{T-1} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) + \beta^T W(\tilde{h}^{(T)}, x_T)$$

Since W belongs to $\mathcal{F}_b(l_+^\infty \times X, \mathbb{R})$, one has taking the limit when $T \to +\infty$, and then by V being the sup of the sum

$$W(\tilde{h}^{(0)}, x_0) \leq \sum_{t=0}^{+\infty} \beta^t F(\tilde{h}^{(t)}, x_t, x_{t+1}) \leq V(x_0, \tilde{h}^{(0)})$$

Let us now show that $V \leq W$. Let $x_0 \in X, \tilde{h}^{(0)} \in l_+^\infty$. For any $\tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$ such that $\mathcal{U}(\tilde{x}) > -\infty$, one has, with $\tilde{h}^{(1)} = (m(x_0, x_1), \tilde{h}^{(0)})$,

$$\begin{aligned} W(\tilde{h}^{(0)}, x_0) &= BW(\tilde{h}^{(0)}, x_0) \\ &\geq F(\tilde{h}^{(0)}, x_0, x_1) + \beta W((m(x_0, x_1), \tilde{h}^{(0)})), x_1) \\ &= F(\tilde{h}^{(0)}, x_0, x_1) + \beta W(\tilde{h}^{(1)}, x_1) \end{aligned}$$

and so by induction, with $\tilde{h}^{(t)} = (m(x_{t-1}, x_t), m(x_{t-2}, x_{t-1}), ..., m(x_1, x_2), m(x_0, x_1), \tilde{h}^{(0)})$,

$$\begin{aligned} W(k_0, \tilde{h}^{(0)}) &\geq \sum_{t=0}^{T} F(\tilde{h}^{(t)}, x_t, x_{t+1}) + \beta^{T+1} W(\tilde{h}^{(T+1)}, x_{T+1}) \\ &\geq \lim_{T \to +\infty} \sum_{t=0}^{T} F(\tilde{h}^{(t)}, x_t, x_{t+1}) + \lim_{T \to +\infty} \beta^{T+1} W(\tilde{h}^{(T+1)}, x_{T+1}) \\ &= \mathcal{U}(\tilde{x}) \end{aligned}$$

which implies that $W(\tilde{h}^{(0)}, x_0) \geq V(\tilde{h}^{(0)}, x_0)$ (since for any $\tilde{x} \in \Pi(\tilde{h}^{(0)}, x_0)$, one has $W(\tilde{h}^{(0)}, x_0) \geq \mathcal{U}(\tilde{x})$ and $V(\tilde{h}^{(0)}, x_0)$ is the sup of $\mathcal{U}(\tilde{x})$ for $\tilde{x}$ in $\Pi(\tilde{h}^{(0)}, x_0)$).

Finally, this shows that dynamic programming tools can be used to deal with general history-dependent optimal growth models.

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