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A Comprehensive Analysis of Production Efficiency: A Tax Reform Perspective

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Abstract

Policies that impact the production sector, such as intermediate goods taxation (e.g. taxing robots) and trade liberalization create winners and losers. When do we need to integrate pre-distribution concerns in the design of these production policies? Should we consider the endogenous changes of factor prices in tax formulas? We show that the answers to these two questions depend only on the features of the income tax system. More precisely, can the tax system distinguish incomes from each factor of production? Can it be reformed along the so-called "GE-replicating directions", reproducing the impact of factor price adjustments on taxpayers' utility? If the answer to either question, or both, is "no", the design of production policies should also take into account its pre-distributive role and all formulas reveal novel, empirically implementable "GE multipliers". These multipliers shape tax systems to correct for market failures as well as for the price incidence effects. In contrast, if the answer to both questions is "yes", it is Pareto-improving to design production policies solely to enlarge production possibilities and the "GE multipliers" shape the income tax system only to account for market failures. We illustrate these insights with realistic tax systems and practical examples of production policies.

Keywords: Production efficiency, Nonlinear income taxation, Several income sources, Endogenous prices.

JEL codes: H21, H22, H23, H24, L5, F13.

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I Introduction

Over the last decade, popular demand for better income redistribution has risen in developed countries, as exemplified by movements such as Occupy Wall Street in the US. However, how to reform the tax system remains a complex issue, all the more since individuals differ in their abilities to supply different production factors, each with its own price, thereby creating a pre-distribution problem.[1] Moreover, production factors can be imperfect substitutes and imperfect competition within the production sector can occur. In this context, tax reforms do not only affect the supply of factors. Changes in the supply of factors also change factor prices through demand responses of the production sector. Changes in prices in turn affect the supply of factors, and so on. Addressing how these General Equilibrium (GE) adjustments modify the desirability of tax reforms and the quantitative evaluation of their effects is the first question addressed in this paper. Second, there exist a variety of policies that exclusively impact the production sector,[2] e.g. taxation of intermediate goods (including taxation of robots and taxation of AI), public production, commodity taxation, trade openness, competition policy or business-focused environmental regulations. The second research question addressed in this paper is whether these "production policies" should be designed solely to improve production possibilities, following what we call the "production efficiency principle", or whether their design should also encompass their pre-distribution effects.[3]

To address these questions, we consider a model in which taxpayers differ along multiple dimensions of unobserved heterogeneity, supply different production factors, hence receive income from a variety of sources.[4] The supply side of the model is described by utility-maximizing taxpayers who take factor prices as given. Conversely, the production sector is represented through inverse demand functions. This reduced-form description of the demand side of the model allows us to derive results that are robust to various market frictions and a large set of microfoundations for the production sector and production policies.

Adopting a tax perturbation approach, we identify the existence of specific directions of tax reforms that have the same effects as changes in factor prices at the GE. We outline, for any tax system, how to easily characterize these directions, which we call "GE-replicating directions". Surprisingly, the answers to our two seemingly independent research questions depend both on whether the tax system distinguishes incomes from each factor of production and whether it already contains or can be improved by tax reforms in the GE-replicating directions.

[1] Pre-distribution is the way in which the market distributed rewards, such as factor prices. (Hacker, 2011, Stiglitz, 2018)

[2] Our analysis of production policies extends to examining the impact of various shocks that alter the production set, such as technological advancements or expanded trade opportunities.

[3] Drawing from the policy matrix introduced by Rodrik and Stantcheva (2021), this question can be reframed as whether policies ought to intervene at the *production stage*, specifically on factor prices.

[4] In tax theory, it is usually assumed that the taxpayer earns a single type of income, see for instance Mirrlees (1971), Diamond (1998), Saez (2001) or Saez (2002a). Tax models with multidimensional types and multiple incomes are studied in (Mirrlees, 1976, Kleven et al., 2007, Golosov et al., 2014, Spiritus et al., 2023, Boerma et al., 2022, Golosov and Krasikov, 2023).

First, we show that the effects of a tax reform can be decomposed into its usual Partial Equilibrium (PE) effects, holding factor prices constant and the effects of price adjustments at the GE. Consequently, the formulas describing the impact of tax reforms and the optimal tax formulas use novel statistics called GE multipliers, along with familiar elasticity concepts. For each factor production, the corresponding GE multiplier captures, in two empirical components, the effects of price adjustments (i) on the factor supply, consumption and utility of taxpayers and (ii) on tax revenue, holding constant factor supply and consumption of taxpayers. In tax formulas, the first component of the GE-multiplier differs from zero once the tax system is not optimal along all the GE replicating directions. The second component differs from zero once there is a market failure. It is equal to the percentage difference between the marginal product and the price of the production factor, regardless of the type of market friction. GE-multipliers, can be calculated from existing data on behavioral responses (including cross-base responses and income effects) and from a calibration of inverse demand elasticities, which use elasticities of substitution across production factors, potential markups or measures of externalities within the production sector.

Our tax formulas and GE-multipliers can then be viewed as a practical guide for tax reforms and a synthesis of prior tax formulas, clarifying their discrepancies. In Diamond and Mirrlees (1971), in the long-run model of Saez (2004), or in Saez and Zucman (2023), the fact that the tax systems distinguish each production factors and incorporate their GE-replicating directions clarifies why optimal tax formulas do not depend on the degree of substitutability between production factors.[5] Conversely, tax systems in Stiglitz (1982), Naito (1999), in the short-run model of Saez (2004), in Naito (2004), Rothschild and Scheuer (2013, 2016), Jacobs (2015), Ales et al. (2015), Ales and Sleet (2016), Sachs et al. (2020) and Schultz et al. (2023) do not differentiate the type of labor generating income. Consequently, the degree of substitutability between factors shapes their tax formulas which corresponds to our GE-multipliers accounting for price adjustments. Should we introduce imperfect competition into these models, our GE-multipliers would also shape the tax systems to correct for the market failures.

In many countries, the tax system consists of many schedules, depending on a single tax base and with restricted forms. Thus our analysis goes beyond characterizing optimal tax systems without any restrictions on their forms (employing ordinary differential equations). We also provide optimal tax formulas for schedular tax systems, where the tax system is the sum of several (possibly non-linear) income-specific functions,[6] as well as for comprehensive tax systems, which depend on the sum of different income sources.[7] When the tax system distinguishes incomes from each factor of production and

[5]There is (linear) tax rate for each income factor in Diamond and Mirrlees (1971) and there is a specific tax rate for each occupation and each occupation corresponds to a specific factor of production in the long-run model of Saez (2004). In this case, the GE-replicating directions are linear and inherent to the tax systems.

[6]Costa Rica, Denmark, Finland, Greece, Hungary, Iceland, Israel, Italy, Latvia, Lithuania, Netherlands, Norway, Poland Slovenia, Spain, Sweden, Türkiye have schedular tax systems, as detailed in Hourani et al. (2023, Table A1).

[7]The tax systems in Switzerland, the United Kingdom and the United States can be viewed as close approximations to comprehensive ones, as highlighted in Hourani et al. (2023, Table A1).

is either unrestricted or schedular, we show that the system can be reformed along all the GE-replicating directions. Consequently, these tax systems are shaped by GE-multipliers solely to address market failures. When such failures are absent, as in Scheuer (2014) with a tax system distinguishing salary from entrepreneurial incomes, there is no need to pre-distribute income through factor prices (i.e. our GE-multipliers are nil). In contrast, when wage and entrepreneurial income are comprehensively taxed, factor prices play a pre-distributive role and GE-price adjustments (captured by our GE-multipliers) increase or decrease optimal marginal income tax rates along the income distribution. This arises from the fact that incomes are not predominantly generated by the same factors of production at different income levels. This sheds light on the findings of Rothschild and Scheuer (2013, Figure II) and Sachs et al. (2020, Figure 4), where GE price adjustments decrease optimal marginal tax rates at high income levels and increase them at low income levels.

Answering our first research question, we also identify conditions for the Pareto efficiency of a tax system, hence contributing to a rich literature that assesses whether a given reform direction is Pareto-improving (Werning, 2007, Bourguignon and Spadaro, 2012, Bargain et al., 2014, Lorenz and Sachs, 2016, Jacobs et al., 2017, Hendren, 2020, Bierbrauer et al., 2023, Bergstrom and Dodds, 2023). With a single source of income, Lorenz and Sachs (2016) provides a formula to test whether a cut in the marginal tax rate at one level of income, combined with a uniform reduction in tax liabilities above that level, is self-financing. Assuming multidimensional types and a single source of income, Bierbrauer et al. (2023) shows that this condition is equivalent to negative revealed welfare weights,[8] a result confirmed with multiple income sources in Spiritus et al. (2023). We demonstrate that combining the Pareto-improving tax reform obtained at the PE with tax reforms along the GE-replicating directions leads to a Pareto-improvement at the GE.[9]

Our second key contribution is characterizing the impact of production policy reforms. We describe results in terms of production policies although they also apply to any shock that modifies the production set, such as technological shocks. Production policies directly affect the economy through changes in factor prices. These price changes, in turn, induce taxpayers to change their supply of factors, thereby generating further price adjustments. We show that these further GE adjustments are equivalent to those induced by an adequate combination of tax reforms in GE-replicating directions. Therefore, whether we need to integrate pre-distribution concerns into the design of production policies depends only on the features of the income tax system. If it differentiates incomes from each factor of production and contains all the GE-replicating directions, implementing a production policy reform that enhances the possibility

[8] Welfare weights are revealed from data in e.g. Bourguignon and Spadaro (2012), Bargain et al. (2014), Jacobs et al. (2017). For the literature on the inverse approach to indirect taxation, see Christiansen and Jansen (1978) and Ahmad and Stern (1984).

[9] Saez and Zucman (2023) focuses on a "distributional current-tax analysis" which aims to provide information on the current distribution of income and tax payments by income groups. In contrast, we focus on a "distributional tax reform analysis" describing how small tax reforms impact different socio-economic groups.

frontier, combined with adequate tax reforms in the GE replicating directions, is Pareto-improving. This is what we call the production efficiency principle. In such cases, the income tax system is rich enough to mitigate negative pre-distributive effects. In contrast, when a given tax system fails to distinguish incomes from each factor or does not include some of the GE-replicating directions, we should depart from the production efficiency principle. In this case, production policies should be designed to induce pre-distribution through prices. The formulas for a tax reform or an optimal tax system then include GE-multipliers that mitigate both market failures and price adjustments. We derive a formula to quantify the impact of any change in production policies on welfare and tax revenue. This formula relies on the same statistics and estimates used to calculate the GE-multipliers.

We then apply our findings to several examples of production policies, demonstrating how our reduced-form description of the production sector encompasses a wide array of micro-founded production frameworks. We consider, as a first production policy, taxes on intermediate good. These goods are used by firms under constant or decreasing returns to scale and we allow for a large variety of tax systems, which may or may not be reformed along the GE-replicating directions. This example is particularly relevant to the discussion surrounding the taxation of robots and AI, treating them as specific types of intermediate goods. Using the same framework, we study a second production policy: the demands of factors and goods by a public firm. In all these examples, we retrieve Diamond and Mirrlees (1971) result that intermediate goods should not be taxed when the income tax system can distinguish the income from each factor and can be reformed along the GE-replicating directions. Moreover, public firms should face the same prices of factors as the private firms, as shown in Little and Mirrlees (1974) but not in the framework of Naito (1999) where the tax system cannot distinguish between different labor types. We contribute to this large literature that builds upon Diamond and Mirrlees (1971). First, while Diamond and Mirrlees (1971) considers only linear taxes, we allow for nonlinear taxes. So doing, we clarify that their result does not rely on the tax system being optimal but only on being optimal along the GE-replicating directions– a condition we show always satisfied with optimal linear income tax schedules. Interestingly, when the income tax system is nonlinear, the GE-replicating directions are only a very small subset of the possible directions. We also stress that recommending to tax or not robots, as carefully discussed in Guerreiro et al. (2021), Costinot and Werning (2022), Thuemmel (2023) and in Beraja and Zorzi (2024), depends solely on the features of the income tax system. Another specific production policy we examine is commodity taxation. We apply Atkinson and Stiglitz (1976) to our framework, and clarify that taxing or not commodities depends entirely on the features of the income tax system. Fourth, we show how our findings extend to trade liberalization (as in e.g. Diamond and Mirrlees (1971), Dixit and Norman (1980), Costinot and Werning (2022), Antràs et al. (2017)), and which income tax systems mitigate trade-induced inequality. To illustrate the relevance of our framework for competition regulation, we study a Cournot duopoly model and show that reducing markups is always desirable when the tax system can be reformed along its GE-replicating directions. Our last example tackles with firm's carbon emissions and the optimal carbon tax.

The paper is organized as follows. The model is presented in the next section. In Section III, we investigate whether tax reforms have the same effects at the PE and at the GE, we derive tax incidence formulas as well as optimal tax formulas under unrestricted tax systems, under schedular tax systems and comprehensive ones. In our GE framework, we also provide an empirical test for the existence of Pareto-improving tax reforms. In Section IV, we examine the validity of the production efficiency principle and show how various micro-founded production policies are seamlessly and easily integrated into our reduced-forms representation of the production sector. These examples span a range of policies, including taxation of intermediate goods, robots, and AI, as well as public production, commodity taxation, trade policies, and competition policies. The last section concludes.[10]

II The Economy

II.1 Taxpayers

We consider an economy with a unit mass of taxpayers and a production sector that produces a numeraire good using n factors with $n \geq 2$. Taxpayers are endowed with varying characteristics summarized by an m-dimensional vector $\mathbf{w} = (w_1, ..., w_m)$, with $m \geq n$. These types are distributed over a closed and convex type space $W \subset \mathbb{R}^m$ according to a continuously differentiable density function $f(\cdot)$ which is positive over W and a CDF $F(\cdot)$.

Each taxpayer supplies $x_i \geq 0$ units of the i^{th} factor and her supply is denoted by $\mathbf{x} = (x_1, ..., x_n)$. For instance, a taxpayer can supply effective units of labor x_1 in a routine job, effective units of labor x_2 in a creative job, effective units of labor x_3 as entrepreneur, investment units in capital x_4, investment units in a specific asset x_5, etc. Each supply of factor x_i incurs an effort or a utility cost that depends on the individual type $\mathbf{w}$, as illustrated in the examples of Appendix A.2.

The income generated by supplying factor x_i is denoted by $y_i = p_i\, x_i$. For the taxpayers, p_i represents the private return on the i^{th} factor they supply and is taken as given. For the firm, it is the price of this factor. These factor prices are summarized in the vector $\mathbf{p} = (p_1, ..., p_n)$. For instance, if x_1 represents effective labor, then p_1 denotes the wage per unit of effective labor, and y_1 stands for labor income. Similarly, if x_2 corresponds to savings, p_2 represents the gross return on savings, y_2 signifies capital income, and so forth. The various sources of income are concisely represented by the vector $\mathbf{y} = (y_1, ..., y_n)$.

[10] Formal proofs are relegated to the online appendix, where we also emphasize our framework's ability to capture the essential mechanisms by which taxation affects individual behaviors in most macroeconomic models, using a two-period model with labor and savings. Additionally, we demonstrate how our framework accommodates economies characterized by different sectors, occupations, or industries, building up on Roy (1951), and also addresses phenomena like income-shifting.

The preferences of type-$\mathbf{w}$ taxpayer are represented by the utility function $(c, \mathbf{x}; \mathbf{w}) \mapsto \mathscr{U}(c, \mathbf{x}; \mathbf{w})$, which is assumed to be twice continuously differentiable over $\mathbb{R}_+^{n+1} \times W$, increasing in the after-tax income c (with partial derivative denoted $\mathscr{U}_c > 0$) and decreasing in the supply of each factor (with partial derivative denoted $\mathscr{U}_{x_i} < 0$). The government enforces taxes based on a tax schedule that depends nonlinearly on all sources of income, denoted as:

$$\mathcal{T} : \mathbf{y} = (y_1, ..., y_n) \mapsto \mathcal{T}(\mathbf{y}) = \mathcal{T}(y_1, ..., y_n), \tag{1}$$

After-tax income, hereafter refereed to as consumption, is $c = \sum_{i=1}^n y_i - \mathcal{T}(y_1, ..., y_n)$. Our framework offers applicability to a wide range of tax-related problems where taxpayers earn different types of income. To illustrate the generality of our model, we present in Appendix A.2 three applications that can be readily solved using our approach: a two-period model with endogenous labor and savings, a model where taxpayers choose their sectors of work and a model with income-shifting.[11]

The marginal rate of substitution between the supply of factor x_i and consumption for a taxpayer with type $\mathbf{w}$, at any bundle $(c, \mathbf{x})$, is given by:

$$\mathcal{S}^i(c, \mathbf{x}; \mathbf{w}) \stackrel{\text{def}}{\equiv} -\frac{\mathscr{U}_{x_i}(c, \mathbf{x}; \mathbf{w})}{\mathscr{U}_c(c, \mathbf{x}; \mathbf{w})}. \tag{2}$$

We assume that the utility function $\mathscr{U}(c, \mathbf{x}; \mathbf{w})$ is weakly concave in $(c, \mathbf{x})$ and that the indifference sets are convex in $(c, \mathbf{x})$ for all utility levels and all types $\mathbf{w}$. This implies that matrix $\left[\mathcal{S}^i_{x_j} + \mathcal{S}^i_c \mathcal{S}^j\right]_{i,j}$ is positive definite, as shown in Appendix A.1.[12] A $\mathbf{w}$-taxpayer chooses her supply of factors $\mathbf{x}$ to solve:

$$U(\mathbf{w}) \stackrel{\text{def}}{\equiv} \max_{\mathbf{x}=(x_1,...,x_n)} \quad \mathscr{U}\left(\sum_{k=1}^n p_k\, x_k - \mathcal{T}(p_1\, x_1, ..., p_n\, x_n), \mathbf{x}; \mathbf{w}\right) \tag{3}$$

This is equivalent to choosing incomes $\mathbf{y}$ to solve:

$$U(\mathbf{w}) \stackrel{\text{def}}{\equiv} \max_{\mathbf{y}=(y_1,...,y_n)} \quad \mathscr{U}\left(\sum_{k=1}^n y_k - \mathcal{T}(y_1, ..., y_n), \frac{y_1}{p_1}, ..., \frac{y_n}{p_n}; \mathbf{w}\right) \tag{4}$$

Both of these formulations will prove useful later on. We assume (See Assumption 1 in Section III below) that, for each taxpayer of type $\mathbf{w} \in W$, these programs admit a single solution with supplies of factors denoted by $\mathbf{X}(\mathbf{w}) \stackrel{\text{def}}{\equiv} (X_1(\mathbf{w}), ..., X_n(\mathbf{w}))$ and incomes denoted by $\mathbf{Y}(\mathbf{w}) \stackrel{\text{def}}{\equiv} (Y_1(\mathbf{w}), ..., Y_n(\mathbf{w}))$ where $Y_i(\mathbf{w}) = p_i\, X_i(\mathbf{w})$. By aggregating the individual factor supplies of $X_i(\mathbf{w})$, we obtain its total quantity, $\mathcal{X}_i$, used in the production process, i.e. $\mathcal{X}_i \stackrel{\text{def}}{\equiv} \int_W X_i(\mathbf{w})\, \mathrm{d}F(\mathbf{w})$. The utility achieved by $\mathbf{w}$-taxpayers is $U(\mathbf{w}) = \mathscr{U}(C(\mathbf{w}), \mathbf{X}(\mathbf{w}); \mathbf{w})$ where $C(\mathbf{w}) \stackrel{\text{def}}{\equiv} \sum_{i=1}^n Y_i(\mathbf{w}) - \mathcal{T}(\mathbf{Y}(\mathbf{w}))$ is their consumption. The first-order conditions are:

$$\forall i \in \{1, ..., n\}: \quad 1 - \mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w})) = \frac{1}{p_i} \mathcal{S}^i\left(C(\mathbf{w}), \frac{Y_1(\mathbf{w})}{p_1}, ..., \frac{Y_n(\mathbf{w})}{p_n}; \mathbf{w}\right). \tag{5}$$

[11] In this paper, we consider only intensive margin decisions. However, the model can easily be extended to include extensive margin decisions such as migration, as in Lehmann et al. (2014) or Janeba and Schulz (2023).

[12] $A_{i,j}$ is a term of matrix A for which the row is i and the column is j.

For each kind $i = 1, ..., n$ of income, the left-hand side is the marginal net-of-tax rate of the i^{th} income. It corresponds to the marginal gain, in terms of after-tax income, of the i^{th} pretax income y_i. The right-hand side is the marginal rate of substitution between supply of factor x_i and after-tax income. It corresponds to the marginal cost of supplying the i^{th} pretax income, in monetary terms.

II.2 Production sector

The production sector can be made of different firms with potential vertical relations and horizontal competition. Firms' market power, rent-seeking behaviors, and production externalities, among other phenomena, can prevail. The production side is presented in reduced-form. We adopt a highly flexible specification to describe how private returns depend on factors $(\mathcal{X}_1, ..., \mathcal{X}_n)$ through the following twice differentiable inverse demand functions:

$$\forall i \in \{1, ..., n\} : \qquad p_i = \mathscr{P}_i \left(\mathcal{X}_1, ..., \mathcal{X}_n; \boldsymbol{\alpha}\right). \tag{6}$$

where $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_L) \in \mathcal{A} \subset \mathbb{R}^L$ is a vector of policies of dimension L that solely impact the economy through the production sector, hereafter *production policies*. These policies influence only the behavior and interactions across firms, consequently impacting the size of aggregate production and the prices of various production factors. However, given a specific vector of factor prices, these policies do not modify taxpayers' behavior. Examples of such policies encompass taxes on intermediate goods, public production of the consumption good, trade policies, competition policies, business laws, financial market regulations, intellectual property protection, among others. The set of production policies is denoted $\mathcal{A}$ and is a convex subset of $\mathbb{R}^L$.

The production function is given by the national accounting equation:

$$\forall (\mathcal{X}_1, ..., \mathcal{X}_n; \boldsymbol{\alpha}) : \qquad \mathcal{F}(\mathcal{X}_1, ..., \mathcal{X}_n; \boldsymbol{\alpha}) \stackrel{\text{def}}{\equiv} \sum_{i=1}^{n} \mathscr{P}_i(\mathcal{X}_1, ..., \mathcal{X}_n; \boldsymbol{\alpha})\, \mathcal{X}_i, \tag{7}$$

i.e., the GDP on the left-hand side equals the sum of incomes derived from each factor on the right-hand side.

A specific case arises under perfect competition where the price (equivalently the *private* return of factor i) p_i coincides with the marginal productivity of the i^{th} factor (equivalently the *social* return of factor i), $\mathcal{F}_{\mathcal{X}_i}$:

$$\forall i \in \{1, ..., n\}, \forall \left(\mathcal{X}_1, ..., \mathcal{X}_n, \boldsymbol{\alpha}\right) : \qquad \mathscr{P}_i(\mathcal{X}_1, ..., \mathcal{X}_n; \boldsymbol{\alpha}) = \mathcal{F}_{\mathcal{X}_i}(\mathcal{X}_1, ..., \mathcal{X}_n; \boldsymbol{\alpha}) \tag{8}$$

Prices are then endogenous whenever factors are imperfect substitutes.

Under perfect competition, profits may occur if the production function has decreasing returns to scale.[13] In such a case, or under imperfect competition, to retrieve the national accounting equation (7),

[13] If function $(\mathcal{X}_1, ..., \mathcal{X}_n) \mapsto \mathcal{F}\left(\mathcal{X}_1, ..., \mathcal{X}_n\right)$ is increasing in each argument and exhibits decreasing returns to scale, then function $(\mathcal{X}_1, ..., \mathcal{X}_{n+1}) \mapsto \mathcal{X}_{n+1}\ \mathcal{F}\left(\mathcal{X}_1/\mathcal{X}_{n+1}, ..., \mathcal{X}_n/\mathcal{X}_{n+1}\right)$ is increasing in each argument and has constant returns to scale.

let $X_{n+1}(\mathbf{w})$ denote the share of profits received by taxpayers of type $\mathbf{w}$ with $\mathcal{X}_{n+1} = \int_{\mathbf{w}} X_{n+1}(\mathbf{w})\, \mathrm{d}F(\mathbf{w}) = 1$ and aggregate profits earned by all taxpayers being equal to $p_{n+1}\mathcal{X}_{n+1} = p_{n+1}$. This additional production factor $X_{n+1}(\mathbf{w})$ can be interpreted as an "entrepreneurial factor" which is inelastically supplied (McKenzie (1959), and Mas-Colell et al. (1995, pp. 134-135)). Equation (7) then still holds, provided that i is summed from 1 to $n+1$ instead of 1 to n.

Two examples will be especially interesting to discuss.

Example 1. *There are two factors ($n = 2$): labor and capital which are imperfect substitutes.*

Example 2. *There are two factors ($n = 2$): low and high skilled labor which are imperfect substitutes.*

II.3 Equilibrium

We employ two distinct equilibrium concepts: partial equilibrium (PE) with exogenous prices and general equilibrium (GE) with endogenous prices. The GE is defined by:

Definition 1 (General Equilibrium (GE))**.** *Given a tax schedule $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y})$ and production policies $\boldsymbol{\alpha}$, a GE is a set of prices $\mathbf{p} = (p_1, ..., p_n)$, of incomes $\mathbf{Y}(\mathbf{w})$ for each type $\mathbf{w}$ of taxpayers, of aggregate factors $(\mathcal{X}_1, ..., \mathcal{X}_n)$ and of aggregate incomes $(\mathcal{Y}_1, ..., \mathcal{Y}_n)$ such that:*

i) Incomes $\mathbf{Y}(\mathbf{w})$ maximize $\mathbf{w}$-taxpayers utility according to (4), taking prices $\mathbf{p}$ as given.

ii) Prices are given by inverse demand functions (6), where aggregate factors and incomes are related by $\mathcal{X}_i = \mathcal{Y}_i/p_i$ and where aggregate incomes $(\mathcal{Y}_1, ..., \mathcal{Y}_n)$ sum up individual incomes according to:

$$\mathcal{Y}_i \overset{def}{\equiv} \int_{\mathcal{W}} Y_i(\mathbf{w})\, \mathrm{d}F(\mathbf{w}) = p_i\, \mathcal{X}_i. \tag{9}$$

The PE takes prices as given and thereby omits part *ii*) of Definition 1, as follows.

Definition 2 (Partial Equilibrium (PE))**.** *Given a tax schedule $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y})$, production policies $\boldsymbol{\alpha}$ and a set of prices $\mathbf{p} = (p_1, ..., p_n)$, a PE is a set of incomes $\mathbf{Y}(\mathbf{w})$ for each type $\mathbf{w}$ of taxpayers that maximize $\mathbf{w}$-taxpayers utility according to (4), taking prices as given.*

At the PE, when a tax reform affects factor prices, we take the determination of prices through the mapping $t \mapsto (p_1^R(t), ..., p_n^R(t))$ as given. At the GE, on the other hand, the mapping $t \mapsto (p_1^R(t), ..., p_n^R(t))$ is endogenous and determined by (6). Throughout this paper, we assume PE and GE exist and are unique.

II.4 Government

The government chooses the tax policy taking into account how its choice impacts the GE. It faces the following budget constraint:[14]

$$E \leq \mathscr{B} \overset{\text{def}}{\equiv} \int_{\mathcal{W}} \mathcal{T}\left(\mathbf{Y}(\mathbf{w})\right)\, \mathrm{d}F(\mathbf{w}) \tag{10}$$

[14] According to the national accounting equation (7), the production function $\mathcal{F}$ represents the production net of the budgetary costs of product sector policies.

where $\mathscr{B}$ stands for the tax revenue and where $E \geq 0$ is an exogenous amount of public expenditure. The social objective is an increasing transformation Φ of taxpayers' individual utility $U(\mathbf{w})$ that may be concave and type-dependent:

$$\mathscr{W} \stackrel{\text{def}}{\equiv} \int_{\mathcal{W}} \Phi\left(U(\mathbf{w}); \mathbf{w}\right) \, \mathrm{d}F(\mathbf{w}). \tag{11}$$

where $\Phi : (u, \mathbf{w}) \mapsto \Phi(u, \mathbf{w})$ is increasing in individual utility u and twice continuously differentiable. This specification includes many different social objectives. The objective is utilitarian when $\Phi(U, \mathbf{w}) = U$ and weighted utilitarian when $\Phi(U; \mathbf{w}) = \gamma(\mathbf{w})\, U$. One obtains maximin when $\gamma(\mathbf{w})$ equal zero for every taxpayer except those with the lowest utility level. When $\Phi(U, \mathbf{w})$ does not depend on type and is concave in U, one has Bergson-Samuelson preferences. The utility function $\mathscr{U}(\cdot, \cdot; \mathbf{w})$ is only one possible cardinal representation of type-$\mathbf{w}$ taxpayers' preferences. Other representations are obtained using an increasing transformation of $\mathscr{U}(\cdot, \cdot; \mathbf{w})$ such as $\Phi(\mathscr{U}(\cdot; \mathbf{w}); \mathbf{w})$. Therefore, the right-hand side of (11) can be interpreted as a utilitarian objective following a recardinalization of individual utility. The government maximizes a linear combination of tax revenue $\mathscr{B}$ and social welfare $\mathscr{W}$ that we refer as the government's Lagrangian:

$$\mathscr{L} \stackrel{\text{def}}{\equiv} \mathscr{B} + \frac{1}{\lambda} \mathscr{W} \tag{12}$$

where the Lagrange multiplier $\lambda > 0$ represents the social value of public funds. We choose to express the Lagrangian in monetary units instead of utility units.

III Which tax reforms are desirable?

In this section, we start by investigating the impacts of tax reforms, and analyze the behavioral and price responses to these reforms at both the PE and at the GE. We define a tax reform as follows.

Definition 3. *Let $R(\cdot)$ be a twice-continuously differentiable function. A tax reform of direction $R(\cdot)$ and magnitude $t \lesseqgtr 0$ replaces the tax schedule $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y})$ by a new tax schedule $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y}) - t\, R(\mathbf{y})$. zing*

For a given income vector $\mathbf{y}$, the change in the tax burden due to the reform is, therefore, given by $t\, R(\mathbf{y})$. After a tax reform, the i^{th} price depends on the direction $R(\cdot)$ and the magnitude t of the tax reform and is denoted by $p_i^R(t)$. When subjected to a tax reform in the direction $R(\cdot)$, the utility of $\mathbf{w}$-taxpayers becomes a function of magnitude $t \lesseqgtr 0$ through:

$$U^R(\mathbf{w}, t) \stackrel{\text{def}}{\equiv} \max_{\mathbf{y}=(y_1,\ldots,y_n)} \mathscr{U}\left(\sum_{i=1}^{n} y_i - \mathcal{T}(\mathbf{y}) + t\, R(\mathbf{y}), \frac{y_1}{p_1^R(t)}, \ldots, \frac{y_n}{p_n^R(t)}; \mathbf{w}\right). \tag{13}$$

Under the perturbed tax schedule $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y}) - t\, R(\mathbf{y})$, $\mathbf{Y}^R(\mathbf{w}, t) = (Y_1^R(\mathbf{w}, t), \ldots, Y_n^R(\mathbf{w}, t))$ denotes the $\mathbf{w}$-taxpayers' incomes, $\mathscr{B}^R(t)$ the government's tax revenue (10), $\mathscr{W}^R(t)$ its social objective (11) and $\mathscr{L}^R(t) \stackrel{\text{def}}{\equiv} \mathscr{B}^R(t) + \frac{1}{\lambda}\mathscr{W}^R(t)$ the Lagrangian (12).[15]

We first investigate the desirability of tax reforms at the PE, i.e. taking as given the determination of factor prices through the mapping $t \mapsto (p_1^R(t), ..., p_n^R(t))$ in Subsection III.1. Second, in Subsection III.2, we study the desirability of these reforms at the GE where the mapping $t \mapsto (p_1^R(t), ..., p_n^R(t))$ is endogenously dictated by the inverse demand equations (6), as detailed in Definition 1. We show that behavioral responses are valued based on marginal tax rates alone at the PE, while they are valued based on the sum of marginal tax rates and new terms, called GE multipliers, at the GE. In Subsection III.3, we compute GE multipliers thanks to tax reforms in "GE-replicating" directions. These reforms replicate the effects of price changes on taxpayers' utility and factor supply and can be simply deduced from the tax system. In Subsection III.4, we provide the optimal tax formula under an unrestricted tax system and use this formula to infer the revealed welfare weights. We then proof the existence of Pareto-improving tax reforms when competition is perfect and when some revealed welfare weights are negative. We finally provide the optimal tax schedule when the tax system is schedular in Subsection III.5 and when the tax system is comprehensive in Subsection III.6.

III.1 Tax reforms at Partial Equilibrium

For each $\mathbf{w}$-taxpayer, we define $\partial Y_i(\mathbf{w})/\partial \tau_j$, the compensated responses of the i^{th} income with respect to the j^{th} marginal net-of-tax rate and $\partial Y_i(\mathbf{w})/\partial \rho$ the income effect on their i^{th} income. Formally, we use "compensated" tax reforms of direction $R(\mathbf{y}) = y_j - Y_j(\mathbf{w})$ and magnitude τ_j to calculate $\partial Y_i(\mathbf{w})/\partial \tau_j$ that captures only substitution effects.[16] We use "lump sum" tax reforms of direction $R(\mathbf{y}) = 1$ and magnitude ρ to calculate $\partial Y_i(\mathbf{w})/\partial \rho$ that captures income effects. Compensated and incomes effects are defined holding prices fixed.[17] Finally, we denote $\partial Y_i(\mathbf{w})/\partial \log p_j$, the responses of the i^{th} income to the j^{th} price log-change. All these responses are computed taking into account the nonlinearity of the tax schedule, as in Jacquet et al. (2013).

Unlike previous works in the literature, we refrain from assuming that individuals respond smoothly to tax reforms. Instead, we apply the implicit function theorem to taxpayers first-order conditions (5). To do so, we make the following assumption on the initial tax schedule and taxpayers' preferences discussed in Appendix B.1:

Assumption 1. *The initial tax schedule $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y})$ is such that:*

i) $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y})$ *is twice continuously differentiable,*

[15] Note that when we define the perturbed Lagrangian $\mathscr{L}^R(t)$, we keep $1/\lambda$ at its value before the tax reform. This will be convenient in Proposition 1.

[16] The reform and the responses of w-taxpayers, around income $Y_j(\mathbf{w})$, are said to be compensated since the j^{th} marginal net-of-tax rate τ_j is modified while the level of tax is unchanged at $\mathbf{y} = \mathbf{Y}(\mathbf{w})$.

[17] To estimate compensated responses and income effects, the empirical literature relies on differences in how much taxpayers are "treated" by tax reforms, while changes in prices are the same across taxpayers. Hence, it is PE responses that are estimated.

ii) the second-order condition associated to the individual maximization program (4) *holds strictly, i.e. the matrix* $\left[\frac{\mathcal{S}^i_{x_j}+\mathcal{S}^i_c\mathcal{S}^j}{p_i\,p_j}+\mathcal{T}_{y_iy_j}\right]_{i,j}$ *is positive definite at* $c=C(\mathbf{w})$, $\mathbf{x}=\mathbf{X}(\mathbf{w})$ *and at* $\mathbf{y}=\mathbf{Y}(\mathbf{w})$, *for each type* $\mathbf{w}\in W$,

iii) for each type $\mathbf{w}\in W$, *program* (4) *admits a unique global maximum.*

Assumption 1 prevents any jump in taxpayer's choices after a small tax reform of magnitude $\mathrm{d}t$. Such a reform affects taxpayers' decisions, hence taxpayers' first-order conditions (5), because of changes in the n marginal tax rates by $R_{y_j}(\mathbf{Y}(\mathbf{w}))\mathrm{d}t$ (for $j=1,...,n$) or because of a change in the level of tax by $R(\mathbf{Y}(\mathbf{w}))\mathrm{d}t$ or because of changes in relative prices by $\partial\log p^R_j(t)/\partial t$, which are taken as given at the PE and will be endogenized at the GE. Changes in the n marginal tax rates induce compensated responses of the n incomes $\partial Y_i(\mathbf{w})/\partial\tau_j$. A change in the level of tax generates income effects $\partial Y_i(\mathbf{w})/\partial\rho$ and changes in relative prices imply potential responses of the n incomes $\partial Y_i(\mathbf{w})/\partial\log p_j$. More formally, when faced with a tax reform, $\mathbf{w}$-taxpayers modify their incomes by (see Appendix B.1):

$$\frac{\partial Y_i^R(\mathbf{w},t)}{\partial t}=\underbrace{\sum_{j=1}^n\frac{\partial Y_i(\mathbf{w})}{\partial\tau_j}R_{y_j}(\mathbf{Y}(\mathbf{w}))}_{\text{Compensated responses}}+\underbrace{\frac{\partial Y_i(\mathbf{w})}{\partial\rho}R(\mathbf{Y}(\mathbf{w}))}_{\text{Income effects}}+\underbrace{\sum_{j=1}^n\frac{\partial Y_i(\mathbf{w})}{\partial\log p_j}\frac{\partial\log p^R_j(t)}{\partial t}}_{\text{Responses to price changes}}. \qquad (14)$$

By applying the envelope theorem to (13), we obtain, in monetary terms, the effects of a tax reform on the social welfare of a $\mathbf{w}$-taxpayer (see Appendix B.1):

$$\frac{1}{\lambda}\frac{\partial\Phi\left(\widetilde{U}^R(\mathbf{w},t);\mathbf{w}\right)}{\partial t}=\left(R(\mathbf{Y}(\mathbf{w}))+\sum_{j=1}^n\left(1-\mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w}))\right)Y_j(\mathbf{w})\,\frac{\partial\log\widetilde{p}^R_j(t)}{\partial t}\right)g(\mathbf{w}), \qquad (15)$$

where social marginal welfare weights $g(\mathbf{w})$ are defined by:

$$g(\mathbf{w})\stackrel{\text{def}}{\equiv}\frac{\Phi_U\left(U(\mathbf{w});\mathbf{w}\right)\mathscr{U}_c(C(\mathbf{w}),\mathbf{X}(\mathbf{w});\mathbf{w})}{\lambda}. \qquad (16)$$

Equation (15) gives changes in utility driven by the mechanical effect in tax liability $R(\mathbf{Y}(\mathbf{w}))$ and by the effects of reform-induced changes in prices, while taxpayers' decisions do not appear in it. The reason for this is that taxpayers' decisions are perturbed from their optimum and taxpayers are indifferent to small changes in their decisions to a first-order approximation. This envelope argument is well understood since Saez (2001). It however does not apply to prices changes since taxpayers take prices as given. Applying the envelope theorem to (3), a log change in the j^{th} price has the same impact on utility as a mechanical increase in consumption of $(1-\mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w})))Y_j(\mathbf{w})$. Multiplying the mechanical effect and the effects of prices changes on utility by the welfare weight $g(\mathbf{w})$ leads to the right-hand side of (15). From (14)-(16), we obtain the following lemma, which is proved in Appendix B.1.

Lemma 1. *At the PE, the impact of a tax reform on the government's perturbed Lagrangian is:*

$$\frac{\partial \mathscr{L}^R(t)}{\partial t} = \int_{\mathcal{W}} \Bigg\{ -\left[1 - g(\mathbf{w}) - \sum_{i=1}^{n} \mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w}))\, \frac{\partial Y_i(\mathbf{w})}{\partial \rho}\right] R(\mathbf{Y}(\mathbf{w})) \tag{17}$$

$$+ \sum_{1 \le i,j \le n} \mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w}))\, \frac{\partial Y_i(\mathbf{w})}{\partial \tau_j}\, R_{y_j}(\mathbf{Y}(\mathbf{w})) \Bigg\} \mathrm{d}F(\mathbf{w}) + \sum_{j=1}^{n} \frac{\partial \mathscr{L}}{\partial \log p_j}\, \frac{\partial \log p_j^R(t)}{\partial t}$$

where:

$$\frac{\partial \mathscr{L}}{\partial \log p_j} \overset{def}{\equiv} \int_{\mathcal{W}} \left\{ \left(1 - \mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w}))\right)\, Y_j(\mathbf{w})\, g(\mathbf{w}) + \sum_{i=1}^{n} \mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w}))\, \frac{\partial Y_i(\mathbf{w})}{\partial \log p_j} \right\} \mathrm{d}F(\mathbf{w}). \tag{18}$$

Equation (17) has three terms. The first term captures the impact of changes in tax liabilities $R(\mathbf{Y}(\mathbf{w}))$ on the Lagrangian. This term includes the mechanical effects on government revenue and social welfare, $1 - g(\mathbf{w})$, as well as the changes in tax revenue due to income effects, which are equal to the sum of each income effect $\partial Y_i(\mathbf{w})/\partial \rho$ multiplied by the corresponding marginal tax rate $\mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w}))$. The second term measures the impact of changes in marginal tax rates $R_{y_j}(\mathbf{Y}(\mathbf{w}))$ on tax revenue through substitution effects. It is the sum of the compensated response of each income $\partial Y_i(\mathbf{w})/\partial \tau_j$ times the corresponding marginal tax rate $\mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w}))$. The third term is novel and captures the impact on the Lagrangian of changes in factor prices due to the tax reform. To understand this third term, note that a log change in the price of j^{th} factor affects the Lagrangian via changes in welfare and tax revenue, as detailed in (18). First, it increases the welfare contribution of $\mathbf{w}$-taxpayers by $\left(1 - \mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w}))\right)\, Y_j(\mathbf{w})\, g(\mathbf{w})$, as indicated by (15). In addition, it leads to responses $\partial Y_i(\mathbf{w})/\partial \log p_j$ for each i^{th} income, as shown in (14). The impact on tax revenue can be calculated by summing up the product of each income source's response $\partial Y_i(\mathbf{w})/\partial \log p_j$ and the corresponding marginal tax rate $\mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w}))$.

The incidence formula above is expressed in terms of variables that can be empirically measured. There is a large empirical literature estimating compensated responses $\partial Y_i(\mathbf{w})/\partial \tau_j$ and income effects $\partial Y_i(\mathbf{w})/\partial \rho$ (e.g., Saez et al. (2012)). Welfare weights $g(\mathbf{w})$ can be calibrated either from normative assumptions (Saez and Stantcheva, 2016) or from survey data (Kuziemko et al., 2015). The i^{th} income response to the change in the j^{th} log price $\partial Y_i(\mathbf{w})/\partial \log p_j$ can then be obtained from Equation (19) derived in Appendix B.1:

$$\frac{\partial Y_i(\mathbf{w})}{\partial \log p_j} = \underbrace{\mathbb{1}_{i=j}\, Y_i(\mathbf{w}) + (1 - \mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w})))\, \frac{\partial Y_i^u(\mathbf{w})}{\partial \tau_j}}_{\text{Uncompensated term}} \underbrace{- Y_j(\mathbf{w}) \sum_{k=1}^{n} \frac{\partial Y_i(\mathbf{w})}{\partial \tau_k}\, \mathcal{T}_{y_k y_j}(\mathbf{Y}(\mathbf{w}))}_{\text{Bracket creep terms}}, \tag{19}$$

where $\partial Y_i^u(\mathbf{w})/\partial \tau_j$ denotes the uncompensated responses of the i^{th} income to the j^{th} marginal tax rate, holding price fixed, using reforms of direction $R(\mathbf{y}) = y_j$. These uncompensated responses are given by the Slutsky equations:

$$\frac{\partial Y_i^u(\mathbf{w})}{\partial \tau_j} = \frac{\partial Y_i(\mathbf{w})}{\partial \tau_j} + Y_j(\mathbf{w})\, \frac{\partial Y_i(\mathbf{w})}{\partial \rho}. \tag{20}$$

To grasp the intuition behind (19), we reconsider taxpayer's first-order conditions (5), which equalizes marginal gains and costs resulting from a marginal change in the i^{th} income y_i:

$$p_i(1 - \mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w}))) = \mathcal{S}^i \left(\sum_{k=1}^{n} p_k X_k(\mathbf{w}) - \mathcal{T}\left(p_1 X_1(\mathbf{w}), ..., p_n X_n(\mathbf{w})\right), \mathbf{X}(\mathbf{w}); \mathbf{w} \right). \tag{21}$$

Equation (21) balances marginal gains and costs induced by a marginal change in the supply of the i^{th} factor x_i.

The alteration of the j^{th} price induces three effects on these first-order conditions. First, a log change in the j^{th} price affects the LHS of the j^{th} first-order condition (21) as much as a log change in the j^{th} marginal net-of-tax rate $1 - \mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w}))$. Second, a log change in the j^{th} price affects consumption in the marginal rates of substitution in the RHS of (21) by $Y_j(\mathbf{w})(1 - \mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w})))$, i.e, as much as an uncompensated change of the j^{th} marginal net-of-tax rate. The combination of these first and second effects is therefore equivalent to an uncompensated change of the j^{th} marginal net-of-tax rate by $\mathrm{d}\tau_j = \left((1 - \mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w}))/p_j\right) \mathrm{d}p_j$. It leads to change in factor supplies equal to $\left((1 - \mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w})))/p_j\right) (\partial X_i^u/\partial\tau_j)$, which shows up in the RHS of (22) as the "uncompensated term". Third, whenever the tax schedule is nonlinear, a log change in the j^{th} price holding the j^{th} factor fixed triggers a further change of marginal tax rates in the LHS of (21) by $X_j(\mathbf{w})\, \mathcal{T}_{y_j y_k}(Y(\mathbf{w}))$. These modification induce compensated responses of the supply of i^{th} factor equal to $-X_j(\mathbf{w}) \sum_{k=1}^{n} \left(\partial X_i(\mathbf{w})/\partial\tau_k\right)\; \mathcal{T}_{y_k y_j}(\mathbf{Y}(\mathbf{w}))$, which appear in the RHS of (22) as the "bracket creep term".[18] Adding all of these effects leads to:

$$\frac{\partial X_i(\mathbf{w})}{\partial p_j} = \underbrace{\frac{1 - \mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w})}{p_j} \frac{\partial X_i^u(\mathbf{w})}{\partial\tau_j}}_{\text{Uncompensated term}} - \underbrace{X_j(\mathbf{w}) \sum_{k=1}^{n} \frac{\partial X_i(\mathbf{w})}{\partial\tau_k} \mathcal{T}_{y_k y_j}(\mathbf{Y}(\mathbf{w})),}_{\text{Bracket creep terms}} \tag{22}$$

which eventually leads to (19).

III.2 Tax reforms at General Equilibrium

To analyze the impact of tax reforms on prices in GE, we compute the terms $\partial \log p_j^R(t)/\partial t$ in equations (14), (15), and (17). To achieve this, we use the inverse demand Equations (6) to determine prices in the GE analysis. According to Definition 1 and Equation (6), for each magnitude t, the prices $(p_1^R(t), ..., p_n^R(t))$, after the tax reform of magnitude t and direction R verify the following fixed-point conditions:

$$\forall t, \forall i \in \{1, ..., n\} \qquad p_i^R(t) = \mathscr{P}_i \left(\frac{\mathcal{Y}_1^R(t)}{p_1^R(t)}, ..., \frac{\mathcal{Y}_n^R(t)}{p_n^R(t)} \right) \tag{23}$$

where the i^{th} aggregate income $\mathcal{Y}_i^R(t)$ is defined from individual i^{th} incomes $Y_i^R(\mathbf{w}, t)$ thanks to (9). Let Ξ denote the matrix where the term in the i^{th} line and j^{th} column is the inverse factor's demand elasticity

[18] This theoretical mechanism is at the core of the identification strategy of compensated responses by Saez (2003) where he uses fixed tax schedules (in nominal terms) with high inflation. The empirical strategy relies in comparing, in a period of high inflation, changes in income of taxpayers near the top-end of a tax bracket, who are likely to creep to the next bracket, to changes in income of other taxpayers far from this top-end.

of the i^{th} price p_i with respect to the aggregate supply of the j^{th} factor denoted $\mathcal{X}_j$:

$$\Xi_{i,j} \stackrel{\text{def}}{\equiv} \frac{\mathcal{X}_j}{\mathscr{P}_i} \frac{\partial \mathscr{P}_i}{\partial \mathcal{X}_j}. \tag{24a}$$

We denote Γ the matrix of factor supply elasticities, where the term $\Gamma_{i,j}$ in the i^{th} row and j^{th} columns corresponds to the elasticity of the aggregate supply of the i^{th} production factor with respect to the price of the j^{th} production factor,

$$\Gamma_{i,j} \stackrel{\text{def}}{\equiv} \left.\frac{\partial \log \mathcal{X}_i}{\partial \log p_j}\right|_{t=0} = \frac{p_j}{\mathcal{X}_i} \int_{\mathcal{W}} \frac{\partial X_i(\mathbf{w})}{\partial p_j} \, \mathrm{d}F(\mathbf{w}). \tag{24b}$$

We denote I_n the n-identity matrix and make the following assumption:

Assumption 2. *The matrix $I_n - \Xi \cdot \Gamma$ is invertible.*

The matrix $I_n - \Xi \cdot \Gamma$ shows up when one log-differentiates (23). Thanks to Assumption 2, one can apply the implicit function theorem to ensure that equilibrium prices are differentiable with respect to t. Under perfect competition and when the production function is linear, i.e. $\mathcal{F}(\mathcal{X}_1, ..., \mathcal{X}_n) = \sum_{i=1}^{n} \mathcal{X}_i$, matrix Ξ is nil. Assumption 2 is then verified. Therefore, by continuity, Assumption 2 remains satisfied as long as the elasticities of substitution between factors are sufficiently high and competition is not too imperfect.[19] In this context, in Appendix B.2.a, we show how, following a tax reform in the direction R, responses of aggregate incomes at PE result in log prices changes at the GE. This incidence is detailed in the following lemma.

Lemma 2. *After a tax reform in direction R, the vector $\partial \log \mathbf{p}^R / \partial t$ of log-price changes at the GE is given by:*

$$\frac{\partial \log \boldsymbol{p}^R}{\partial t} = (I_n - \Xi \cdot \Gamma)^{-1} \cdot \Xi \cdot \frac{\partial \log \boldsymbol{Y}^{R,PE}}{\partial t}, \tag{25}$$

where $\partial \log \mathbf{Y}^{R,PE} / \partial t$ is the vector for which the i^{th} row is:

$$\frac{\partial \mathcal{Y}_i^{R,PE}(t)}{\partial t} = \int_{\mathcal{W}} \left\{ \frac{\partial Y_i(\mathbf{w})}{\partial \rho} \, R(\mathbf{Y}(\mathbf{w})) + \sum_{j=1}^{n} \frac{\partial Y_i(\mathbf{w})}{\partial \tau_j} \, R_{y_j}(\mathbf{Y}(\mathbf{w})) \right\} \mathrm{d}F(\mathbf{w}), \tag{26}$$

which measures how aggregate income i reacts to a tax reform in the direction R at the PE.

Tax reforms generate responses in supplies and demands described in Figure 1. After a tax reform, the initial taxpayers' responses at PE impacts the supplies of production factors (through matrix Γ) which modifies prices (as determined by the inverse demand equations (6)). These price changes, in turn, affect supplies of production factor (through matrix Γ), creating an ongoing loop of interdependence between prices of production factors and their supplies. Equations (25)-(26) formalize this process of price adjustments illustrated in Figure 1.

[19] Using the contracting mapping theorem, the existence and uniqueness of the GE can be shown under the assumption that for all out-of-equilibrium price $\mathbf{p}$ and factor vectors $\mathcal{X}_1, ..., \mathcal{X}_n$., matrices $\Xi \cdot \Gamma$ have all eigenvalues with a modulus below a bound strictly lower than 1.

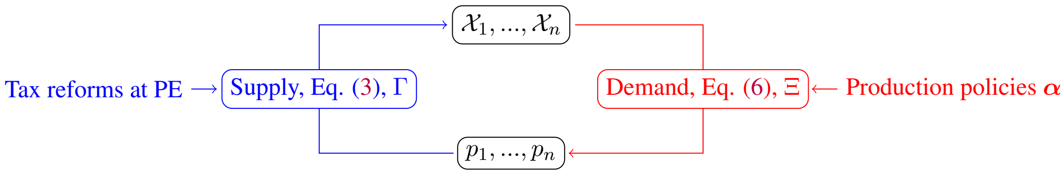

Figure 1: The GE adjustments following a tax or a production policy reform.

We now want to compute the effects on the Lagrangian resulting from these price adjustments following a tax reform in the direction $R(\cdot)$, which are equal to $\sum_{j=1}^{n} (\partial \mathscr{L}/\partial \log p_j) \left(\partial \log p_j^R/\partial t\right)$ according to Equation (17) in Lemma 1. For this purpose, let us denote $\boldsymbol{\partial \mathscr{L}/\partial \log p}$ the row vector where the j^{th} term is $\partial \mathscr{L}/\partial \log p_j$. Using Equation (25) from Lemma 2, we obtain:

Lemma 3. *At the GE, the effects of a tax reform on the Lagrangian, through prices, are given by:*

$$\sum_{j=1}^{n} \frac{\partial \mathscr{L}}{\partial \log p_j} \frac{\partial \log p_j^R}{\partial t} = \sum_{i=1}^{n} \eta_i \frac{\partial \log \mathcal{Y}_i^{R,PE}(t)}{\partial t} = \sum_{i=1}^{n} \mu_i \frac{\partial \mathcal{Y}_i^{R,PE}(t)}{\partial t} \tag{27}$$

*where for any $i \in \{1, ..., n\}$, we define the i^{th} **General Equilibrium multiplier** as*

$$\mu_i \stackrel{def}{\equiv} \frac{\eta_i}{\mathcal{Y}_i} \tag{28}$$

where the row vector $\boldsymbol{\eta} = (\eta_1, ..., \eta_n)$ is defined by:

$$\boldsymbol{\eta} \stackrel{def}{\equiv} \frac{\boldsymbol{\partial \mathscr{L}}}{\boldsymbol{\partial \log p}} \cdot (I_n - \Xi \cdot \Gamma)^{-1} \cdot \Xi. \tag{29}$$

The GE multipliers depend neither on the direction $R(\cdot)$ nor on the size t of the reform.

Lemma 3 introduces new empirically meaningful variables, the GE multipliers. For each factor, the associated GE multiplier μ_i gives the impact of the price adjustments on the Lagrangian when the i^{th} factor's supply is modified. This multiplier is the ratio of η_i over aggregate income $\mathcal{Y}_i$. This i^{th} component of vector $\boldsymbol{\eta}$ indicates the effects on the Lagrangian of the price adjustments induced by a log change in the i^{th} factor. It sums for all income factors j, the product of two components: the Lagrangian change caused by a relative change in the j^{th} price, $\partial \mathscr{L}/\partial \log p_j$ and the relative change in the j^{th} price in response to a PE log change in the i^{th} factor. The latter is equal to the term in the j^{th} row and i^{th} column of matrix $(I_n - \Xi \cdot \Gamma)^{-1} \cdot \Xi$, according to (25).

Lemmas 1 and 3 yield the following proposition (proofed in Appendix B.2), which characterizes the incidence of any arbitrary tax reform on the Lagrangian at the GE.

Proposition 1. *i) At the GE, the impact of a tax reform on the Lagrangian (12) is:*

$$\begin{aligned} \frac{\partial \mathscr{L}^R(t)}{\partial t} \quad = \quad & \int_{\mathcal{W}} \left\{ - \left[1 - g(\mathbf{w}) - \sum_{i=1}^{n} (\mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w})) + \mu_i) \frac{\partial Y_i(\mathbf{w})}{\partial \rho} \right] R(\mathbf{Y}(\mathbf{w})) \right. \\ & \left. + \sum_{1 \le i,j \le n} (\mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w})) + \mu_i) \frac{\partial Y_i(\mathbf{w})}{\partial \tau_j} R_{y_j}(\mathbf{Y}(\mathbf{w})) \right\} \mathrm{d}F(\mathbf{w}). \end{aligned} \tag{30}$$

ii) If the social value of public funds λ verifies:

$$0 = \int_{\mathcal{W}} \left[1 - g(\mathbf{w}) - \sum_{i=1}^{n} \left(\mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w})) + \mu_i \right) \frac{\partial Y_i(\mathbf{w})}{\partial \rho} \right] \mathrm{d}F(\mathbf{w}), \tag{31}$$

and if $\frac{\partial \mathscr{L}^R}{\partial t} > 0$ (< 0), then reforming the tax schedule in the direction $R(\cdot)$ with a small positive magnitude t (a small negative t) and rebating the net budget surplus in a lump-sum way is a budget-balanced reform that is socially desirable.

Part *i)* of Proposition 1 enables one to evaluate the desirability of a tax reform at the GE. A reform of the tax system in the direction $R(\cdot)$ with positive (negative) magnitude t improves welfare when $\partial \mathscr{L}^R / \partial t > 0$ (< 0). According to Part *ii)* of the proposition, the effect of any admissible perturbation on the government's Lagrangian has the same sign as the effect on the social objective of that perturbation combined with a lump-sum transfer that ensures budget balance. This result remains valid also outside of the optimum, as long as the weight λ assigned to government's revenue verifies (31). This result is noteworthy because it allows for the assessment of the desirability of a tax reform in the direction $R(\cdot)$, combined with the adequate lump-sum transfer to maintain budget neutrality. Notice that the social value of public funds λ scales the social welfare weights in (16). Condition (31) therefore states that welfare weights are normalized to ensure that the revenue loss from a lump-sum transfer is offset by the welfare gains.[20]

Equation (30) differs from its PE version (17) only by the inclusion of GE multipliers μ_i.[21] The presence of endogenous prices and the implied GE effects modify prices along the process described in Equation (25) and in Figure 1. The GE multipliers μ_i, defined in Equation (29), indicate how an increase in aggregate income $\mathcal{Y}_i$ at the PE impacts the government Lagrangian through price changes at the GE. When $\mu_i > 0$ (resp $\mu_i < 0$), a rise in the i^{th} aggregate income at the PE improves (deters) the Lagrangian via the GE adjustments of prices. Hence, adding GE multipliers μ_i to marginal tax rates $T_i'(Y_i(\mathbf{w}))$ in the right-hand side of (30) suffices to capture how behavioral responses $\partial Y_i(\mathbf{w})/\partial \tau_j$ and $\partial Y_i(\mathbf{w})/\partial \rho$ impact the Lagrangian though the GE price adjustments. Therefore, the incidence of a tax reform at the GE can be broken down into the incidence of the same reform at the PE level, augmented by the cumulative impact of this reform on each aggregate income at the PE, times the respective GE multiplier,

[20] The right-hand side of (31) is obtained from (30) by using tax reforms in the direction $R(\mathbf{y}) = -1$, which implies $R_{y_j}(\mathbf{Y}(\mathbf{w})) = 0$. We assume that:

$$1 - \sum_{k=1}^{n} \int_{\mathbf{w} \in W} \left(\mathcal{T}_{y_k}(\mathbf{Y}(\mathbf{w})) + \mu_k \right) \frac{\partial Y_k(\mathbf{w})}{\partial \rho} \, \mathrm{d}F(\mathbf{w}) > 0$$

i.e. that a positive lump-sum transfer to taxpayers reduces government's tax revenue, despite income responses. Otherwise, a lump-sum transfer would simultaneously increase taxpayers' well being and the government's revenue so that the initial economy would be Pareto-dominated.

[21] The role of GE multipliers μ_i in our tax perturbation approach is akin to the role of consistency constraint multipliers in Rothschild and Scheuer (2013, 2014)'s mechanism design approach.

as described by:

$$\frac{\partial \mathscr{L}^R}{\partial t} = \frac{\partial \mathscr{L}^{R,PE}}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial \mathcal{Y}_i^{R,PE}}{\partial t}. \tag{32}$$

III.3 The role of GE-replicating tax reforms

In this section, we give a streamlined calculation of the GE multiplier, simplifying it to the assessment of specific reforms' incidence. To achieve this, we first demonstrate, in Appendix B.3, that the impact of any log-change in price on welfare, consumption and factor supplies at the GE can be replicated, at the PE, through reforms in the directions:

$$\forall \mathbf{y}: \qquad \mathcal{R}^j(\mathbf{y}) \overset{\text{def}}{\equiv} \left(1 - \mathcal{T}_{y_j}(\mathbf{y})\right) \; y_j, \tag{33}$$

which we refer to as the j^{th} GE-replicating directions. As demonstrated in Appendix B.3, the impact on taxpayers' consumption and factors supply of either a log-change in the j^{th} price at the GE or of a reform of magnitude $t = dp_j/p_j$ in the j^{th} replicating direction at the PE are identical, as both equally modify $\mathbf{x} \mapsto \mathscr{U}\left(\sum_{i=1}^n p_i\, x_i - \mathcal{T}(p_1 x_1, ..., p_n x_n) + t\, \mathcal{R}^j(p_1 x_1, ..., p_n x_n), \mathbf{x}; \mathbf{w}\right)$. In terms of tax policy, this implies that whatever the initial tax reform with direction $R(\cdot)$, the tax authority can always combine it with reforms in the GE-replicating directions $\mathcal{R}^j(\cdot)$ (for $j = 1, ..., n$) to maintain factor supplies, consumption and utility at their PE outcomes. This annihilates the incidence (on taxpayer's behavior and welfare) of the price endogeneity. Formally, we show:

Proposition 2. *For any given direction $R(\cdot)$, a tax reform of magnitude t in the direction $R^N(\cdot)$ defined by:*

$$\forall \mathbf{y}: \qquad R^N(\mathbf{y}) \overset{def}{\equiv} R(\mathbf{y}) - \sum_{j=1}^{n} \gamma_j^R \, \mathcal{R}^j(\mathbf{y}), \tag{34}$$

where $\mathcal{R}^j(\cdot)$ is defined by (33) and:

$$\gamma_j^R \overset{def}{\equiv} \sum_{i=1}^{n} \frac{\Xi_{j,i}}{\mathcal{Y}_i} \frac{\partial \mathcal{Y}_i^{R,PE}}{\partial t} = \frac{\partial \log p_j^{R^N}}{\partial t}, \tag{35}$$

has the same impact at the GE on taxpayers' factor supplies $\mathbf{X}(\mathbf{w}) = X_1(\mathbf{w}), ..., X_n(\mathbf{w})$, consumption $C(\mathbf{w})$ and utility level $U(\mathbf{w})$ as does a reform in the direction R and magnitude t at the PE.

To neutralize the impact of price adjustments induced by reforms in the direction $R(\cdot)$, we construct in (34) a new direction of tax reforms denoted as $R^N(\cdot)$. This new direction subtracts a combination of GE-replicating directions $\mathcal{R}^j(\cdot)$ from the initial direction $R(\cdot)$, with adequate scale factors γ_j^R. In Figure 1, a reform in the direction $R^N(\cdot)$ impacts the economy in several stages. First, it affects taxpayers' factor supplies through the PE effects of a reform in the direction $R(\cdot)$. Second, it modifies factor supplies via the PE effects of a reform in the direction $-\sum_{j=1}^n \gamma_j^R \, \mathcal{R}^j(\cdot)$. Third, these PE effects trigger demand-driven responses, leading to modifications in prices, that induce further responses in factor supplies, and so on. The scale factors γ_j^R in (34) are set so that the second and third effects cancel each other, which leads to: $\gamma_j^R = \partial \log p_j^{R^N} / \partial t$ for $j = 1, ..., N$, as explained in Appendix B.3. In Figure 1, this implies that the effects of reforms in the direction $R^N(\cdot)$ cease after the first demand-driven responses.[22]

Following Proposition 2, a reform in the direction $R^N(\cdot)$ at the GE and a reform in the direction $R(\cdot)$ at the PE have the same impact on taxpayers' utility levels, factor supplies and consumption. Rewriting taxpayers' liabilities as $T(\mathbf{Y}(\mathbf{w})) = \sum_{j=1}^n p_j \; X_j(\mathbf{w}) - C(\mathbf{w})$, the effects on the government's Lagrangian of reforms in the direction $R^N(\cdot)$ at the GE differ from the effects of reforms in the direction $R(\cdot)$ at the PE only due to price changes in the former and not the latter. This leads to:

$$\frac{\partial \mathscr{L}^{R^N}}{\partial t} = \frac{\partial \mathscr{L}^{R,PE}}{\partial t} + \sum_{j=1}^n \frac{\partial \log p_j^{R^N}}{\partial t} \mathcal{Y}_j. \tag{36}$$

Combining (34), (35) and (36) the following lemma provides the difference between the effects of a tax reform on the Lagrangian at the PE and at the GE:

Lemma 4. *At the GE, the effects of a tax reform on the Lagrangian are given by:*

$$\frac{\partial \mathscr{L}^{R}}{\partial t} = \frac{\partial \mathscr{L}^{R,PE}}{\partial t} + \sum_{j=1}^n \left(\sum_{i=1}^n \frac{\Xi_{j,i}}{\mathcal{Y}_i} \frac{\partial \mathcal{Y}_i^{R,PE}}{\partial t} \right) \left(\mathcal{Y}_j + \frac{\partial \mathscr{L}^{\mathcal{R}^j}}{\partial t} \right). \tag{37}$$

Lemma 4 indicates the differences in the impact of a given tax reform at the PE and at the GE. Equation (32) defines the GE multipliers from these differences. Comparing Equations (32) and (37) leads to the following expression for the GE multipliers (see Appendix B.4 for a formal proof):

Proposition 3. *The effects of tax reforms on government's Lagrangian are:*

$$\frac{\partial \mathscr{L}^{R}}{\partial t} = \int \left\{ (g(\mathbf{w}) - 1) \;\; R(\mathbf{Y}(\mathbf{w})) + \sum_{i=1}^n (\mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w})) + \mu_i) \;\; \frac{\partial Y_i^{R,PE}(\mathbf{w})}{\partial t} \right\} \mathrm{d}F(\mathbf{w})$$

where the GE multipliers are given by:

$$\forall i \in \{1, ..., n\} : \qquad \mu_i = \frac{\mathcal{F}_{\mathcal{X}_i} - p_i}{p_i} + \sum_{j=1}^n \frac{\partial \mathscr{L}^{\mathcal{R}^j}}{\partial t} \frac{\Xi_{j,i}}{\mathcal{Y}_i}. \tag{38a}$$

The GE multiplier μ_i consists of two elements: a corrective term for market failures and a corrective term for the suboptimality of the tax system. The corrective term for market failure $(\mathcal{F}_{\mathcal{X}_i} - p_i)/p_i$ assesses whether the marginal product of factor $\mathcal{F}_{\mathcal{X}_i}$ differs from its private return p_i, indicating the absence of perfect competition. In a perfectly competitive setting, this term equals zero and (38a) simplifies to:

$$\forall i \in \{1, ..., n\} : \qquad \mu_i = \sum_{j=1}^n \frac{\partial \mathscr{L}^{\mathcal{R}^j}}{\partial t} \frac{\Xi_{j,i}}{\mathcal{Y}_i}. \tag{38b}$$

[22] In practice, the effects of tax reforms are uncertain. To take this into account, one can assume that the matrix Ξ of inverse demand functions or that the PE effects $\partial \mathcal{Y}_i^{R,PE} / \partial t$ of tax reforms are state-dependent. Proposition 2 is robust to such uncertainty provided that for for each direction $R(\cdot)$, the direction $R^N(\cdot)$, defined in (34), becomes state-dependent since the γ_j^R's defined in (35) become state-dependent.

The second term in (38a) corrects for the suboptimality of the tax system along the GE-replicating directions, indicating whether the tax schedule could be improved along the GE-replicating directions of tax reforms $\mathcal{R}^j$, in which case we have $\partial \mathscr{L}^{\mathcal{R}^j}/\partial t \neq 0$ for at least one $j \in \{1, ..., n\}$. To understand why this second term shows up, notice that tax reforms generate price changes, impacting taxpayers similarly to reforms in the GE-replicating directions $\mathcal{R}^j(\cdot)$. The term $\sum_{j=1}^{n} (\partial \mathscr{L}^{\mathcal{R}^j}/\partial t)\,(\Xi_{j,i}/\mathcal{Y}_i)$ therefore captures that the price changes induced by reform in direction $R(\cdot)$ replicate the effects on taxpayers' factor supply, consumption and utility of reforms in the GE-replicating directions $\mathcal{R}^j$ for $j = 1, ..., n$. To compute the GE multipliers, one must solve (30) for the n GE-replicating directions of tax reforms $\mathcal{R}^j$ ($j = 1, ..., n$) along with the n equations (38a) for all production factors. These computations yield the n GE multipliers $\mu_1, ..., \mu_n$ as well as the effects of reforms along the GE-replicating directions $\partial \mathscr{L}^{\mathcal{R}^j}/\partial t$.

We now explore the possibility of improving an optimal tax system through GE-replicating directions of tax reforms. On the one hand, a tax system which is optimal along the GE-replicating directions $\mathcal{R}^j(\cdot)$ is characterized by $\partial \mathscr{L}^{\mathcal{R}^j}/\partial t = 0$ for all $j \in \{1, ..., n\}$. Equation (38a) then simplifies to:

$$\forall i \in \{1, ..., n\} \qquad \mu_i = \frac{\mathcal{F}_{\mathcal{X}_i} - p_i}{p_i}. \tag{38c}$$

Note that in such a case, the GE multipliers generalize to any kind of market imperfections the corrective term described by Pigou (1920) in the presence of externalities. Finally, if the tax system is optimal along GE-replicating directions and if there is perfect competition, Equation (38a) simplifies to:

$$\forall i \in \{1, ..., n\} \qquad \mu_i = 0. \tag{38d}$$

However, there exist different reasons why reforms in the GE-replicating directions $\mathcal{R}^j(\cdot)$ might not be admissible (i.e. are not part of the available tax instruments), generically leading to $\partial \mathscr{L}^{\mathcal{R}^j}/\partial t \neq 0$ for some $j \in \{1, ..., n\}$ at the optimum. First, the government may be unable to distinguish different incomes from different factors. In such a case, we say that the tax system is not *exhaustive*. In Example 1 with labor and capital, the tax system is exhaustive. Conversely, in Example 2 with low and high skill labor, the tax system is not exhaustive and the government cannot distinguish between different labor income sources. Typical examples of these different labor types are Example 2 with low and high-skilled labor or a case with routine, manual and conceptual labor. To grasp why the optimal tax system may be suboptimal along GE-replicating tax directions in such cases, index $(L, 1), ..., (L, q)$ these different factors behind labor income where L encompasses all types of labor and $i = 1, ..., q$ denotes the specific types of labor. Denote $y_L = y_{L,1} + ... + y_{L,q}$ the total individual labor income and $T_L(\cdot)$ the labor income tax schedule. According to (33), the directions of GE-replicating tax reforms for each type of labor income are:

$$\forall j \in \{1, ..., q\} : \qquad \mathcal{R}^{L,j}(\mathbf{y}) = (1 - T_L'(y_L))\, y_{L,j}. \tag{39a}$$

Since the government cannot distinguish the different types of labor income $y_{L,j}$ from the total labor income y_L, it is unable to reform the tax system in these GE-replicating directions. That is, the tax system is typically suboptimal in GE-replicating directions, i.e. $\partial \mathscr{L}^{\mathcal{R}^j}/\partial t \neq 0$ for some $j \in \{1, ..., n\}$.

Second, the government may choose a tax system within a restricted set of functional forms, e.g. a comprehensive tax schedule of the form $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y}) = T(y_1 + ... + y_n)$ or a schedular system of the form $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y}) = T_1(y_1) + ... + T_n(y_n)$. Various reasons may explain this restricted choice including political considerations, implementability constraints and legal arguments (Haig, 1921, Simons, 1938, Nielsen and Sørensen, 1997, Boadway, 2004). In this case, restrictions on the structure of the tax system might hinder its optimization along all GE-replicating directions. Then, even if the tax system is exhaustive, it might be too restrictive to allow for improvements through tax reforms in the GE-replicating directions. For instance, if the tax schedule is restricted to be comprehensive, the GE-replicating directions defined by (33) are given by:

$$\mathcal{R}^j(\mathbf{y}) = \big(1 - T'(y_1 + ... + y_n)\big) \; y_j \tag{39b}$$

which do not belong to the set of comprehensive tax schedules since y_j is not specifically observed by the tax authority. Consequently, the optimal comprehensive tax system is (generically) not optimal along some of the GE-replicating directions. In Example 2 where the government cannot distinguish incomes from high-skilled labor and from low-skilled labor incomes, the tax schedule must be comprehensive, so the optimal tax schedule does not optimize specifically all GE-replicating directions. However, if the tax schedule is restricted to be schedular, the GE-replicating directions are given by:

$$\mathcal{R}^j(\mathbf{y}) = \big(1 - T_j'(y_j)\big) \; y_j \tag{39c}$$

and are part of a schedular tax system since y_j is specifically observed by the tax authority. The optimal schedular tax system is therefore optimal along all GE-replicating directions and GE multipliers are then given by (38c). In Example 1, the government can apply different schedules to labor and capital incomes, i.e. applies a schedular tax system. Note that when the tax system is schedular and the j^{th} income is taxed at a linear tax rate τ_j, the j^{th} GE-replicating direction simplifies to a linear direction through:

$$\mathcal{R}^j(\mathbf{y}) = (1 - \tau_j) \; y_j. \tag{39d}$$

There are other situations where the government considers some restrictions in the tax system which imply that the GE-replicating directions of tax reforms do not respect these constraints. For example, let consider the case where the tax system is the sum of a comprehensive income tax schedule and income-specific tax schedules, i.e., $\mathbf{y} \mapsto T_0(y_1 + ... + y_n) + T_1(y_1) + ... + T_n(y_n)$. We call it the mixed tax system. From (33), the GE-replicating directions of tax reforms are then given by:

$$\mathcal{R}^j(\mathbf{y}) = \left(1 - T_0'(y_1 + ... + y_n) - T_j'(y_j)\right)\ y_j, \tag{39e}$$

and imply changes in marginal tax rates that depend on both total income $y_1 + ... + y_n$ and on the j^{th} specific income y_j. However, under a mixed tax system, available tax instruments must be functions either of each income y_j or of total income $y_1 + ...y_n$. While $\left(1 - T_j'(y_j)\right)\ y_j$ in (39e) is a part of available tax instruments, $-T_0'(y_1 + ... + y_n)\ y_j$ in (39e) is not, since it depends on both total income $y_1 + ... + y_n$ and j^{th} income y_j. Consequently, the optimal tax system in this case does not optimize along GE-replicating directions of tax reforms. In other words, adding income-specific schedules to a comprehensive income tax schedule does not ensure the optimality of the tax system with respect to reforms in all GE-replicating directions.

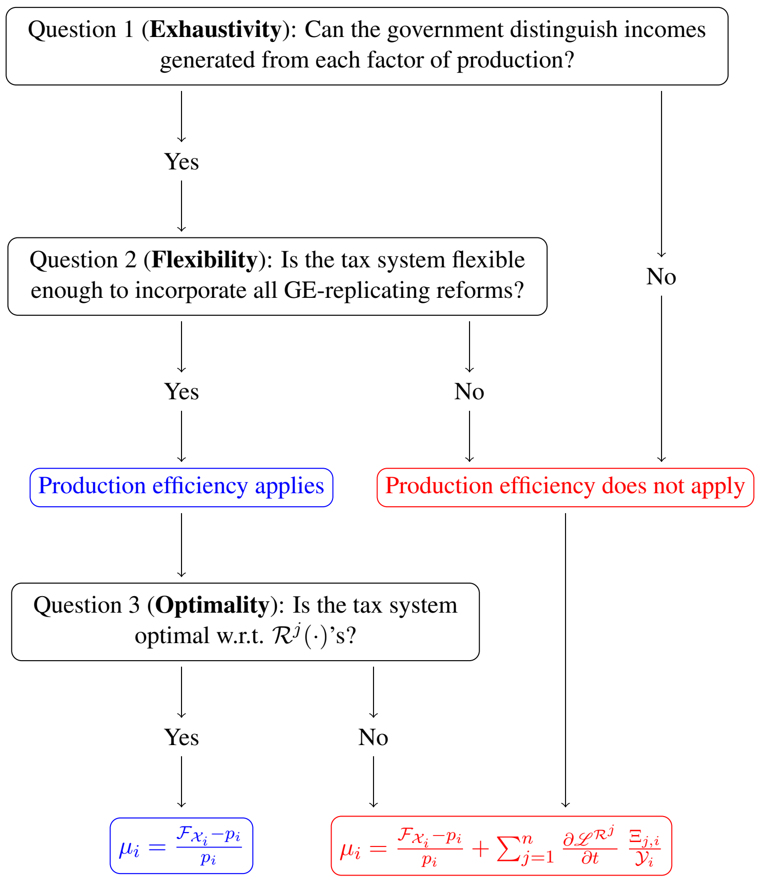

Figure 2: A test on tax systems to determine GE-multipliers and assess production efficiency

In Figure 2, we outline the three key questions guiding the investigation into whether an optimal tax schedule can be improved along the GE-replicating directions of a tax reform.

First, the exhaustivity question verifies whether the government can observe separately the incomes from each production factor. If this this not possible, then the government lacks the information to implement reforms in the GE-replicating direction specific to each factor and, typically, $\partial \mathscr{L}^{\mathcal{R}^j} / \partial t \neq 0$.

Otherwise, if the system is exhaustive, the government may be constrained to adopt a specific functional form that prevents him from reforming the tax system in the GE-replicating directions. In the latter case, once again, typically, $\partial \mathscr{L}^{\mathcal{R}^j}/\partial t \neq 0$. Even if the tax system is exhaustive and flexible, the tax system may not be optimal, which again imply $\partial \mathscr{L}^{\mathcal{R}^j}/\partial t \neq 0$. However, if the tax system is exhaustive, flexible and optimal, one has $\partial \mathscr{L}^{\mathcal{R}^j}/\partial t = 0$ for all GE- replicating directions and the GE multipliers are solely given by market frictions through (38c).

We are now in position to discuss why, there are in the literature contrasting results about whether GE price adjustments affects the optimal tax formula. In Diamond and Mirrlees (1971) and in Saez and Zucman (2023), the optimal tax system is linear, i.e. $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y}) = \sum_{i=1}^{n} t_i \ y_i$, schedular and exhaustive. In such a linear tax system, there is a coincidence between GE-replicating directions, which are $\mathcal{R}^j(\mathbf{y}) = (1 - t_j) y_j$ according to (33), and the directions which change each linear tax rates. Their optimal tax system is therefore also optimal along GE-replicating tax reforms. Since they also assume perfect competition, the GE multipliers are given by (38d) in their context, i.e. they are nil. Hence, their optimal tax formula does not depend on the parameters of the production function.

Saez (2004) proposes two models. In his long run model, agents select an occupation, corresponding to a specific production factor, with a single income level. The tax schedule is then occupation-specific, making it both exhaustive and not too constrained. Therefore, his optimal tax schedule is optimal along GE-replicating tax reforms. Translated within our framework, due to perfect competition, this implies that GE multipliers are nil in (38d), which explains that his optimal tax formula does not depend on the production function parameters.

In Stiglitz (1982), Naito (1999, 2004), Jacobs (2015), Ales and Sleet (2016), Sachs et al. (2020) or Schultz et al. (2023), the tax system is not exhaustive due to the existence of different imperfectly substitutable types of labor that the government cannot distinguish. In the short-run model of Saez (2004), the nonlinear tax schedule is also non-exhaustive. In our framework, this lack of exhaustivity prevents the government from optimizing along GE-replicating directions, which are given by (39a). Hence GE multipliers are typically different from zero, even under competition.

In Rothschild and Scheuer (2013, 2016) and Scheuer (2014) workers choose a sector in which to work. They are endowed with different skills, which are perfect substitutes within each sector. In Rothschild and Scheuer (2013, 2016), the income tax is not sector-specific, leading to a lack of exhaustivity in the tax system. This implies GE price adjustments that modify their optimal tax formulas. In Scheuer (2014), two sectors co-exist: salary workers and entrepreneurs. Scheuer (2014) contrasts two cases. Under comprehensive taxation, the optimal tax system does not optimize along GE-replicating reforms, which are given by (39b), and the optimal tax formula is again affected by GE adjustments. Conversely, if the government can tax differently salary workers and entrepreneurs, the optimal tax system optimizes along GE-replicating reforms described in (39c). As a result, the optimal tax system does not depend on the parameters of the production sector.

III.4 Pareto-improving tax reforms

In this section, we provide conditions for the existence of Pareto-improving directions of tax reform in the presence of GE adjustments. We show how to test whether a given tax system can be Pareto improved and whether a given tax reform is Pareto-improving. As a preamble to this exercise, we must establish the optimal tax system when it is exhaustive and there is no restriction on its form.

For this purpose, we denote W_Y the income space, ∂W_Y its smooth boundary. Let $\partial\widehat{Y}_i(\mathbf{y})/\partial\tau_j$, $\partial\widehat{Y}_i(\mathbf{y})/\partial\rho$ and $\widehat{g}(\mathbf{y})$ denote the mean values of $\partial Y_i(\mathbf{w})/\partial\tau_j$, $\partial Y_i(\mathbf{w})/\partial\rho$ and $g(\mathbf{w})$, respectively, among taxpayers with earnings $\mathbf{Y}(\mathbf{w}) = \mathbf{y}$. The following proposition, proved in Appendix B.5 characterizes the optimal tax system when it is exhaustive and unrestricted.

Proposition 4. *When the tax system is exhaustive and there is no restriction on its form, the optimal tax system has to verify, $\forall y \in W_Y$:*

$$\left[1 - \widehat{g}(\mathbf{y}) - \sum_{i=1}^{n} \left(\mathcal{T}_{y_i}(\mathbf{y}) + \mu_i\right) \frac{\partial\widehat{Y}_i(\mathbf{y})}{\partial\rho}\right] h(\mathbf{y}) = -\sum_{j=1}^{n} \frac{\partial\left[\sum_{i=1}^{n} \left(\mathcal{T}_{y_i}(\mathbf{y}) + \mu_i\right) \frac{\partial\widehat{Y}_i(\mathbf{y})}{\partial\tau_j}\, h(\mathbf{y})\right]}{\partial y_j}, \tag{40}$$

and the boundary conditions:

$$\forall y \in \partial W_Y : \sum_{1\leq i,j\leq n} \left(\mathcal{T}_{y_i}(\mathbf{y}) + \mu_i\right) \frac{\partial\widehat{Y}_i(\mathbf{y})}{\partial\tau_j} h(\mathbf{y})\phi_j(\mathbf{y}) = 0 \tag{41}$$

where $\phi(\mathbf{y}) = (\phi_1(\mathbf{y}), ..., \phi_n(\mathbf{y}))$ is the outward unit vector normal to the boundary at $\mathbf{y}$, where the GE multipliers are given by (38c). Under perfect competition, $\mu_i = 0$, for $i = 1, ..., n$.

The Partial Differential Equation (40) is a divergence equation that must hold for any income $\mathbf{y}$. Equations (41) are boundary conditions that must hold at any income $\mathbf{y} \in W_Y$ in the boundary of W_Y.[23] The aforementioned tax formulas describe the optimal tax system which is unconstrained on its form, across a large spectrum of economic environments (e.g., with any type of market failure or under perfect competition, with a production factors which are imperfect substitutes or not). Since the system is optimized and not restricted at all on its form, we have $\partial\mathscr{L}^{\mathcal{R}^j}/\partial t = 0$ for all j, i.e. the tax system is optimized along the GE-replicating directions defined in (33). Hence, according to (38a), GE multipliers are given by (38c) to correct for market failures, if any. Under perfect competition, GE multipliers are nil as given by (38d).

We now explore the identification of Pareto-improving directions of tax reforms, under the assumption of perfect competition. To do so, based on (40), one needs to calculate revealed welfare weights, as

[23] Proposition 4 extends to a context with GE effects and market failures the optimal tax formulas of Mirrlees (1976), Golosov et al. (2014), Spiritus et al. (2023), Boerma et al. (2022) and Golosov and Krasikov (2023).

detailed in Appendix B.5. The literature on the inverse tax problem solves for these weights for which an observed tax system satisfies the first-order conditions of an optimal tax problem, with one source of income.[24] With multiple incomes, as shown in Appendix B.5, revealed welfare weights are defined as:

$$\widetilde{g}(\mathbf{y}) \stackrel{\text{def}}{\equiv} 1 - \sum_{i=1}^{n} \mathcal{T}_{y_i}(\mathbf{y}) \frac{\partial \widehat{Y}_i(\mathbf{y})}{\partial \rho} + \frac{1}{h(\mathbf{y})} \sum_{j=1}^{n} \frac{\partial \left[\sum_{i=1}^{n} \mathcal{T}_{y_i}(\mathbf{y}) \frac{\partial \widehat{Y}_i(\mathbf{y})}{\partial \tau_j} h(\mathbf{y}) \right]}{\partial y_j} \tag{42}$$

The function $h(\cdot)$ denotes the joint income density. With real data, these revealed welfare weights can be computed using (42) and estimations of compensated responses $\partial \widehat{Y}_i(\mathbf{y})/\partial \tau_j$, of income responses $\partial \widehat{Y}_i(\mathbf{y})/\partial \rho$ and income density $h(\mathbf{y})$.

Lorenz and Sachs (2016), Hendren (2020), Bierbrauer et al. (2023) and Bergstrom and Dodds (2023) show that negative revealed welfare weights indicate a Pareto inefficiency in the observed tax system, when taxpayers earn a single income ($n = 1$). In a PE framework with multiple income sources, Spiritus et al. (2023, Proposition 3) also show that a reform in the direction $R(\cdot)$ such that:[25]

$$R(\mathbf{y}) = 0 \quad \text{if} \quad \widetilde{g}(\mathbf{y}) \geq 0 \qquad\qquad R(\mathbf{y}) \geq 0 \quad \text{if} \quad \widetilde{g}(\mathbf{y}) < 0 \tag{43}$$

and a small positive t is Pareto-improving at the PE. In Appendix B.6, we show that combining this result with Proposition 2 yields the following proposition.

Proposition 5. *Under perfect competition, if $\widetilde{g}(\mathbf{y}) < 0$ for some income bundles $\mathbf{y}$ within the interior of the income bundle space, then there exist directions $R(\cdot)$ and $R^N(\cdot)$ that verify Equations (34), (35) and (43) such that a reform in the direction R^N with a magnitude $t > 0$ is Pareto-improving at the GE.*

According to Proposition 2, the GE-replicating reform $R^N(\cdot)$ defined by Equations (34) and (35) and a reform in the direction $R(\cdot)$ at the PE exerts the same impact on taxpayers' factor supplies and utilities at the GE. Since the former is Pareto-improving, the latter achieves Pareto improvement only if the change in price does not reduce tax revenue. Rewriting tax liabilities as $T(\mathbf{Y}(\mathbf{w})) = \sum_{j=1}^{n} p_j\, X_j(\mathbf{w}) - C(\mathbf{w})$, the difference between the effects on tax revenue of a reform in the direction $R^N(\cdot)$ at the GE and a reform in the direction $R(\cdot)$ at the PE is therefore equal to $\sum_{j=1}^{n} \mathcal{X}_j \; \mathrm{d}p_j = \sum_{j=1}^{n} \mathcal{Y}_j \; \mathrm{d}p_j / p_j$. This difference is zero under perfect competition. Note that the Pareto-improving tax reform described in Proposition 5 may not be implementable if the tax schedule is non exhaustive or must have a constrained form.

The following proposition, proofed in Appendix B.7 establishes that positive welfare weights are both necessary and sufficient for the non-existence of a Pareto-improving direction.

[24] See, for instance, Bourguignon and Spadaro (2012), Bargain et al. (2014), Lorenz and Sachs (2016), Jacobs et al. (2017), Hendren (2020), Bierbrauer et al. (2023) and Bergstrom and Dodds (2023).

[25] We obviously exclude a zero direction where $R(\mathbf{y}) = 0$ for all $\mathbf{y} \in \mathcal{W}_Y$.

Proposition 6. *Under perfect competition, if $\widetilde{g}(\mathbf{y}) \geq 0$ almost everywhere for income bundles $\mathbf{y}$ within the interior of the income bundle space, then there is no Pareto-improving direction neither at the PE, nor at the GE.*

It is noteworthy that, as in Bierbrauer et al. (2023) with a single income, Proposition 6 does not exclude the existence of a Pareto-improving reform which would be non-infinitesimal, i.e. a Pareto improvement resulting from a large magnitude t.

III.5 Optimal schedular taxation

Proposition 4 characterizes the optimal tax function when the tax system is exhaustive and there is no restriction on its form. In the presence of numerous income types and sources of income, the lack of restrictions on the form of the tax system results in an optimal tax formula expressed as a partial differential equation. However, we argue that in reality, the tax code combines many functions (schedules), each of them depending on a single argument (tax base). The imposition of such a realistic restriction on the tax system takes our exploration a step further, revealing that with numerous types and income sources, the optimal tax system must now conform to a system of ordinary differential equations, adopting the ABC form introduced by Diamond (1998) and Saez (2001). This transformation not only enhances the mathematical tractability of the model but, critically, introduces a more realistic framework leading to more intuitive optimal tax formulas.

In this subsection, we investigate the case where the tax system is *schedular*,[26] i.e. is the sum of n income-specific functions $T_i(\cdot)$, so that:

$$\mathcal{T}(y_1, ..., y_n) = \sum_{i=1}^{n} T_i(y_i).$$

Moreover, we introduce the possibility that for some incomes, say those for $i > n'$, with $1 \leq n' \leq n$, the corresponding tax schedule is linear i.e. $T_i(y_i) = t_i \ y_i$ where t_i is a real number. Let then denote $h_i(\cdot)$ the density of the i^{th} income and $H_i(\cdot)$ the corresponding CDF. For any variable $Z(\mathbf{w})$ and for any $i = 0, ..., n$, we denote $\overline{Z(\mathbf{w})}|_{Y_i(\mathbf{w})=y_i}$ the mean of $Z(\mathbf{w})$ among types $\mathbf{w}$ for which $Y_i(\mathbf{w}) = y_i$. The notation $\varepsilon_i(y_i)$ refers to the compensated elasticity of the i^{th} income with respect to its own marginal net-of-tax rate. The corresponding uncompensated elasticity is denoted $\varepsilon_i^u(y_i)$. These means of elasticities are calculated among $\mathbf{w}$-taxpayers who earn their i^{th} income equal to y_i:

$$\varepsilon_i(y_i) \stackrel{\text{def}}{\equiv} \frac{1 - T_i'(y_i)}{y_i} \overline{\frac{\partial Y_i}{\partial \tau_i}}\Bigg|_{Y_i(\mathbf{w})=y_i} \qquad \text{and :} \qquad \varepsilon_i^u(y_i) \stackrel{\text{def}}{\equiv} \frac{1 - T_i'(y_i)}{y_i} \overline{\frac{\partial Y_i^u}{\partial \tau_i}}\Bigg|_{Y_i(\mathbf{w})=y_i}. \tag{44}$$

We thus get the following Proposition, which is proved in Appendix B.8:

[26] Costa Rica, Denmark, Finland, Greece, Hungary, Iceland, Israel, Italy, Latvia, Lithuania, Netherlands, Norway, Poland Slovenia, Spain, Sweden, Türkiye have schedular tax systems according to Hourani et al. (2023, Table A1).

Proposition 7. *When the tax system is exhaustive and schedular, the GE multipliers $\mu_1, ..., \mu_n$ are given by* (38c) *at the optimum which has also to verify:*

a) When the i^{th} schedule is nonlinear, i.e. for $i = 1, ..., n'$:

$$\frac{T_i'(y_i) + \mu_i}{1 - T_i'(y_i)} \, \varepsilon_i(y_i) \, y_i \, h_i(y_i) + \sum_{1 \le k \le n, k \neq i} \overline{\left(T_k'(Y_k(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k(\mathbf{w})}{\partial \tau_i}}\Bigg|_{Y_i(\mathbf{w}) = y_i} h_i(y_i)$$

$$= \int_{z = y_i}^{\infty} \left\{ 1 - \overline{g(\mathbf{w})}\big|_{Y_i(\mathbf{w}) = z} - \sum_{k=1}^{n} \overline{\left(T_k'(Y_k(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k(\mathbf{w})}{\partial \rho}}\Bigg|_{Y_i(\mathbf{w}) = z} \right\} \mathrm{d}H_i(z). \quad (45a)$$

b) When the i^{th} schedule is linear, i.e. for $i = n' + 1, ..., n$:

$$\frac{t_i + \mu_i}{1 - t_i} \int_{\mathcal{W}} \varepsilon_i^u(\mathbf{w}) \, Y_i(\mathbf{w}) \, \mathrm{d}F(\mathbf{w}) + \int_{\mathcal{W}} \sum_{k=1, k \neq i}^{n} \left(T_k'(Y_k(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k^u(\mathbf{w})}{\partial \tau_i} \, \mathrm{d}F(\mathbf{w})$$

$$= \int_{\mathcal{W}} \left[1 - g(\mathbf{w})\right] Y_i(\mathbf{w}) \, \mathrm{d}F(\mathbf{w}). \quad (45b)$$

To grasp the economic intuitions behind (45a), consider a small increase in the i^{th} marginal tax rate around income y_i and a uniform increase in tax liabilities for all taxpayers with their i^{th} income above y_i. Given the other tax schedules, the tax schedule specific to the i^{th} income is optimal only if these reforms do not imply any first-order effects on the Lagrangian. In Equation (45a), the costs and gains resulting from these reforms are equated.

A rise in the i^{th} marginal tax rate around y_i implies direct compensated responses, $\partial Y_i(\mathbf{w}) / \partial \tau_i$, of the i^{th} income which is proportional to the mean compensated elasticity ε_i of the i^{th} income with respect to its own marginal net-of-tax rate (as emphasized in Equation (44)). A first difference with the one income ABC tax formula is that all behavioral responses have to be averaged across taxpayers who earn the same i^{th} income y_i. Composition effects then take place (Jacquet and Lehmann, 2021). A second difference arises due to the GE price adjustments. Under imperfect competition, the optimal tax formulas include a corrective term which corresponds to the GE multipliers $\mu_1, ..., \mu_n$ given by (38c). Under perfect competition, these corrective terms are nil, as in Saez (2001). This arises from the fact that according to (39c), the GE-replicating directions $\mathcal{R}^j(\cdot)$ are part of a schedular tax system . Consequently, an optimal schedular tax system optimizes along all GE-replicating directions, i.e. $\partial \mathscr{L}^{\mathcal{R}^j} / \partial t = 0$. The GE multipliers are then given by (38c) under imperfect competition and are nil under perfect competition, as established in Proposition 3. A third difference occurs because a rise in the i^{th} marginal tax rate triggers (compensated) cross-base responses of all other tax bases $\partial Y_k(\mathbf{w}) / \partial \tau_i$ for $k \in \{1, ..., n\} \setminus \{i\}$ (see the second term of the left-hand side of (45a)). For example, taxpayers can report some of their i^{th} income as k^{th} income, with $k \neq i$, when the i^{th} marginal tax rate rises (i.e. the i^{th} marginal net of tax rate τ_i declines), a phenomenon known as income shifting. The compensated increase in the k^{th} income due to income-shifting, i.e. $\partial Y_k(\mathbf{w}) / \partial \tau_i < 0$, can partly offset the loss due to the compensated responses of the i^{th} income. Conversely, positive cross base responses ($\partial Y_k/\partial \tau_i > 0$), as in the two-period example of Section A.2, can exacerbate the loss due to compensated responses of the i^{th} income.

As usual, a rise in the tax liability above income y_i implies mechanical gains in terms of tax revenue and mechanical welfare losses that are emphasized by the aggregation of $1 - \overline{g(\mathbf{w})|}_{Y_i(\mathbf{w})=z}$ for all $z \geq y_i$ in the right-hand side of (45a). It also creates income effects $\partial y_i(\mathbf{w})/\partial \rho$ in the right-hand side of (45a). Again, compared to the one income optimal income tax formula, welfare weights and incomes responses have first to be aggregated for all income earners with income above y. Second, if competition is imperfect, income responses may be attenuated or exacerbated by GE price adjustments. Third, income response matter for all income sources y_k for $k = 1, ..., n$.

From (45b), we see that, when the tax schedule on the i^{th} income is restricted to be linear, with no restriction on the other tax schedules, similar intuitions than under nonlinear tax schedule apply. There are however several particularities. First, under a linear tax schedule, income effects and compensated effects can be combined and are equivalent to uncompensated responses, as can been verified using the Slutsky Equation (20). Replacing the sum of income and compensated effects by the uncompensated ones implies fewer terms in the right-hand side of (45b) compared to (45a). Second, in the optimal linear tax formula (45b), integrals emphasize that means of sufficient statistics over the whole population need to be estimated instead of means of sufficient statistics at each income level. Third, as expected from the optimal linear tax formula (see e.g. Piketty and Saez (2013)), the mean of welfare weights and uncompensated elasticities are income-weighted. Conversely, the mean of uncompensated cross-base responses $\partial Y_k^u(\mathbf{w})/\partial \tau_i$ for $k \neq i$ are not income-weighted since they are expressed in derivatives rather than elasticities.

Finally, we provide an order of magnitude of how important GE effects are from a back-to-the envelope calculation. For this exercise, assume there are no cross-base or income responses and fix the right-hand sides of (45a)-(45b). For simplicity, assume there is neither cross base response nor income responses and fix the right-hand sides of (45a)-(45b). Let $T_i^{\prime,PE}$ denote the optimal marginal tax rate from the right-hand sides of (45a)-(45b), when the GE multipliers are erroneously ignored. The optimal marginal tax rates that take into account GE price adjustments are related to $T_i^{\prime,PE}$ and to the GE multipliers by:[27]

$$T_i' = T_i^{\prime,PE} - \mu_i \left(1 - T_i^{\prime,PE}\right)$$

For example, if $T_i^{\prime,PE} = 0$, the optimal marginal tax rate is equal to minus the GE multipliers. In the absence of a redistributive motive, the marginal tax rate deviates from zero only to correct for market

[27] Put differently T_i', $T_i^{\prime,PE}$ and μ_i are related by:

$$\frac{T_i' + \mu_i}{1 - T_i'} = \frac{T_i^{\prime,PE}}{1 - T_i^{\prime,PE}}$$

where these ratios are equal to the right-hand side of (45a) or (45b).

inefficiencies in a Pigou (1920) way. Marginal tax rates then vary one to one with the value of the GE multiplier. However, if the redistributive motive is high enough (which implies larger $T_i^{\prime,PE}$), the effect of the GE multiplier on the optimal marginal tax rate is of a smaller order of magnitude. To illustrate this point, Table 1 shows that the higher the marginal tax rate at the PE (i.e. the higher the redistributive motive) in the first column, the lower the effect of GE multiplier (in the top row) on optimal tax rates.

		μ_i				
		-0.10	-0.05	0	0.05	0.10
$T_i^{\prime,PE}$	20%	28%	24%	20%	16%	12%
	40%	46%	43%	40%	37%	34%
	60%	64%	62%	60%	58%	56%
	80%	82%	81%	80%	79%	78%

Table 1: How much GE multipliers matter?

III.6 Optimal Comprehensive Taxation

Building upon Haig (1921) and Simons (1938), we now turn our attention to comprehensive tax schedules, wherein the tax function depends on the sum of all incomes, so-called taxable income. Formally, the tax schedule takes the form

$$\mathcal{T}(\mathbf{y}) = T_0\left(y_1 + ... + y_n\right)$$

where $y_0 \stackrel{\text{def}}{\equiv} y_1 + ... + y_n$ and $Y_0(\mathbf{w}) = Y_1(\mathbf{w}) + ... + Y_n(\mathbf{w})$. Tax systems of countries such as Australia, Canada, Chile, Luxembourg, New Zealand, Switzerland, the United Kingdom and the United States can be viewed as close approximations to comprehensive tax systems, see Hourani et al. (2023, Table A1).[28]

We denote $h_0(\cdot)$ the density of taxable income and $H_0(\cdot)$ the associated CDF. Since marginal tax rate on all incomes is equal to $T_0'(y_1 + ..., y_n)$, the compensated responses with respect to the marginal net of tax rate is given by:

$$\forall i \in \{0, ..., n\} \qquad \frac{\partial Y_i}{\partial \tau_0} = \sum_{j=1}^{n} \frac{\partial Y_i(\mathbf{w})}{\partial \tau_j}, \tag{46}$$

the compensated elasticity of taxable income is:

$$\varepsilon_0(y_0) = \frac{1 - T_0'(y_0)}{y_0} \sum_{1 \leq i,j \leq n} \overline{\frac{\partial Y_i(\mathbf{w})}{\partial \tau_j}}\Big|_{Y_0(\mathbf{w})=y_0} \tag{47}$$

which is positive,[29] and the income response of taxable income are given by:

$$\frac{\partial Y_0(y_0)}{\partial \rho} = \sum_{k=1}^{n} \overline{\frac{\partial Y_k(\mathbf{w})}{\partial \rho}}\Big|_{Y_0(\mathbf{w})=y_0} \tag{48}$$

[28]In the United Kingdom, income is taxed comprehensively, yet distinct tax rates are applied to capital gains and dividend income. Different allowances exist for savings, dividends, capital gains, and property. Similarly, the United States taxes income comprehensively but employs varied tax rates for long-term capital gains and certain types of dividends.

[29]Since the matrix $\left[\frac{\partial Y_i(\mathbf{w})}{\partial \tau_j}\right]_{i,j}$ is positive definite, taxable income's compensated elasticity is positive.

This elasticity depends on every compensated responses $\partial Y_i(\mathbf{w})/\partial\tau_j$ to changes in every net-of-marginal tax rate τ_j for $i, j \in \{1, ..., n\}$. The following proposition, which is proved in Appendix B.9, characterizes the optimal comprehensive income tax schedule.

Proposition 8. *When the tax system is comprehensive, the GE multipliers $\mu_1, ..., \mu_n$ are given by (38a) at the optimum which has also to verify:*

$$\frac{T_0'(y_0)}{1-T_0'(y_0)}\,\varepsilon_0(y_0)\,y_0\,h_0(y_0)+\sum_{1\le k\le n}\mu_k\,\overline{\frac{\partial Y_k(\mathbf{w})}{\partial\tau_0}}\Big|_{Y_0(\mathbf{w})=y_i}h_0(y_0) \tag{49}$$

$$=\int_{z=y_0}^{\infty}\left\{1-\overline{g(\mathbf{w})}\big|_{Y_0(\mathbf{w})=z}-T_0'(z)\,\frac{\partial Y_0(z)}{\partial\rho}-\sum_{k=1}^{n}\mu_k\,\overline{\frac{\partial Y_k(\mathbf{w})}{\partial\rho}}\Big|_{Y_0(\mathbf{w})=z}\right\}\mathrm{d}H_0(z).$$

This optimal income tax formula differs from the usual ABC formula only by the presence of GE multipliers. Under a comprehensive tax system, the GE-replicating directions of tax reforms are given by (39b) and do not belong to the set of comprehensive tax schedules. Hence the optimal comprehensive tax function does not optimize along all GE-replicating directions. This is due to the non-exhaustiveness nature of the tax system, where only the sum of all income y_0 determines tax liabilities. Hence the optimal tax system has to solve (38) for all $k = 1, ..., n$ together with (49) for all income levels.

To better understand how GE price adjustments affect the optimal comprehensive tax schedule, we consider the simple case with two production factors $n = 2$ and perfect competition. In this case, the GE-replicating directions (39b) simplify to $\mathcal{R}^1(y_1, y_2) = (1 - T_0'(y_1 + y_2))\, y_1$ and $\mathcal{R}^2(y_1, y_2) = (1 - T_0'(y_1+y_2))\, y_2$. The optimal comprehensive tax system optimizes along all comprehensive tax directions, including $(1 - T_0'(y_1 + y_2))\,(y_1 + y_2) = \mathcal{R}^1(y_1, y_2) + \mathcal{R}^2(y_1, y_2)$, but does (generically) not optimize along $\mathcal{R}^1$ or $\mathcal{R}^2$ separately. Optimizing along $(1 - T_0'(y_1 + y_2))\,(y_1 + y_2) = \mathcal{R}^1(y_1, y_2) + \mathcal{R}^2(y_1, y_2)$ leads to $\partial\mathscr{L}^{\mathcal{R}^1}/\partial t + \partial\mathscr{L}^{\mathcal{R}^2}/\partial t = 0$ by Gateaux differentiability of the Lagrangian with respect to the tax reforms. Denoting σ the elasticity of substitution between the two production factors, one obtains:

$$\mu_1=-\frac{1}{\sigma\,\mathcal{Y}_1}\frac{\partial\mathscr{L}^{\mathcal{R}^1}}{\partial t}\qquad \mu_2=-\frac{1}{\sigma\,\mathcal{Y}_2}\frac{\partial\mathscr{L}^{\mathcal{R}^2}}{\partial t} \tag{50}$$

These two GE multipliers have opposite signs. Let $\mathscr{Y}_k(y_0)$ denote the mean k^{th} income earned by taxpayers with taxable income y_0. Define:

$$\varepsilon_k^0(y_0)\stackrel{\text{def}}{\equiv}\frac{1-T_0'(y_0)}{\mathscr{Y}_k(y_0)}\overline{\frac{\partial Y_k(\mathbf{w})}{\partial\tau_0}}\Big|_{Y_0(\mathbf{w})=y_0}$$

as the elasticity of the mean of the k^{th} income, with respect to the net-of-marginal tax rate of taxable income, among taxpayers earning taxable income y_0. Fixing the right-hand side of (49), Equation (50) indicates that GE price adjustments affect the optimal marginal tax rate at taxable income y_0 in proportion to:

$$\sum_{1\le k\le n}\mu_k\,\overline{\frac{\partial Y_k(\mathbf{w})}{\partial\tau_0}}\Big|_{Y_0(\mathbf{w})=y_i}=\frac{1}{\sigma\,(1-T_0'(y_0))}\,\frac{\partial\mathscr{L}^{\mathcal{R}^1}}{\partial t}\left[\frac{\mathscr{Y}_2(y_0)}{\mathcal{Y}_2}\,\varepsilon_2^0(y_0)-\frac{\mathscr{Y}_1(y_0)}{\mathcal{Y}_1}\,\varepsilon_1^0(y_0)\right]$$

The impact of GE adjustments on the optimal marginal tax rates at taxable income y_0 relies on the sign of $\partial \mathscr{L}^{\mathcal{R}^1}/\partial t$ which is the same across the taxable income distribution. Conversely, the term in square brackets may vary with taxable income. This term compares the two elasticities with respect to the net-of-marginal tax rate scaled by the ratios of average k^{th} income at taxable income y_0 over aggregate k^{th} income $\mathcal{Y}_k$. In particular, if the two elasticities are identical, as is the case for instance in Rothschild and Scheuer (2013) or in Sachs et al. (2020), then the impact of GE price adjustments on the optimal marginal tax rates may be positive at low taxable income levels y_0 and negative at high taxable income levels, as in Figure 2 of Rothschild and Scheuer (2013) and Figure 4 of Sachs et al. (2020). In our framework, this outcome occurs when $\partial \mathscr{L}^{\mathcal{R}^1}/\partial t > 0$ and if taxpayers with low (high) taxable income y_0 earn relatively more (less) income 2 and relatively less (more) income 1 than in the overall population.

IV Production policies

In this section, we investigate the effects of production policies. Our analysis of production policy changes extends to any shock that modifies the production set like technological shocks or expanded trade opportunities. First, we determine how production policy reforms affect the economy and when they are desirable. To achieve this, we use a reduced-form description of the production sector employing inverse demand functions $\mathscr{P}_i(\cdot)$. Second, we provide different examples with specific microfoundations tailored to our reduced-form for the production sector. With these examples, we illustrate how the use of a reduced-form enables us to derive results that cover a large set of production policies. This includes, but is not limited to, the taxation of intermediate goods (such as robot and AI taxation), public production, commodity taxation, competition policies, trade policies and modifications in business-focused environmental regulations.

IV.1 The effects of production policy reforms

Production policy reforms impact the economy exclusively through the production sector thereby, affecting the taxpayers indirectly through the induced changes in prices. These price changes lead taxpayers to modify their incomes and to experience a change in welfare that are respectively given by (see Appendix B.1):

$$\frac{\partial Y_i(\mathbf{w})}{\partial \alpha_\ell} = \sum_{j=1}^{n} \frac{\partial Y_i(\mathbf{w})}{\partial \log p_j} \frac{\partial \log p_j}{\partial \alpha_\ell} \tag{51a}$$

$$\frac{1}{\lambda}\frac{\partial \Phi\left(U(\mathbf{w});\mathbf{w}\right)}{\partial \alpha_\ell} = \left(\sum_{j=1}^{n} \left(1-\mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w}))\right) Y_j(\mathbf{w}) \frac{\partial \log p_j}{\partial \alpha_\ell}\right) g(\mathbf{w}), \tag{51b}$$

for each component α_ℓ of the vector of production policies. Equation (51a) is obtained by differentiating taxpayers' first-order conditions (5), while Equation (51b) is obtained by applying the envelope theorem on (3).

The impact of production policies on the Lagrangian is thus equal to the sum, for each factor, of the product of the log change in price induced by the production policy reform and the impact of log price change on the Lagrangian.

$$\frac{\partial \mathscr{L}}{\partial \alpha_\ell} = \sum_{j=1}^{n} \frac{\partial \mathscr{L}}{\partial \log p_j} \frac{\partial \log p_j}{\partial \alpha_\ell} \qquad \forall \ell \in \{1, ..., L\} \tag{51c}$$

To compute the GE effects of production policies on prices $\partial \log p_j / \partial \alpha_\ell$, we combine the log differentiation of inverse demand equations (6) with respect to factor supplies $\mathcal{X}_i$ and production policies $\boldsymbol{\alpha}$ using (24a) and the log differentiation of the factor supplies with respect to price using (24b) leads to the following lemma (see Figure 1).

Lemma 5. *After a production policy reform, the vectors $\partial \log \boldsymbol{p} / \partial \alpha_\ell$ of log-price changes at the GE are:*

$$\forall \ell \in \{1, ..., L\} : \qquad \frac{\partial \log \boldsymbol{p}}{\partial \alpha_\ell} = (I_n - \Xi \cdot \Gamma)^{-1} \cdot \frac{\partial \log \mathscr{P}}{\partial \alpha_\ell} \tag{52}$$

where $\partial \log \mathscr{P} / \partial \alpha_\ell$ is the vector for which the i^{th} term, $\partial \mathscr{P}_i / \partial \alpha_\ell$, describes how the ℓ^{th} production policy reform modifies the i^{th} price absent any change in factor supplies.

Differentiating both sides of (7) with respect to the strength of the ℓ^{th} production policies leads to:

$$\forall \ell \in \{1, ..., L\} : \qquad \mathcal{F}_{\alpha_\ell} = \sum_{j=1}^{n} \mathcal{Y}_j \frac{\partial \log \mathscr{P}_j}{\partial \alpha_\ell}. \tag{53}$$

In the following proposition, which is proved in Appendix C.1, we present the impact on the Lagrangian of any production policy reform.

Proposition 9. *The effects of production policies on the Lagrangian are given by:*

$$\forall \ell \in \{1, ..., L\} : \qquad \frac{\partial \mathscr{L}}{\partial \alpha_\ell} = \mathcal{F}_{\alpha_\ell} + \sum_{j=1}^{n} \frac{\partial \mathscr{L}^{\mathcal{R}^j}}{\partial t} \frac{\partial \log \mathscr{P}_j}{\partial \alpha_\ell}. \tag{54}$$

Proposition 9 characterizes the social desirability of any production policy reform or technological shock, $d\alpha_\ell$. The effect of a production policy reform on the Lagrangian can be decomposed into two components. Firstly, there is a direct mechanical effect of the production policy reform itself, absent any behavioral responses. According to (53), this mechanical effect coincides with the effect of the production policy reform on the production function, capturing its impact on production efficiency. Secondly, the production policy reform induces price changes $\partial \log \mathscr{P}_j / \partial \alpha_\ell$, which trigger behavioral responses and GE effects. These changes in factor prices generate distributional consequences, as captured by the second term of (54), which measures the pre-distributive effect of production policy reforms (or technological shocks). When the production policies α_ℓ, for $\ell = 1, ..., L$ are optimal, the net impact in terms of efficiency and pre-distribution is null, resulting in Equation (54) being equal to zero. More generally, the social desirability of any production policy reform can be tested using Equation (54). First, when the tax system is exhaustive and not overly restricted, its optimality along the GE-replicating directions $\mathcal{R}^j(\cdot)$– i.e. $\partial \mathscr{L}^{\mathcal{R}^j}/\partial t = 0$ for all $j = 1, ..., n$– implies that the desirability of any policy reform depends only on the sign of the mechanical efficiency effect $\mathcal{F}_{\alpha_\ell}(\cdot)$. Second, when the tax system is either not exhaustive or too restricted, assessing the desirability of reforms or shocks requires evaluating both the mechanical efficiency effect and the pre-distributive effect, ie. both terms in the right-hand side of (54). This requires to determine the magnitude of both the mechanical effect of the production policy and its pre-distributive effect. To compute $\partial \mathscr{L}^{\mathcal{R}^j}/\partial t$, one must solve (30) for the n GE-replicating directions of tax reforms $\mathcal{R}^j$ ($j = 1, ..., n$) along with the n GE-multiplier equations (38a) for all production factors.

Equation (54) highlights that a production policy reform aimed at enhancing production efficiency (i.e. $\mathcal{F}_{\alpha_\ell}(\cdot) > 0$) may be undermined by too negative predistributive effects, resulting in a negative right-hand side. In an economy where the tax system is not exhaustive or too restricted, simply labeling a production policy reform as efficiency-improving is insufficient; it is also crucial to ensure that negative pre-distributive effects do not offset these efficiency gains.

We now consider the effect of a multidimensional production policy reform, denoted $t \mapsto (\alpha_1(t),$ $..., \alpha_L(t))$, where t represents the magnitude of the production policy reform and the $\alpha_\ell(\cdot)$ are continuously differentiable functions. Following (7), we define production efficient policy reforms as:

Definition 4. *The (multidimensional) production policy reform $(\alpha_1(t), ..., \alpha_L(t))$ is production efficient if:*

$$\sum_{\ell=1}^{L} \mathcal{F}_{\alpha_\ell}\, \alpha'_\ell(t) > 0.$$

In the following proposition, as demonstrated in Appendix C.2, we establish that for any production efficient policy reform, there exists a tax reform direction denoted R^N, such that combining the production policy reform with a tax reform in the direction R^N is Pareto-improving.

Proposition 10. ***Generalized Production Efficiency Principle***

Combining a production efficient policy reform $(\alpha_1(t), \ldots, \alpha_L(t))$ with a tax reform in the direction $R^N(\cdot) = \sum \gamma_j\, \mathcal{R}^j(\cdot)$ such that:

$$\forall j \in \{1, \ldots, n\} : \quad \gamma_j = -\sum_{\ell=1}^{L} \frac{\partial \log \mathscr{P}_j}{\partial \alpha_\ell} \alpha'_\ell(t) \tag{55}$$

is Pareto-improving.

To make everyone better off without deteriorating public finances, it is enough to combine a production efficient policy with the tax reforms outlined in Proposition 10. A production efficient policy reform triggers adjustments in prices, which in turn, modify taxpayers' factor supplies and utilities. However, Proposition 2 states that these effects on taxpayers' factor supplies and utilities can be nullified by using the adequate combination of GE-replicating tax reforms with directions $\mathcal{R}^j$, for $j = 1, ..., n$. The details of this adequate combination are provided in Proposition 10 and Equation (55). Consequently, combining the production efficient policy reform $\alpha_1(\cdot), ...\alpha_L(\cdot)$ with the tax reform $\sum_{j=1}^{n} \gamma_j \; \mathcal{R}^j$ results in no impact on taxpayers' factor supplies and utility. These combined production policy and tax reforms affect each price by $\sum_{\ell=1}^{L} (\partial \mathscr{P}_j / \partial \alpha_\ell) \; \alpha'_\ell(t)$. As a result, tax revenues are modified by $\sum_{j=1}^{n} \sum_{\ell=1}^{L} \mathcal{X}_j (\partial \mathscr{P}_j / \partial \alpha_\ell) \alpha'_\ell(t)$, which is equal to $\sum_{\ell=1}^{L} \mathcal{F}_{\alpha_\ell} \alpha'_\ell(t)$, based on (53). Given the efficiency of the production policy reform, its combination with the aforementioned tax reform yields a Pareto improvement.

According to Proposition 10, the choice of production policies should be solely driven by efficiency considerations if all GE-replicating directions $\mathcal{R}^j(\cdot)$ of tax reforms are feasible. This is what we call the "production efficiency principle". Another way of putting it is that we do not need pre-distribution through production policy when the tax system is exhaustive enough to incorporates all its GE-replicating reforms with directions $\mathcal{R}^j(\cdot)$. In this case, redistribution should only take place thanks to taxation and only efficiency concerns should guide the design of production policies. The production efficiency principle can be regarded as some form of "Tinbergen principle": production policies should not be concerned with redistribution, as that role falls within tax policy. The potentially detrimental effects of production efficient policy reforms on certain categories of taxpayers can be offset by tax reforms in the GE-replicating directions when they are admissible. Proposition 10 thus generalizes, to a context with nonlinear taxes, the production efficiency theorem of Diamond and Mirrlees (1971).

However, as previously discussed, there are various reasons why reforms in the GE-replicating directions $\mathcal{R}^j(\cdot)$ might not belong to set of available tax instruments, generically leading to $\partial \mathscr{L}^{\mathcal{R}^j} / \partial t \neq 0$ for some $j \in \{1, ..., n\}$. The tax schedule may not be exhaustive (see Question 1 in Figure 2) or the tax system may be too restricted (see Question 1 in Figure 2). Note that Proposition 10 does not require the tax system to be optimal along all GE-replicating directions, i.e. $\partial \mathscr{L}^{\mathcal{R}^j} / \partial t =$ for $j = 1, ..., n$, (see Question 3 in Figure 2). It requires that the government is able to reform the tax system in all GE-replicating directions. When the tax system prevents production efficiency, Proposition 9 describes how to optimally deviate from the production efficiency principle. In addition to their efficiency effects $\mathcal{F}_{\alpha_\ell}(\cdot)$, production policies, by generating price changes, induce effects on tax revenue and taxpayer welfare – i.e. pre-distributive effects – that replicate the effects of tax reforms in the GE-replicating directions, as described in Equation (54).

Our analysis extends to the welfare compensation problem, aimed at designing tax reforms that offset the welfare losses, induced by economic disruptions, by redistributing the winners' gains. To address this, reinterpret changes in $\boldsymbol{\alpha}$ as exogenous technological shocks instead of production policy reforms.

According to Proposition 10, if the tax system is exhaustive and flexible enough to enable reforms of the tax system along all GE-replicating directions, the welfare effects of a technological shocks can be nullified using Equation (55) without implying any changes in factor supplies. In such a case the Kaldor (1939), Hicks (1939, 1940) logic applies. If otherwise the tax system can not be reformed in some of GE replicating directions, the compensation problem is more complex and implies distortions in factor supplies, as in Schultz et al. (2023). Solving the compensation problem under such constraints falls beyond the scope of the present paper.

IV.2 Examples of production policies

We derive Proposition 10 by describing the production sector only in terms of inverse demand functions $\mathscr{P}_i(\cdot)$. Relying on the simplicity of these reduced-forms allows us to demonstrate Proposition 10 without any precise specification of the production sector. However, this simplicity hides the large set of problems that can be described by these reduced-forms. To address this concern, we now describe how our reduced-form description of the production sector thoroughly incorporates examples with explicit micro-foundations. In all our examples, we rely on fewer structural assumptions compared to those commonly found in the existing literature that studies similar scenarios.

To do this, we focus on economies with many intermediate goods and sectors and adopt the following notations. There are one final good and S intermediate goods therefore, $S+1$ sectors, indexed by $s = 0, ..., S$. Within each sector s, there exist N_s firms. In sector $s > 0$, firm $\varphi = 1, ..., N_s$ produces the s^{th} intermediate good, employing factors $\boldsymbol{\mathcal{X}}^{\varphi,s} \stackrel{\text{def}}{\equiv} (\mathcal{X}_1^{\varphi,s}, ..., \mathcal{X}_n^{\varphi,s})$ and goods $\mathbf{z}^{\varphi,s} \stackrel{\text{def}}{\equiv} (z_0^{\varphi,s}, ..., z_{s-1}^{\varphi,s}, z_{s+1}^{\varphi,s}, ..., z_S^{\varphi,s})$ with the production function $\mathcal{F}^{\varphi,s}(\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s})$. Firm $\varphi = 1, ..., N_0$ in sector $s = 0$ produces the final good using factors $\boldsymbol{\mathcal{X}}^{\varphi,0} \stackrel{\text{def}}{\equiv} \left(\mathcal{X}_1^{\varphi,0}, ..., \mathcal{X}_n^{\varphi,0}\right)$ and goods $\mathbf{z}^{\varphi,0} \stackrel{\text{def}}{\equiv} (z_1^{\varphi,0}, ..., z_S^{\varphi,0})$ with the production function $\mathcal{F}^{\varphi,0}\left(\boldsymbol{\mathcal{X}}^{\varphi,0}, \mathbf{z}^{\varphi,0}\right)$. The production functions are differentiable with non-negative partial derivatives and well-behaved. The market clearing condition in the intermediate goods sector s can be written as:

$$\forall s \in \{1, ..., S\}: \qquad \sum_{\varphi=1}^{N_s} \mathcal{F}^{\varphi,s}(\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s}) = \sum_{\substack{s'=0 \\ s'\neq s}}^{S} \sum_{\varphi=1}^{N_{s'}} z_s^{\varphi,s'} \tag{56}$$

i.e., the total production of firms in sector s on the left-hand side is equal to the sum of the demands for good s by firms in all sectors s' other than s on the RHS. The market clearing condition for the final goods is:

$$\sum_{\varphi=1}^{N_0} \mathcal{F}^{\varphi,0}\left(\boldsymbol{\mathcal{X}}^{\varphi,0}, \mathbf{z}^{\varphi,0}\right) = \sum_{s=1}^{S} \sum_{\varphi=1}^{N_s} z_0^{\varphi,s} + \int_{\mathcal{W}} C(\mathbf{w}) \, \mathrm{d}F(\mathbf{w}) + E. \tag{57}$$

It equalizes the total production of firms in sector $s = 0$ in the left-hand side to the demands for the final good $s = 0$ by intermediate goods producers, taxpayers and the government, in the right-hand side.

Finally, the market clearing condition for factor $i = 1, ..., n$ can be expressed as:

$$\forall i \in \{1, ..., n\}: \qquad \mathcal{X}_i = \sum_{s=0}^{S} \sum_{\varphi=1}^{N_s} \mathcal{X}_i^{\varphi,s} \tag{58}$$

i.e., the total supply of the i^{th} factor by taxpayers in the left-hand side is equal to the sum of demands of factors by all firms in all sectors in the right-hand side.

IV.2.a Taxation of intermediate goods and taxing robots and AI

Production policies can be taxes on intermediate goods. Consider that all firms operate under constant or decreasing returns to scale, and intermediate goods are subject to the sector-specific ad-valorem tax rates α_s, for $s = 1, ..., S$, with the normalization $\alpha_0 = 0$ for the final good. Let q_s denote the purchasing price of good s, with the normalization $q_0 = 1$ for the final good. In this scenario, firm $\varphi = 1, ..., N_s$ in sector $s = 0, ..., S$ solves:

$$\pi^{\varphi,s} \stackrel{\text{def}}{\equiv} \max_{\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s}} \quad q_s(1-\alpha_s)\, \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s}\right) - \sum_{i=1}^{n} p_i\, \mathcal{X}_i^{\varphi,s} - \sum_{\substack{s'=0 \\ s' \neq s}}^{S} q_{s'}\, z_{s'}^{\varphi,s}, \tag{59}$$

where $\pi^{\varphi,s}$ denotes the profit of firm φ in sector s. Since firms operate under perfect competition, profit $\pi^{\varphi,s}$ is positive if the production function of the firm φ in sector s has decreasing returns to scale. Let $X_{n+1}(\mathbf{w})$ denote the exogenous share of firms' profits earned by $\mathbf{w}$-taxpayers and $p_{n+1}X_{n+1}(\mathbf{w})$ the profits earned by $\mathbf{w}$-taxpayers. Program (59) leads to the following conditions:

$$\forall i \in \{1, ..., n\}: \quad p_i = q_s(1-\alpha_s)\mathcal{F}_{\mathcal{X}_i}^{\varphi,s} \qquad \text{and} \qquad \forall s' \neq s: \quad q_{s'} = q_s(1-\alpha_s)\mathcal{F}_{z_{s'}}^{\varphi,s}. \tag{60}$$

The competitive allocation of the production resources is a vector $(\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s})$ for all firms $\varphi = 1, ..., N_s$ in sector $s = 0, ..., S$, a vector of intermediate goods' prices $(q_1, ..., q_S)$ (with normalization $q_0 = 1$) and a vector of factor prices $(p_1, ..., p_n)$. These vectors must verify the market clearing conditions (56) and (58) as well as the optimality conditions (60), for all firms, in all sectors.

We show, in Appendix C.3, that the competitive allocation of production resources coincides with the choice of an hypothetical "production coordinator". This reformulation will prove useful to easily retrieve the reduced-forms $\mathcal{F}(\cdot)$ and the inverse demand equations $\mathscr{P}_i(\cdot)$. The production coordinator's objective is the total production of the final good net of the final good demands from the firms producing intermediate goods. According to (57), this coincides with the total consumption of final good by taxpayers and the government. The production coordinator's program has to verify resource constraints on production factors (58) and on intermediate goods (56). Crucially, instead of Equation (56), the production coordinator considers a reformulation of the resource constraints on intermediate goods. This reformulation acknowledges that the government collects a fraction α_s of the production of the s^{th} intermediate good. The program for the production coordinator is:

$$\max_{\{\boldsymbol{\mathcal{X}}^{\varphi,s},\mathbf{z}^{\varphi,s}\}_{s=0,\dots,S}^{\varphi=1,..,N_s}} \qquad \sum_{\varphi=1}^{N_0} \mathcal{F}^{\varphi,0}\left(\boldsymbol{\mathcal{X}}^{\varphi,0},\mathbf{z}^{\varphi,0}\right) - \sum_{s=1}^{S}\sum_{\varphi=1}^{N_s} z_0^{\varphi,s} \tag{61a}$$

$$\forall i \in \{1,...,n\} \quad : \quad \mathcal{X}_i = \sum_{s=0}^{S}\sum_{\varphi=1}^{N_s} \mathcal{X}_i^{\varphi,s} \tag{61b}$$

$$\forall s \in \{1,...,S\} \quad : \quad \alpha_s\,\overline{Z}_s + (1-\alpha_s)\sum_{\varphi=1}^{N_s} \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s},\mathbf{z}^{\varphi,s}\right) = \sum_{\substack{s'=0\\ s'\neq s}}^{S}\sum_{\varphi=1}^{N_{s'}} z_s^{\varphi,s'} \tag{61c}$$

where:

$$\forall s \in \{1,...,S\}: \quad \overline{Z}_s \stackrel{\text{def}}{\equiv} \sum_{\varphi=1}^{N_s} \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s},\mathbf{z}^{\varphi,s}\right) \tag{61d}$$

is taken as given by the production coordinator. Importantly, the combination of (61c) and (61d) recovers the resource constraint (56) on the intermediate goods $s = 1,...,S$. With this reformulation, the production coordinator is induced to mimic the firms' behavior in the competitive allocation when there exists a wedge α_s between the purchasing and the selling price of the s^{th} intermediate good. Let p_i denote the Lagrange multiplier associated with (61b) and let q_s represent the Lagrange multiplier associated with (61c) in the coordinator's program. The first-order conditions with respect to $\mathcal{X}_i^{\varphi,s}$ and $z_{s'}^{\varphi,s}$ associated to (61a)-(61c) then coincide with the first-order conditions (60) in the competitive scenario. Since the allocation of production resources chosen by the production coordinator verifies the market clearing conditions (56) and (58), the production coordinator's chosen allocation aligns with the competitive allocation of production resources. By reformulating the competitive allocation of production resources through the program of this hypothetical production coordinator, we can directly retrieve the inverse demand functions $\mathscr{P}_i(\cdot)$ (in (6)) and the production function $\mathcal{F}(\cdot)$. For each value of $(\mathcal{X}_1,...,\mathcal{X}_n)$ of factor supplies, of intermediate good tax rates $(\alpha_1,...,\alpha_S)$ and intermediate good tax revenues $(\overline{Z}_1,...,\overline{Z}_S)$, the production function $\mathcal{F}(\cdot)$ is defined as the value of the program (61a)-(61c), and the inverse demands $\mathscr{P}_i(\cdot)$ are defined as the Lagrange multipliers associated to the factor constraints (61b) in the Program (61a)-(61c).[30]

The most efficient allocation of production resources is obtained when the total production of the final good net of its consumption by intermediate producers is maximal, given the resources constraints

[30] In other words, the allocation of production resources solves a fixed point problem, since Program (61) takes $\overline{Z}_s$ for $s = 1,...,S$ as given and the solution to this program in turn determines the $\overline{Z}_s$ for $s = 1,...,S$.

In appendix C.3, we show that the sum of income factors $\sum_{i=1}^{n} p_i\ \mathcal{X}_i$, profits $\sum_{s=0}^{S}\sum_{\varphi=1}^{N_s}\pi^{\varphi,s}$ and revenue from intermediate good taxation $\sum_{s=1}^{S}\sum_{\varphi=1}^{N_{s'}}\alpha_s q_s \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s},\mathbf{z}^{\varphi,s}\right)$ is equal to the total production of the final good $\sum_{\varphi=1}^{N_0}\mathcal{F}^{\varphi,0}\left(\boldsymbol{\mathcal{X}}^{\varphi,0},\mathbf{z}^{\varphi,0}\right)$ net of the consumption of the final good by intermediate good producers, i.e. $\sum_{s=1}^{S}\sum_{\varphi=1}^{N_s} z_0^{\varphi,s}$, as expected from the Walras Law.

To retrieve the national accounting equation (7), we normalize $\mathcal{X}_{n+1} = \int_{\mathcal{W}} X_{n+1}(\mathbf{w})\mathrm{d}F(\mathbf{w}) = 1$, so that the aggregate profits earned by all taxpayers are $\mathcal{Y}_{n+1} = p_{n+1}\mathcal{X}_{n+1} = p_{n+1}$. Symmetrically, we denote $X_{n+2}(\mathbf{w}) = 1$ the allocation of tax revenue from intermediate good taxation and p_{n+2}, the Lagrange multiplier associated with (61b), the total tax revenue from the taxation of intermediate goods. So doing, the production function coincides with the value function of the production coordinator's Program (61a)-(61d) and verifies the accounting equation (7), provided that the sum on the right-hand side of (7) runs for i from 1 to $n+2$ instead of n.

on intermediate goods (56) and on the production factors (58). The program defining the most efficient allocation of production resources therefore coincides with (61) whenever $\alpha_1 = ... = \alpha_S = 0$. Thus, production efficiency is achieved if intermediate goods are not taxed, which is an important implication of Diamond and Mirrlees (1971)'s production efficiency theorem.

The model of Diamond and Mirrlees (1971) is a sub-case of the present example where all production functions have constant returns to scale and where taxation is assumed to be exhaustive, schedular and linear. In this case, the optimal tax system is also optimal along all GE-replicating directions (which are here given by (39d)). Consequently, our Proposition 10 implies that not taxing intermediate goods is Pareto-optimal. Extending this framework to exhaustive and nonlinear taxation, our Proposition 10 states that the desirability of not taxing intermediate goods does not require a fully optimal tax system, but rather one that is optimal only along the n GE-replicating directions.

Another sub-case of our example occurs where some production functions exhibit decreasing returns to scale and taxation is exhaustive, schedular and linear, as in Dasgupta and Stiglitz (1971, 1972). In such a case, for the tax system to be optimal along all GE-replicating directions, the government has to be free to tax profits. This requirement, on top of the exhaustivity and unrestrictiveness of the tax system, ensures the tax system to be optimal along the $n + 1$ GE-replicating directions (including the one corresponding to the $n + 1^{\text{th}}$ entrepreneurial factor), thereby ensuring that Proposition 10 applies.

This formulation of our framework also allows us to address the question of taxing robots and AI, as Koizumi (2020), Guerreiro et al. (2021), Costinot and Werning (2022) and Thuemmel (2023) do, by simply considering them as particular intermediate goods. If the tax function is exhaustive, encompassing GE-replicating directions, and taxation is optimized along these directions, it is optimal not to tax robots. However, the literature on robot taxation assumes a tax authority unable to distinguish between various (imperfectly substitutable) types of labor, such as routine and non-routine tasks. The assumption of the non-exhaustiveness of the tax function implies that the tax system is not optimized along its GE-replicating directions. In such an environment, it becomes optimal to tax robots.[31]

IV.2.b Public production

Consider the government owns the public firm $\varphi^\star$ in sector $s^\star$. Within this framework, the production policies are the public firm's demand of factors and the demand of goods, i.e. $\boldsymbol{\alpha} \stackrel{\text{def}}{\equiv} (\boldsymbol{\mathcal{X}}^{\varphi^\star, s^\star}, \mathbf{z}^{\varphi^\star, s^\star})$, instead of tax rates on intermediate goods, as in the previous example. The private firms solve (59) and their behaviors are described by (60). Therefore, the competitive allocation of production resources

[31] Through GE price adjustments, the taxation of robots, assumed to substitute routine labor, indirectly reduces the wage gap between routine and non-routine tasks. This leads to an increase in the wage rate for routine labor, exclusively earned by routine workers. The optimal robot tax balances the equity gains from wage compression with the efficiency losses stemming from distorted production decisions.

coincides with the solution of our production coordinator's program, which is now:

$$\max_{\{\boldsymbol{\mathcal{X}}^{\varphi,s},\mathbf{z}^{\varphi,s}\}_{s=0,\dots,S}^{\varphi=1,\dots,N_s,(\varphi,s)\neq(\varphi^\star,s^\star)}} \qquad \sum_{\varphi=1}^{N_0} \mathcal{F}^{\varphi,0}\left(\boldsymbol{\mathcal{X}}^{\varphi,0},\mathbf{z}^{\varphi,0}\right) - \sum_{s=1}^{S}\sum_{\varphi=1}^{N_s} z_0^{\varphi,s} \tag{62a}$$

$$\forall i \in \{1,\dots,n\} \quad : \qquad \mathcal{X}_i = \sum_{s=0}^{S}\sum_{\varphi=1}^{N_s} \mathcal{X}_i^{\varphi,s} \tag{62b}$$

$$\forall s \in \{1,\dots,S\} \quad : \qquad \sum_{\varphi=1}^{N_s} \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s},\mathbf{z}^{\varphi,s}\right) = \sum_{\substack{s'=0\\ s'\neq s}}^{S}\sum_{\varphi=1}^{N_{s'}} z_s^{\varphi,s'}. \tag{62c}$$

instead of (61a)-(61c). Again, for each vector of factor supply $(\mathcal{X}_1,\dots,\mathcal{X}_n)$ and each vector of production policy, $(\boldsymbol{\mathcal{X}}^{\varphi^\star,s^\star},\mathbf{z}^{\varphi^\star,s^\star})$, the inverse demands $\mathscr{P}_i(\cdot)$ are defined as the Lagrange multipliers associated to constraints (62b) and the production function $\mathcal{F}(\cdot)$ is the value function associated to program (62a)-(62c).

According to Proposition 10, if the tax system can be reformed along all the GE-replicating directions $\mathcal{R}_j(\cdot)$ (for $j = 1,\dots,n+1$), the government sets the production plan $(\boldsymbol{\mathcal{X}}^{\varphi^\star,s^\star},\mathbf{z}^{\varphi^\star,s^\star})$ of the public firm $\varphi^\star$ in sector $s^\star$ to maximize the total production of the final good (minus its consumption by producers of intermediate goods), as detailed in (62a). This amounts to solving program (62a)-(62c) with respect to the production plan of private firms (as in (62a)-(62c)) and of the public firm $\varphi^\star$ in sector $s^\star$. In such a case, private and public firms face the same first-order conditions:

$$\forall i \in \{1,\dots,n\}: \quad p_i = q_s\, \mathcal{F}^{\varphi,s}_{\mathcal{X}_i} \qquad \text{and} \qquad \forall s' \neq s: \quad q_{s'} = q_s\, \mathcal{F}^{\varphi,s}_{z_{s'}}.$$

Put differently, whenever the tax system can be modified along all GE replicating directions, Proposition 10 implies that the public firms should face the same factor prices as the private firms. We retrieve the result of Diamond and Mirrlees (1971). This has the implication, exploited by Little and Mirrlees (1974), that in evaluating public projects prices used to value factors purchased (or sold) in the market by the public sector should be producer prices. Again, we do not need to assume optimality of the tax schedule, i.e. optimality with respect to all directions $R(\cdot)$. We only need that the tax system can be reformed along the n GE-replicating directions $\mathcal{R}_j(\cdot)$ for $j = 1,\dots,n$ if there is no profit and and along $\mathcal{R}_j(\cdot)$ for $j = 1,\dots,n+1$ in the case of profits. However, as soon as the tax system can not be reformed along the GE-replicating directions $\mathcal{R}_j$, it is desirable to use a different price system for public firms as emphasized in Naito (1999).

IV.2.c Commodity taxation

The framework employed to analyze the taxation of intermediate goods, in Section IV.2.a, can be applied to consider whether the taxation of final goods should be uniform when the utility is weakly separable in leisure and consumption, as examined by Atkinson and Stiglitz (1976). Their theorem considers that each taxpayer has preference over factor $\mathbf{x}$ and commodities $\mathbf{z} = (z_1, ..., z_S)$, according to a weakly separable utility function of the form $\mathcal{U}(\mathscr{V}(z_1, ..., z_S), \mathbf{x}; \mathbf{w})$. We can align our model with theirs by interpreting our intermediate goods within our framework of Section IV.2.a as their commodities $(z_1, ..., z_s)$. Additionally, assume that all taxpayers in Section IV.2.a produce and consume one final good z_0 using the same production function $z_0 = \mathscr{V}(z_1, ..., z_n)$ so that this corresponds to the sub-utility obtained from commodities in Atkinson and Stiglitz (1976). We assume constant returns to scale in the production functions of the intermediate good sectors $s \in 1, ..., S$ and that final goods are not employed as production factors (thus, $z_0^{\varphi,s} = 0$ for all firms $\varphi \in 1, ..., N_s$ in sectors $s = 1, ..., S$). Upon this reinterpretation, our taxation of intermediate goods in Section IV.2.a is taxation of commodities in Atkinson and Stiglitz (1976). Therefore, the no-tax result on intermediate goods discussed in Section IV.2.a translates to a no-tax result on commodities, or equivalently, uniform commodity tax rates, in Atkinson and Stiglitz (1976).

Thanks to this reinterpretation of the model, our Proposition 10 implies that the no-commodity taxation result of Atkinson and Stiglitz (1976) remains robust to endogenous producer prices, whenever the tax system can be reformed along the GE-replicating directions. This holds e.g. in the long-run model of Saez (2004) where taxation is occupation-specific hence exhaustive. Conversely, in e.g. Naito (1999), the short-run model of Saez (2004) or in Jacobs (2015), the income tax system does not discriminate between the different types of labor, i.e. is not exhaustive. The same level of income drives the same tax rate, even when earned by different labor types. In this type of framework, the tax systems can therefore not be reformed along GE-replicating directions specific to each type of labor. Commodity taxation should then not be uniform and may have a pre-distributive role, which is described in Equation (54) in Proposition 9. It is worth mentioning that our reinterpretation of Atkinson and Stiglitz (1976)'s theorem leading to no-tax on intermediate goods does not hold when taxpayers have different preferences $\mathcal{V}(\cdot)$ over commodities, as in e.g. Saez (2002b) and Ferey et al. (2024).

IV.2.d Trade policy

We now adapt our multi-sector framework to discuss the desirability of trade liberalization policies. For this purpose, we assume that, in each sector $s \in \{0, ..., n\}$, certain firms operate abroad. Regardless of whether firms are domestic or foreign, the arguments of the production function refer only to goods or production factors from the home country. Foreign firms $\varphi \in \{1, ..., N_s\}$ in sector $s \in \{0, ..., S\}$ do not use domestic factors of production, so $\mathcal{X}_i^{\varphi,s} = 0$ for all $i \in \{1, ..., n\}$, but these foreign firms import intermediate goods $z_{s'}^{\varphi,s}$ from sector $s' \neq s$. Their imports of goods s are given by $\mathcal{F}^{\varphi,s}(\boldsymbol{z}^{\varphi,s}; \alpha)$, where α captures the impact of trade frictions. In particular, α_s captures various costs associated with the imports or exports of foreign producers in sector s, costs that trade policies can diminish, so that $\mathcal{F}_\alpha^{\varphi,s} < 0$ for foreign firms. Conversely, trade policies do not impact the production possibilities of domestic firms, hence, $\mathcal{F}_\alpha^{\varphi,s} = 0$ for domestic firms. Assuming perfect competition, the competitive allocation of resources within the production sector coincides with the solution of the following production coordinator's program:

$$\max_{\{\boldsymbol{\mathcal{X}}^{\varphi,s},\mathbf{z}^{\varphi,s}\}_{s=0,\dots,S}^{\varphi=1,\dots,N_s}} \quad \sum_{\varphi=1}^{N_0} \mathcal{F}^{\varphi,0}\left(\boldsymbol{\mathcal{X}}^{\varphi,0},\mathbf{z}^{\varphi,0};\alpha\right) - \sum_{s=1}^{S}\sum_{\varphi=1}^{N_s} z_0^{\varphi,s} \tag{63a}$$

$$\forall i \in \{1,...,n\} \quad : \quad \mathcal{X}_i = \sum_{s=0}^{S}\sum_{\varphi=1}^{N_s} \mathcal{X}_i^{\varphi,s} \tag{63b}$$

$$\forall s \in \{1,...,S\} \quad : \quad \sum_{\varphi=1}^{N_s} \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s},\mathbf{z}^{\varphi,s};\alpha_s\right) = \sum_{\substack{s'=0\\ s'\neq s}}^{S}\sum_{\varphi=1}^{N_{s'}} z_s^{\varphi,s'}. \tag{63c}$$

As in Subsection IV.2.a, for each vector of factor supply $(\mathcal{X}_1,...,\mathcal{X}_n)$ and each vector $(\alpha_0,...,\alpha_S)$ of sector-specific trade costs, the inverse demands $\mathscr{P}_i(\cdot)$ are defined as the Lagrange multipliers associated to (63b) and the production function $\mathcal{F}(\cdot)$ is the value function associated to Program (63a)-(63c). A policy that reduces trade costs is therefore unambiguously production efficient. The desirability of trade liberalization policies thus depends solely on whether or not the income tax system can be reformed along the n GE-replicating directions of the tax reforms. This is the case in Diamond and Mirrlees (1971), Dixit and Norman (1980, 1986) where the tax system include sector-specific and linear taxes on labor. Production efficiency and free trade then follow. The multi-country Ricardian model of trade proposed by Hosseini and Shourideh (2018) also aligns with production efficiency. Conversely, in Costinot and Werning (2022), the different types of labor are imperfect substitutes but generate incomes that the tax administration cannot distinguish and therefore must tax comprehensively. Due to this lack of exhaustiveness, the tax system cannot be reformed along all the GE-replicating directions of the tax reforms. In this context, barriers to trade can have a beneficial pre-distibutive role.

IV.2.e Competition policy

The production efficiency principle applies to any policy that affects the economy only by shifting factor demands. It then also applies to corporate law, regulation, competition policy, etc. We illustrate this with an example where a pro-competitive policy reduces oligopolistic rents.

Consider that all firms, within each sector, have the same production function with constant returns to scale. There is perfect competition in the final goods sector $s = 0$ and Cournot competition in the intermediate goods sectors $s \in \{1,...,S\}$. Intermediate goods are produced using factors of production. Conversely, the final good is produced using both intermediate goods and factors of production according to the following Cobb-Douglas production function:

$$\mathcal{F}_0^{\varphi,0}\left(\boldsymbol{\mathcal{X}}^{\varphi,0},\mathbf{z}^{\varphi,0}\right) = \prod_{s=1}^{S}\left(z_s^{\varphi,0}\right)^{\beta_s} \prod_{i=1}^{n}\left(\mathcal{X}_i^{\varphi,0}\right)^{\gamma_i}$$

where $\beta_s \geq 0$, $\gamma_i \geq 0$ and $\sum_{s=1}^{S} \beta_s + \sum_{i=1}^{n} \gamma_i = 1$. In all sectors $s \in \{0, ..., S\}$, let $z_s^{\varphi,s}$ denote the output of firm $\varphi \in \{1, ...N_s\}$, let $z_s \stackrel{\text{def}}{\equiv} \sum_{\varphi=1}^{N_s} z_s^{\varphi,s}$ denote the total output of the intermediate good s and let $z_s^{\varphi,-s} \stackrel{\text{def}}{\equiv} z_s - z_s^{\varphi,s}$ denote the amount of intermediate good s produced by the competitors of firm φ. The program of the final good producer $\varphi \in \{1, ..., N_0\}$ is:

$$\max_{\boldsymbol{\mathcal{X}}^{\varphi,0},\mathbf{z}^{\varphi,0}} \qquad \prod_{s=1}^{S} \left(z_s^{\varphi,0}\right)^{\beta_s} \prod_{i=1}^{n} \left(\mathcal{X}_i^{\varphi,0}\right)^{\gamma_i} - \sum_{s=1}^{S} q_s \, z_s^{\varphi,0} - \sum_{i=1}^{n} p_i \, \mathcal{X}_i^{\varphi,0}.$$

Since, in the final goods sector, all production functions admit constant returns to scale and are identical, the first-order condition of this program leads to the following inverse demand for the s^{th} intermediate good:

$$q_s = \beta_s \, \left(z_s^0\right)^{\beta_s - 1} \prod_{s'=1,s'\neq s}^{S} \left(z_{s'}^0\right)^{\beta_{s'}} \prod_{i=1}^{n} \left(\mathcal{X}_i^0\right)^{\gamma_i}$$

where $\mathcal{X}_i^0 \stackrel{\text{def}}{\equiv} \sum_{\varphi=1}^{N_0} \mathcal{X}_i^{\varphi,s}$ denotes the sum of the i^{th} production factor used in the final good sector. Because there are many firms in the final good sector and because there are many intermediate goods sectors, intermediate goods producers take prices $p_1, ..., p_n$ of production factors as given. They moreover take the output of the other intermediate goods producers as given. Since only final goods producers purchase intermediate goods, one has $z_s^0 = z_s = z_s^{\varphi,s} + z_s^{\varphi,-s}$. The intermediate goods producer φ in sector s thus solves:

$$\begin{aligned} \max_{\boldsymbol{\mathcal{X}}^{\varphi,s},\mathbf{z}^{\varphi,s},q_s} \qquad & q_s \, \mathcal{F}^s \left(\mathcal{X}^{\varphi,s}, \mathbf{z}^{\varphi,s}\right) - \sum_{i=1}^{n} p_i \, \mathcal{X}_i^{\varphi,s} \qquad (64) \\ s.t: \qquad & q_s = \beta_s \, \left(\mathcal{F}^s \left(\mathcal{X}^{\varphi,s}, \mathbf{z}^{\varphi,s}\right) + z_s^{\varphi,-s}\right)^{\beta_s - 1} \prod_{s'=1,s'\neq s}^{S} \left(z_{s'}^0\right)^{\beta_{s'}} \prod_{i=1}^{n} \left(\mathcal{X}_i^0\right)^{\gamma_i}. \end{aligned}$$

At the symmetric Cournot-Nash equilibrium, within each sector, all producers make the same choices, so $z_s = N_s \, z_s^{\varphi,s}$ and $z_s^{\varphi,-s} = (N_s - 1) z_s^{\varphi,s}$. The first-order conditions associated with (64) thus imply:

$$\forall i \in \{1, ..., n\}: \quad p_i = q_s(1 - \alpha_s)\mathcal{F}_{\mathcal{X}_i}^{\varphi,s}$$

where, $\alpha_s \stackrel{\text{def}}{\equiv} (1 - \beta_s)/N_s$ indicates how much output price q_s is overpriced due to imperfect competition. Since the production functions have constant returns to scale, α_s also denotes the profit share in sector s. Under Cournot competition, this profit share is a decreasing function of the number N_s of firms and an increasing function of the elasticity $1 - \beta_s$ of the inverse demand for the s^{th} intermediate good in absolute value. Production policies directly set these sector-specific markups α_s. Taking into account profits, the allocation of resources within the production sector under Cournot competition coincides with the solution of the following production coordinator's program:

$$\max_{\{\boldsymbol{\mathcal{X}}^{\varphi,s},\mathbf{z}^{\varphi,s}\}_{s=0,...,S}^{\varphi=1,..,N_s}} \quad \sum_{\varphi=1}^{N_0} \mathcal{F}^{\varphi,0}\left(\boldsymbol{\mathcal{X}}^{\varphi,0},\mathbf{z}^{\varphi,0}\right) \tag{65a}$$

$$\forall i \in \{1,...,n\} \quad : \quad \mathcal{X}_i = \sum_{s=0}^{S}\sum_{\varphi=1}^{N_s} \mathcal{X}_i^{\varphi,s} \tag{65b}$$

$$\forall s \in \{1,...,S\} \quad : \quad \alpha_s\,\overline{Z}_s + (1-\alpha_s)\sum_{\varphi=1}^{N_s}\mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s},\mathbf{z}^{\varphi,s}\right) = \sum_{\varphi=1}^{N_0} z_s^{\varphi,0} \tag{65c}$$

where:

$$\forall s \in \{1,...,S\}: \quad \overline{Z}_s \overset{\text{def}}{\equiv} \sum_{\varphi=1}^{N_s}\mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s},\mathbf{z}^{\varphi,s}\right) \tag{65d}$$

is taken as given by the production coordinator. As in Subsection IV.2.a, for each vector of factor supply $(\mathcal{X}_1,...,\mathcal{X}_n)$, vectors of sector-specific mark-ups $(\alpha_1,...,\alpha_S)$ and of sector-specific profits $(\overline{Z}_1,...,\overline{Z}_s)$, the inverse demands $\mathscr{P}_i(\cdot)$ are defined as the Lagrange multipliers associated to (65b). The production function $\mathcal{F}(\cdot)$ is the value function associated to Program (65a)-(65c).[32]

Production enhancing competition policies consists in reduction in markups α_s. Therefore, according to Proposition 10, whenever the tax system can be improved along all the GE replicating directions, the production efficiency principle applies, and reducing markups is always desirable. We think this result can be generalized beyond the specific case of Cournot competition. Any competition policy that reduces markups α_s is desirable provided that the tax system aligns with respect to tax reforms in the GE-replicating directions. We posit that this extends to policies like merger regulations in the case of horizontal or vertical integration, as well as to corporate law.

IV.2.f The effects of business-focused environmental regulations

Consider now the scenario where the production sector is polluting, e.g. with carbon emissions and firms have the option to mitigate emissions by adopting cleaner technologies. Production policy consists in taxing carbon emissions. Here, intermediate good producers not only produce intermediate goods according to the production function $\mathcal{F}^{\varphi,s}(\mathcal{X}_1^{\varphi,s},...,\mathcal{X}_n^{\varphi,s};\beta^{\varphi,s})$ but also emit carbon according to $\mathcal{E}^{\varphi,s}(\mathcal{X}_1^{\varphi,s},...,\mathcal{X}_n^{\varphi,s};\beta^{\varphi,s})$ where $\beta^{\varphi,s}$ is the degree of cleanliness in the technology adopted by firm φ in sector s. Employing more production factor increases both production and pollution, thus $\mathcal{F}^{\varphi,s}_{\mathcal{X}_i} > 0$ and $\mathcal{E}^{\varphi,s}_{\mathcal{X}_i} > 0$. Production is concave in $\beta^{\varphi,s}$ with a maximum at a level normalized to zero. Hence $\mathcal{F}^{\varphi,s}_{\beta} < 0$ if $\beta^{\varphi,s} > 0$ and $\mathcal{F}^{\varphi,s}_{\beta} > 0$ if $\beta^{\varphi,s} < 0$. Conversely, carbon emissions decrease when firms adopt greener technology, thus $\mathcal{E}^{\varphi,s}_{\beta} < 0$. We assume that the government can observe each firm's carbon emissions and tax them at a rate denoted by α. Assuming perfect competition and a constant returns to

[32]The allocation of resources actually solves a fixed-point problem since the production coordinator's program solves (65a)-(65c) taking $(\overline{Z}_1,...,\overline{Z}_s)$ as given, while $(\overline{Z}_1,...,\overline{Z}_s)$ is determined by the solution of the production coordinator's program through (65d).

scale production functions, firm $\varphi \in \{1, ..., N_s\}$ in sector s solves:

$$\max_{\mathcal{X}_1^{\varphi,s},...,\mathcal{X}_n^{\varphi,s},\beta^{\varphi,s}} \qquad q_s\, \mathcal{F}^{\varphi,s}\left(\mathcal{X}_1^{\varphi,s}, ..., \mathcal{X}_n^{\varphi,s}; \beta^{\varphi,s}\right) - \sum_{i=1}^{n} p_i\, \mathcal{X}_i^{\varphi,s} - \alpha\, \mathcal{E}^{\varphi,s}\left(\mathcal{X}_1^{\varphi,s}, ..., \mathcal{X}_n^{\varphi,s}; \beta^{\varphi,s}\right).$$

This leads to the following first-order conditions:

$$\forall i \in \{1, ..., n\}: \qquad q_s\, \mathcal{F}_{\mathcal{X}_i}^{\varphi,s} = p_i + \alpha\, \mathcal{E}_{\mathcal{X}_i}^{\varphi,s} \qquad \text{and}: \qquad q_s\, \mathcal{F}_{\beta}^{\varphi,s} = \alpha\, \mathcal{E}_{\beta}^{\varphi,s} \tag{66}$$

As in IV.2.c, each taxpayer produces a final good through the same production function, which is denoted $\mathcal{F}_0(\cdot)$.[33] Moreover, pollution exerts a negative externality. Hence $\mathcal{F}_0$ is decreasing in aggregate emissions $\mathcal{E} \overset{\text{def}}{\equiv} \sum_{s=1}^{S} \sum_{\varphi=1}^{N_s} \mathcal{E}^{\varphi,s}\left(\mathcal{X}^{\varphi,s}; \beta^{\varphi,s}\right)$, so we have $\mathcal{F}^0\left(z_1, ..., z_S, \mathcal{E}\right)$, with $\mathcal{F}_{z_i}^0 > 0 > \mathcal{F}_{\mathcal{E}}^0$. For tractability, we assume that the final good production function exhibits constant returns to scale with respect to intermediate goods consumption $(z_1, ..., z_S)$. This leads to the intermediate goods demand conditions:

$$\forall s \in \{1, ..., S\}: \qquad q_s = \mathcal{F}_{z_s}^0\left(z_1^0, ..., z_S^0, \mathcal{E}\right), \tag{67}$$

where

$$z_s^0 \overset{\text{def}}{\equiv} \sum_{\varphi=1}^{N_s} \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s}, \beta^{\varphi,s}\right)$$

denotes the total production of the s^{th} intermediate good.

The competitive allocation of resources within the production sector is the same as the one chosen by an hypothetical production coordinator whose program consists in:

$$\max_{\{\mathcal{X}^{\varphi,s},\beta^{\varphi,s}\}_{s=1,...,S}^{\varphi=1,..,N_s}, z_1^0,...,z_S^0} \qquad \mathcal{F}^0\left(z_1^0, ..., z_S^0, \overline{\mathcal{E}}\right) - \alpha \sum_{s=1}^{S}\sum_{\varphi=1}^{N_s} \mathcal{E}^{\varphi,s}\left(\mathcal{X}^{\varphi,s}; \beta^{\varphi,s}\right) + \alpha\, \overline{\mathcal{E}} \tag{68a}$$

$$\forall i \in \{1, ..., n\} \quad : \qquad \mathcal{X}_i = \sum_{s=1}^{S}\sum_{\varphi=1}^{N_s} \mathcal{X}_i^{\varphi,s} \tag{68b}$$

$$\forall s \in \{1, ..., S\} \quad : \qquad \sum_{\varphi=1}^{N_s} \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s}, \beta^{\varphi,s}\right) = z_s^0. \tag{68c}$$

where the production coordinator takes aggregate emissions

$$\overline{\mathcal{E}} = \sum_{s=1}^{S}\sum_{\varphi=1}^{N_s} \mathcal{E}^{\varphi,s}\left(\mathcal{X}^{\varphi,s}; \beta^{\varphi,s}\right) \tag{68d}$$

and carbon tax revenue $\alpha\, \overline{\mathcal{E}}$ as given.[34]

[33] Again this production function for final good coincide with a reinterpretation of the subutility function $\mathcal{V}(\cdot)$ à la Atkinson and Stiglitz (1976).

[34] Denoting p_i the Lagrange multiplier associated to the i^{th} equation (68b) and q_s the Lagrange multiplier associated to s^{th} equation (68c), the first-order conditions of (68) with respect to $\mathcal{X}^{\varphi,s}$, $\beta^{\varphi,s}$ and z_s^0 leads to (66) and (67). Since the production coordinator is constrained by the same resource constraints (68c) as the competitive economy, the production allocation chosen by the production coordinator coincides with that of the competitive economy. Finally, since revenue from carbon tax $\alpha\, \overline{\mathcal{E}}$ shows up in the production coordinator's objective (68a), the Walras Law ensures that the value function associated to the production coordinator's program (68) verifies the accounting equation (7).

For each vector of factor supply $(\mathcal{X}_1, ..., \mathcal{X}_n)$, each carbon tax rate α and each carbon tax revenue $\alpha\ \mathcal{E}$, the inverse demands $\mathscr{P}_i(\mathcal{X}_1, ..., \mathcal{X}_n; \alpha, \overline{\mathcal{E}})$ are defined as the Lagrange multipliers associated to constraints (68b) and the production function $\mathcal{F}(\mathcal{X}_1, ..., \mathcal{X}_n; \alpha, \overline{\mathcal{E}})$ is the value function associated to program (68a)-(68c). The carbon tax maximizing this aggregate production verifies[35] the Pigouvian rule $\mathcal{F}^0_{\overline{\mathcal{E}}} = -\alpha$ that correct for the externality.

According to Proposition 10, whenever the government can reform the tax system in all the GE-replicating directions $\mathcal{R}^j$ for $j = 1, ..., n$, the optimal carbon tax verifies $\mathcal{F}^0_{\overline{\mathcal{E}}} = -\alpha$. The redistributive consequences of this carbon tax can then be offset by tax reforms in the GE-replicating directions.

V Conclusion

In a framework with multiple income sources, taxpayers who differ along many unobserved dimensions and potential imperfect competition in the production sector, we have addressed two fundamental questions on appropriate income tax and production policies – such as taxation of intermediate goods, provision of public goods by the government, taxing robots, competition and trade policies, financial markets regulation, business laws, intellectual property protection, immigration policies and technological changes–. First, we have studied how the endogeneity of prices, stemming from factors' supply, influences the optimal tax formulas and the incidence tax formulas. In this General Equilibrium (GE) environment, we have derived an empirical test for Pareto-improving tax reforms. Second, we have analyzed whether policy reforms affecting the production sector (i.e. any change in guidelines, strategies, or government interventions that directly impact production with only indirect consequences for consumers) should exclusively aim at enhancing production efficiency or whether their design should also incorporate redistributive concerns. We have shown that when the tax system allows for directions that can be modified to counteract the impact of prices (so-called GE-replicating directions of tax reforms), production policy reforms should aim only to improve production efficiency. In this case, all taxation formulas diverge from those obtained in Partial Equilibrium solely by a a term that corrects for potential discrepancies between private and social returns of factors. However, when the tax system does not allow for the correction of production inefficiencies, new empirically meaningful statistics, called GE multipliers, have to be considered in the formulas for optimal taxation, tax incidence, Pareto improvements and optimal production policies. We explain when and how the tax system can correct for the inefficiency created by production policy reforms and how the calculation of GE multipliers depends on the type of tax system.

There are several avenues for further research. For instance, we could examine tax incidence and the optimal combination of labor and housing taxation, as well as the optimal taxation of various financial

[35] Applying the envelope theorem to Program (68) with respect to α and taking (68d) into account leads formally to $\mathcal{F}_\alpha = 0$. Applying the envelope theorem with respect to $\overline{\mathcal{E}}$ leads to $\mathcal{F}_{\overline{\mathcal{E}}} = \mathcal{F}^0_{\overline{\mathcal{E}}} + \alpha$.

income sources and capital gains. Another area of potential exploration includes determining the optimal taxation of capital income and wealth. It would also be worthwhile to expand the analysis of the political feasibility of tax reforms carried out by Bierbrauer et al. (2021) to our GE environment, which takes into account GE effects, multidimensional heterogeneity and cross-base responses.

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Online Appendix

A Appendix related to Section II

A.1 Convexity of the Indifference Set

Let $\mathscr{C}(\cdot, \mathbf{x}; \mathbf{w})$ denote the reciprocal of $\mathscr{U}(\cdot, \mathbf{x}; \mathbf{w})$. Taxpayers of type $\mathbf{w}$ who supply factors $\mathbf{x}$ obtain consumption $c = \mathscr{C}(u, \mathbf{x}; \mathbf{w})$ to enjoy utility $u = \mathscr{U}(c, \mathbf{x}; \mathbf{w})$. Using (2), we obtain:

$$\mathscr{C}_u(u, \mathbf{x}; \mathbf{w}) = \frac{1}{\mathscr{U}_c(\mathscr{C}(u, \mathbf{x}; \mathbf{w}), \mathbf{x}; \mathbf{w})} \qquad \mathscr{C}_{x_i}(u, \mathbf{x}; \mathbf{w}) = \mathcal{S}^i(\mathscr{C}(u, \mathbf{x}; \mathbf{w}), \mathbf{x}; \mathbf{w}) \tag{A.1}$$

For each type $\mathbf{w} \in W$ and each utility level u, we assume that the indifference set $\mathbf{y} \mapsto \mathscr{C}(u, y_1/p_1, ..., y_n/p_n; \mathbf{w})$ is strictly convex. The i^{th} partial derivative of $\mathbf{y} \mapsto \mathscr{C}(u, y_1/p_1, ..., y_n/p_n; \mathbf{w})$ being $(1/p_i)\,\mathcal{S}^i(\mathscr{C}(u, y_1/p_1, ..., y_n/p_n; \mathbf{w}), y_1/p_1, ..., y_n/p_n; \mathbf{w})$, the Hessian is matrix:

$$\left[\frac{\mathcal{S}^i_{x_j} + \mathcal{S}^i_c \mathcal{S}^j}{p_i\, p_j}\right]_{i,j} = \left[-\frac{\mathscr{U}_{x_i x_j} + \mathcal{S}^j \mathscr{U}_{c x_i} + \mathcal{S}^i \mathscr{U}_{c x_j} + \mathcal{S}^i \mathcal{S}^j \mathscr{U}_{cc}}{p_i\, p_j\, \mathscr{U}_c}\right]_{i,j}$$

which is symmetric. Finally, the latter matrix is obviously positive definite if and only if matrix $\left[\mathcal{S}^i_{x_j} + \mathcal{S}^i_c \mathcal{S}^j\right]_{i,j}$ is positive definite as well. The first-order condition of (4) is given by:

$$0 = (1 - \mathcal{T}_{y_i}(\mathbf{y}))\, \mathscr{U}_c\left(\sum_{k=1}^n y_k - \mathcal{T}(\mathbf{y}), \frac{y_1}{p_1}, ..., \frac{y_n}{p_n}; \mathbf{w}\right) + \frac{1}{p_i} \mathscr{U}_{x_i}\left(\sum_{k=1}^n y_k - \mathcal{T}(\mathbf{y}), \frac{y_1}{p_1}, ..., \frac{y_n}{p_n}; \mathbf{w}\right).$$

Therefore, using (5), the matrix of the second-order condition is:

$$\left[\frac{\mathscr{U}_{x_i x_j} + \mathcal{S}^j \mathscr{U}_{c x_i} + \mathcal{S}^i \mathscr{U}_{c x_j} + \mathcal{S}^i \mathcal{S}^j \mathscr{U}_{cc}}{p_i\, p_j} - \mathscr{U}_c \mathcal{T}_{y_i y_j}\right]_{i,j} = -\mathscr{U}_c \left[\frac{\mathcal{S}^i_{x_j} + \mathcal{S}^i_c \mathcal{S}^j}{p_i p_j} + \mathcal{T}_{y_i y_j}\right]_{i,j}$$

Hence, for taxpayers of type $\mathbf{w}$, the second-order condition holds strictly if and only if the matrix $\left[\frac{\mathcal{S}^i_{x_j} + \mathcal{S}^i_c \mathcal{S}^j}{p_i\, p_j} + \mathcal{T}_{y_i y_j}\right]_{i,j}$ is positive definite, i.e. if and only if the indifference set $\mathbf{y} \mapsto \mathscr{C}(U(\mathbf{w}), \frac{y_1}{p_1}, ..., \frac{y_n}{p_n}; \mathbf{w})$ is strictly more convex than the budget set $\mathbf{y} \mapsto \sum_{k=1}^n y_k - \mathcal{T}(\mathbf{y})$ at $\mathbf{y} = \mathbf{Y}(\mathbf{w})$.

A.2 Examples of applications

The two-period model with labor and savings

The two-period model has been widely used in the literature to study capital taxation since Atkinson and Stiglitz (1976). Taxpayers are characterized by $\mathbf{w} = (w_1, w_2)$ where w_1 is individual labor productivity and w_2 is initial wealth, which may come from previously saved labor income (Judd, 1985, Chamley, 1986) or inherited wealth.[36] In the first period, taxpayers save x_2 and consume $c^{per.1} = w_2 - x_2$. In the second period, they earn capital income $y_2 = p_2\, x_2$ where p_2 is the (endogenous) return on savings and labor income $y_1 = p_1\, x_1$ where x_1 is the efficient units of labor they supply. Their consumption in the second period is the sum of both their capital and labor incomes minus taxes $T(y_1, y_2)$, i.e. $c^{per.2} = y_1 + y_2 - T(y_1, y_2)$. This corresponds to our definition of after-tax income c in the general framework. We represent the preferences of taxpayers over first period consumption $c^{per.1}$, second period consumption $c^{per.2}$, and efficient units of labor x_1 as $(c^{per.1}, c^{per.2}, x_1) \mapsto \mathcal{U}(c^{per.1}, c^{per.2}, x_1; w_1)$,

[36] We here depart from the usual timing assumption where labor and capital incomes are taxed in distinct periods.

which we use to retrieve the utility function of the general framework through the following change of variables:

$$\mathscr{U}(c, x_1, x_2; \mathbf{w}) \stackrel{\text{def}}{\equiv} \mathcal{U}\left(\underbrace{w_2 - x_2}_{=c^{per.1}}, \underbrace{c}_{=c^{per.2}}, x_1; w_1\right). \tag{A.2}$$

The two-period model is able to capture the essential mechanisms by which taxation affects individual behaviors in most macroeconomic models. Individuals accumulate capital through savings, which requires forgoing consumption. These concepts are crucial for characterizing the steady state(s) in the neoclassical growth model (Ramsey, 1928) and in the overlapping generation model (Diamond, 1965). In addition, unlike models such Judd (1985) and Chamley (1986), which assume an infinite elasticity of supply of capital, our framework allows for an elasticity of supply of any factor with respect to its tax rate that can take any value, including an infinite elasticity. Lastly, our two-period framework can be expanded to accommodate various forms of capital income (such as dividends, interest, realized and unrealized capital gains).

Roy model

Our framework can also encompass economies with different sectors, occupations, or industries, as in Rothschild and Scheuer (2013, 2014, 2016), Scheuer (2014) and Gomes et al. (2017). For each sector $i = 1, ..., n$, taxpayers choose the amount x_i of labor supplied. The production of each sector is equal to $\mathcal{X}_i = \int_W X_i(\mathbf{w})\, \mathrm{d}F(\mathbf{w})$. The consumption good is produced by combining the production of all sectors according to the production function $\mathcal{F}(\mathcal{X}_1, ..., \mathcal{X}_n)$. In Rothschild and Scheuer (2013), Scheuer (2014) and Gomes et al. (2017), workers can supply labor only in one sector. In our model, this consists in assuming that $\mathscr{U}(c, \mathbf{x}; \mathbf{w}) = -\infty$ if more than one supply of factor is positive.

An income-shifting model

Our framework is consistent with income-shifting. Entrepreneurs may relabel some of their labor income as capital for tax avoidance purpose, a central tax policy issue according to Saez and Zucman (2019) after many others (Christiansen and Tuomala, 2008, Selin and Simula, 2020). To follow this literature, we consider income shifting behaviors assuming exogenous prices. This occurs for instance under perfect competition when the production function is given by: $\mathcal{F}(\mathcal{X}_1, ..., \mathcal{X}_n, \boldsymbol{\alpha}) = \sum_{i=1}^n \mathcal{X}_i$,which implies $p_1 = ... = p_n = 1$. To ease the presentation, we focus our attention here on two sources of income ($i = 1, 2$). Let z_1 and z_2 represent the true first and second sources of income for taxpayers, which are unobserved by the government. Let the preferences of a $\mathbf{w}$-taxpayer be described by the utility function $(c, z_1, z_2) \mapsto \mathcal{U}(c, z_1, z_2)$ with $\mathcal{U}_{z_1}, \mathcal{U}_{z_2} < 0 < \mathcal{U}_c$.[37] Let $s \gtrless 0$ depict the amount of the first source of income taxpayers shift to be realized as the other source of income. Income shifting involves a monetary costs $S(s; \mathbf{w})$ with S being convex in s for all $\mathbf{w}$-taxpayers. The reported income sources are then $y_1 = x_1 = z_1 - s$ and $y_2 = x_2 = z_2 + s$. Consumption, that we denote d, is $d = y_1 + y_2 - S(s; \mathbf{w}) - \mathcal{T}(y_1, y_2)$ with $\mathcal{U}_d > 0$. Expressing this equation in terms of the general framework's after-tax income $c = y_1 + y_2 - \mathcal{T}(y_1, y_2)$, it can be rewritten as $d = c - S(s; \mathbf{w})$.

The determination of the amount of shifted income s is a subprogram for which the value function enables us to retrieve the utility function of the general framework as follows:

$$\mathscr{U}(c, x_1, x_2; \mathbf{w}) \stackrel{\text{def}}{\equiv} \max_s \quad \mathcal{U}\left(\underbrace{c - S(s; \mathbf{w})}_{=d}, \underbrace{x_1 + s}_{=z_1}, \underbrace{x_2 - s}_{=z_2}; \mathbf{w}\right) \tag{A.3}$$

The government budget constraint is unaffected since taxes are based on observed income y_1 and y_2.

[37] For instance, taxpayers can be self-employed and business-owners with effective labor income z_1 and income from their business z_2. In this example, w_1 is labor ability and w_2 pertains to the ability to generate return on business.

Naturally, shifted income s could also be investment in tax heavens. In this case, $y_2 = x_2 = z_2 + s$ is reinterpreted as income invested in tax heavens which is unobserved by the government. This income is not taxed under $\mathcal{T}(\cdot)$ yet it does not alter the optimal tax formula for each taxable income source.

In this income-shifting version of our model, we assume exogenous prices. Naturally, one could extend this example to encompass both income-shifting behaviors and endogenous pricing.

B Appendix related to Section III

B.1 Proof of Lemma 1

To be able to apply the implicit function theorem to the first-order condition associated to the individual maximization program, we presume Assumption 1 is verified.

B.1.a Assumption 1

Part (*i*) of Assumption 1 ensures that first-order conditions (5) are continuously differentiable in incomes $\mathbf{y}$. It rules out kinks in the tax function, thereby bunching.[38] Parts (*i*) and (*ii*) of Assumption 1 together enable one to apply the implicit function theorem to first-order conditions (5) to ensure that each local maximum of

$$\mathbf{y} \mapsto \mathscr{U}\left(\sum_{k=1}^{n} y_k - \mathcal{T}(\mathbf{y}) + t\, R(\mathbf{y}), \frac{y_1}{p_1}, ..., \frac{y_n}{p_n}; \mathbf{w}\right)$$

is differentiable in type $\mathbf{w}$, in price $\mathbf{p}$ and in the tax perturbation's magnitude t. Part (*iii*) of Assumption 1 rules out the existence of multiple global maxima. This prevents the incremental tax reform from causing a jump in the taxpayer's choice from one maximum to another. Part (*iii*) also ensures the allocation changes in a differentiable way with the magnitude of the tax reform and with types.

Because the indifference set is convex (see Appendix A.1), Assumption 1 is automatically satisfied when the tax schedule is linear, or when the tax schedule is weakly convex. It is also satisfied when the tax schedule is not "too" concave, so that function $\mathbf{y} \mapsto \sum_{k=1}^{n} y_k - \mathcal{T}(\mathbf{y})$ is either concave, linear or less convex than the indifference set with which it has a tangency point in the $(\mathbf{y}, c)$-space (so that Part *ii)* of Assumption 1 is satisfied). Geometrically, it implies that, for each type $\mathbf{w}$, the indifference set defined by $\mathbf{y} \mapsto \mathscr{C}\left(U(\mathbf{w}), \frac{y_1}{p_1}, ..., \frac{y_n}{p_n}; \mathbf{w}\right)$ admits a single tangency point with the budget set defined by $\mathbf{y} \mapsto \sum_{k=1}^{n} y_k - \mathcal{T}(\mathbf{y})$ and lies strictly above the budget set for all other $\mathbf{y}$.

B.1.b Taxpayers' program

Since w-taxpayers take the prices $\mathbf{p} = (p_1, ..., p_n)$ after any tax reform as given, they solve, under the tax schedule $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y}) - t\, R(\mathbf{y})$, the following program that depends on the direction $R(\cdot)$ and magnitude t of the tax reform and on the price vector $\mathbf{p}$:

$$U^{R,PE}(\mathbf{w}; t, \mathbf{p}) \stackrel{\text{def}}{\equiv} \max_{x_1,...,x_n} \mathscr{U}\left(\sum_{i=1}^{n} p_i x_i - \mathcal{T}(p_1 x_1, ..., p_n x_n) + t\, R(p_1 x_1, ..., p_n x_n), x_1, ..., x_n; \mathbf{w}\right). \tag{A.4}$$

[38] In reality, most of real world tax schedules are piecewise linear. From theory, one should observe bunching at convex kinks and gaps at concave kinks. Empirically, most convex kinks do not cause significant bunching, with the exception of the self-employed in the United States at the first kink point of the EITC Saez (2010). Moreover, no gap is observed at concave kinks. These discrepancies between the theoretical predictions and empirical evidence can be reconciled by assuming that taxpayers do not optimize with respect to the exact tax schedule but with respect to some smooth approximation of it, e.g. $\mathbf{y} \mapsto \int \mathcal{T}(\mathbf{y} + \mathbf{u})\, \mathrm{d}\Psi(\mathbf{u})$ where $\mathbf{u}$ is an n-dimensional random shock on incomes with joint CDF Ψ, which does verify part *i)* of Assumption 1.

Note that (A.6) also holds out of the GE where $\mathbf{p} = (p_1^R(t), ..., p_n^R(t))$. The first-order conditions are:

$$\mathcal{S}^i \left(\sum_{i=1}^n p_i\, x_i - \mathcal{T}(p_1\, x_1, ..., p_n\, x_n) + t\, R(p_1\, x_1, ..., p_n\, x_n), x_1, ..., x_n; \mathbf{w} \right) \tag{A.5}$$
$$= \quad p_i \left[1 - \mathcal{T}_{y_i}(p_1\, x_1, ..., p_n\, x_n) + t\, R_{y_i}(p_1\, x_1, ..., p_n\, x_n) \right], \qquad \forall i \in \{1, ..., n\}.$$

Let $\mathbf{X}^{R,PE}(\mathbf{w}, t, \mathbf{p}) = (X_1^{R,PE}(\mathbf{w}, t, \mathbf{p}), ..., X_n^{R,PE}(\mathbf{w}, t, \mathbf{p}))$ denote the solution of this program and let $Y_i^{R,PE}(\mathbf{w}, t, \mathbf{p}) \stackrel{\text{def}}{\equiv} p_i\, X_i^{R,PE}(\mathbf{w}, t, \mathbf{p})$. At the GE where $p_j = p_j^R(t)$, one obviously has $Y_i^R(\mathbf{w}, t) \equiv Y_i^{R,PE}(\mathbf{w}, \mathbf{p}^R(t))$ for all $i \in \{1, ..., n\}$ and $U^R(\mathbf{w}, t) \equiv U^{R,PE}(\mathbf{w}, \mathbf{p}^R(t))$.

Under Assumption 1, the implicit function theorem ensures that the solution $X^{R,PE}(\mathbf{w}, t, \mathbf{p})$ to program (A.4) is differentiable with respect to t, $\mathbf{p}$ and $\mathbf{w}$. Moreover, its partial derivatives, at $\mathbf{p} = (p_1^R(0), ..., p_n^R(0))$ and $t = 0$, can be obtained by differentiating Equations (A.5) at $x = \mathbf{X}(\mathbf{w})$ and $t = 0$, which leads to:

$$\forall i \in \{1, ..., n\} \qquad : \qquad \sum_{j=1}^n \left[\mathcal{S}^i_{x_j} + \mathcal{S}^i_c \mathcal{S}^j + p_i\, p_j\, \mathcal{T}_{y_i y_j} \right] \mathrm{d}x_j =$$
$$\left[p_i\, R_{y_i}(\mathbf{Y}(\mathbf{w})) - \mathcal{S}^i_c\, R(\mathbf{Y}(\mathbf{w})) \right] \mathrm{d}t + \sum_{j=1}^n \left((1 - \mathcal{T}_{y_j}) \mathbb{1}_{i=j} - (1 - \mathcal{T}_{y_j})\, x_j\, \mathcal{S}^i_c - p_i\, x_j\, \mathcal{T}_{y_i y_j} \right) \mathrm{d}p_j.$$

This differentiation can be rewritten in matrix form as:

$$\begin{aligned} \left[\mathcal{S}^i_{x_j} + \mathcal{S}^i_c \mathcal{S}^j + p_i p_j \mathcal{T}_{y_i y_j} \right]_{i,j} \cdot \mathrm{d}\mathbf{x}^T &= \left[p_i\, R_{y_i}(\mathbf{Y}(\mathbf{w})) - \mathcal{S}^i_c\, R(\mathbf{Y}(\mathbf{w})) \right]_i^T \mathrm{d}t \\ &+ \left[(1 - \mathcal{T}_{y_j}) \left(\mathbb{1}_{i=j} - x_j\, \mathcal{S}^i_c \right) - p_i\, x_j\, \mathcal{T}_{y_i y_j} \right]_{i,j} \cdot \mathrm{d}\mathbf{p}^T. \end{aligned}$$

where superscript T denotes the transpose operator$[A_{i,j}]_{i,j}^T = [A_{j,i}]_{i,j}$ and "$\cdot$" denotes the matrix product. Matrix $\left[\mathcal{S}^i_{x_j} + \mathcal{S}^i_c \mathcal{S}^j + p_i p_j \mathcal{T}_{y_i y_j} \right]_{i,j}$ is the Hessian matrix associated to the maximization program (A.4). It is therefore symmetric and semi-positive definite. From point (*ii*) of Assumption 1, this matrix is positive definite, thereby invertible. Let $H_{i,j}$ denote the term in the i^{th} row and j^{th} column of the inverse of the Hessian matrix. We obtain:

$$\begin{aligned} \mathrm{d}x_i &= \sum_{k=1}^n H_{i,k} \left[p_k\, R_{y_k}(\mathbf{Y}(\mathbf{w})) - \mathcal{S}^k_c\, R(\mathbf{Y}(\mathbf{w})) \right] \mathrm{d}t \\ &+ \sum_{j=1}^n \left\{ \sum_{k=1}^n H_{i,k} \left[(1 - \mathcal{T}_{y_j}) \left(\mathbb{1}_{k=j} - x_j\, \mathcal{S}^k_c \right) - p_k\, x_j\, \mathcal{T}_{y_k y_j} \right] \right\} \mathrm{d}p_j \end{aligned} \tag{A.6}$$

Under a compensated tax reform of the j^{th} marginal tax rate at income $\mathbf{y} = \mathbf{Y}(\mathbf{w})$ where $R(\mathbf{y}) = y_j - Y_j(\mathbf{w})$, one has $R(\mathbf{Y}(\mathbf{w})) = 0$ and $R_{y_k}(\mathbf{Y}(\mathbf{w})) = \mathbb{1}_{k=j}$. Hence, according to (A.6) compensated responses are given by:

$$\frac{\partial X_i(\mathbf{w})}{\partial \tau_j} = p_j\, H_{i,j}, \qquad \frac{\partial Y_i(\mathbf{w})}{\partial \tau_j} = p_i\, p_j\, H_{i,j} \tag{A.7a}$$

with $\frac{\partial Y_i(\mathbf{w})}{\partial \tau_j} = \frac{\partial Y_j(\mathbf{w})}{\partial \tau_i}$ since the Hessian matrix is symmetric.

Under a lump-sum tax reform where $R(\mathbf{y}) = 1$, one has $R(\mathbf{Y}(\mathbf{w})) = 1$ and $R_{y_k}(\mathbf{Y}(\mathbf{w})) = 0$. Hence, according to (A.6), income effects are given by:

$$\frac{\partial X_i(\mathbf{w})}{\partial \rho} = -\sum_{k=1}^n H_{i,k}\, \mathcal{S}^k_c, \qquad \frac{\partial Y_i(\mathbf{w})}{\partial \rho} = -p_i \sum_{k=1}^n H_{i,k}\, \mathcal{S}^k_c. \tag{A.7b}$$

Under an uncompensated tax reform of the j^{th} marginal tax rate where $R(\mathbf{y}) = y_j$, one gets $R(\mathbf{Y}(\mathbf{w})) = Y_j(\mathbf{w})$ and $R_{y_k}(\mathbf{Y}(\mathbf{w})) = \mathbb{1}_{j=k}$. Hence, according to (A.6), uncompensated responses are given by:

$$\frac{\partial X_i^u(\mathbf{w})}{\partial \tau_j} = p_j \left(H_{i,j} - X_j(\mathbf{w}) \sum_{k=1}^{n} H_{i,k}\, \mathcal{S}_c^k \right) \quad \frac{\partial Y_i^u(\mathbf{w})}{\partial \tau_j} = p_i\, p_j\, H_{i,j} - p_i\, Y_j(\mathbf{w}) \sum_{k=1}^{n} H_{i,k}\, \mathcal{S}_c^k. \tag{A.7c}$$

Simplifying Equation (A.7c), with (A.7a) and (A.7b), yields the Slutsky Equations (20) and

$$\frac{\partial X_i^u(\mathbf{w})}{\partial \tau_j} = \frac{\partial X_i(\mathbf{w})}{\partial \tau_j} + Y_j(\mathbf{w}) \frac{\partial X_i(\mathbf{w})}{\partial \rho}. \tag{A.7d}$$

According to (A.6), the responses to changes in prices are given by:

$$\frac{\partial X_i(\mathbf{w})}{\partial p_j} = (1 - \mathcal{T}_{y_j}) \left(H_{i,j} - X_j(\mathbf{w}) \sum_{k=1}^{n} H_{i,k}\, \mathcal{S}_c^k \right) - X_j(\mathbf{w}) \sum_{k=1}^{n} p_k\, H_{i,k}\, \mathcal{T}_{y_k y_j}. \tag{A.7e}$$

Using (A.7a) and (A.7c), the latter equation can be rewritten as:

$$\frac{\partial X_i(\mathbf{w})}{\partial p_j} = \frac{1 - \mathcal{T}_{y_j}}{p_j} \frac{\partial X_i^u(\mathbf{w})}{\partial \tau_j} - X_j(\mathbf{w}) \sum_{k=1}^{n} \frac{\partial X_i(\mathbf{w})}{\partial \tau_k}\, \mathcal{T}_{y_k y_j}. \tag{A.7f}$$

This yields:

$$\frac{\partial Y_i(\mathbf{w})}{\partial p_j} = \mathbb{1}_{i=j}\, X_i(\mathbf{w}) + \frac{1 - \mathcal{T}_{y_j}}{p_j} \frac{\partial Y_i^u(\mathbf{w})}{\partial \tau_j} - X_j(\mathbf{w}) \sum_{k=1}^{n} \frac{\partial Y_i(\mathbf{w})}{\partial \tau_k}\, \mathcal{T}_{y_k y_j}.$$

Multiplying both sides of the previous equation by p_j leads to (19).

B.1.c Taxpayers' responses

Consider a tax perturbation, as detailed in Definition 3, which implies that prices are determined by $t \mapsto (p_1^R(t), ..., p_n^R(t))$. Plugging $\mathrm{d}p_j = \frac{\partial p_j^R(t)}{\partial t}\mathrm{d}t$ into (A.6) leads to:

$$\begin{aligned}\frac{\partial X_i^R(\mathbf{w}, t)}{\partial t} &= \sum_{k=1}^{n} H_{i,k} \left[p_k\, R_{y_k}(\mathbf{Y}(\mathbf{w}), 0) - \mathcal{S}_c^k\, R(\mathbf{Y}(\mathbf{w}), 0) \right] \\ &+ \sum_{j=1}^{n} \left\{ \sum_{k=1}^{n} H_{i,k} \left[(1 - \mathcal{T}_{y_j}) \left(\mathbb{1}_{k=j} - x_j\, \mathcal{S}_c^k \right) - p_k\, x_j\, \mathcal{T}_{y_k y_j} \right] \right\} \frac{\partial p_j^R(t)}{\partial t}\end{aligned}$$

Using (A.7a), (A.7b) and (A.7e), we obtain:

$$\frac{\partial X_i^R(\mathbf{w}, t)}{\partial t} = \sum_{k=1}^{n} \frac{\partial X_i(\mathbf{w})}{\partial \tau_k}\, R_{y_k}(\mathbf{Y}(\mathbf{w})) + \frac{\partial X_i(\mathbf{w})}{\partial \rho}\, R(\mathbf{Y}(\mathbf{w})) + \sum_{j=1}^{n} \frac{\partial X_i(\mathbf{w})}{\partial p_j} \frac{\partial p_j^R(t)}{\partial t} \tag{A.8}$$

which, eventually, leads to (14). Applying the envelope theorem to (A.4), we obtain:

$$\frac{\partial U^{R,PE}(\mathbf{w}, t, \mathbf{p})}{\partial t} = \mathscr{U}_c\left(C(\mathbf{w}), \mathbf{X}(\mathbf{w}); \mathrm{w}\right)\; R\left(\mathbf{Y}(\mathbf{w})\right) \tag{A.9a}$$

$$\frac{\partial U^{R,PE}(\mathbf{w}, t, \mathbf{p})}{\partial p_j} = \mathscr{U}_c\left(C(\mathbf{w}), \mathbf{X}(\mathbf{w}); \mathrm{w}\right) \left(1 - T_{y_j}(\mathbf{Y}(\mathbf{w}))\right)\; X_j(\mathbf{w})$$

$$\frac{\partial U^{R,PE}(\mathbf{w}, t, \mathbf{p})}{\partial \log p_j} = \mathscr{U}_c\left(C(\mathbf{w}), \mathbf{X}(\mathbf{w}); \mathrm{w}\right) \left(1 - T_{y_j}(\mathbf{Y}(\mathbf{w}))\right)\; Y_j(\mathbf{w}) \tag{A.9b}$$

where the last equality follows from (2) and (5). Applying chain rule and using (A.9a) and (A.9b) leads to (15).

Note that for each type $\mathbf{w}$ and price $\mathbf{p}$, the mappings $R \mapsto \partial Y_i^R(\mathbf{w},t)/\partial t$ and $R \mapsto \partial U^R(\mathbf{w},t)/\partial t$ are continuous and linear, i.e., for all functions R_1, R_2 and scalars ζ_1, ζ_2, one has:

$$\frac{\partial Y_i^{\zeta_1 R_1+\zeta_2 R_2}(\mathbf{w},t)}{\partial t} = \zeta_1 \frac{\partial Y_i^{R_1}(\mathbf{w},t)}{\partial t} + \zeta_2 \frac{\partial Y_i^{R_2}(\mathbf{w},t)}{\partial t}$$

and

$$\frac{\partial U^{\zeta_1 R_1+\zeta_2 R_2}(\mathbf{w},t)}{\partial t} = \zeta_1 \frac{\partial U^{R_1}(\mathbf{w},t)}{\partial t} + \zeta_2 \frac{\partial U^{R_2}(\mathbf{w},t)}{\partial t}.$$

In other words, not only $Y_i^R(\mathbf{w},t)$ and $U^R(\mathbf{w},t)$ admit *partial* derivatives with respect to t at $t=0$ for any direction R, but these functions are *Gateaux* differentiable with respect to the tax schedule, since the partial derivatives are linear and continuous functions of the direction $R(\cdot)$.

B.1.d Proof of Lemma 1

A tax reform impacts the tax liability of $\mathbf{w}$-taxpayers $\mathcal{T}\left(\mathbf{Y}^R(\mathbf{w},t)\right) - t\ R\left(\mathbf{Y}^R(\mathbf{w},t)\right)$ through mechanical and behavioral effects as follows:

$$\frac{\partial\left[\mathcal{T}\left(\mathbf{Y}^R(\mathbf{w},t)\right) - t\ R\left(\widetilde{\mathbf{Y}}^R(\mathbf{w},t)\right)\right]}{\partial t} = \underbrace{-R(\mathbf{Y}(\mathbf{w}))}_{\text{Mechanical effects}} + \underbrace{\sum_{i=1}^{n} \mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w}))\frac{\partial \widetilde{Y}_i^R(\mathbf{w},t)}{\partial t}}_{\text{Behavioral effects}}. \tag{A.10}$$

Plugging (14) into (A.10) to decompose the response of the $\mathbf{w}$-taxpayers' i^{th} income, we obtain:

$$\frac{\partial\left[\mathcal{T}\left(\mathbf{Y}^R(\mathbf{w},t)\right) - t\ R\left(\mathbf{Y}^R(\mathbf{w},t)\right)\right]}{\partial t} = -\left[1-\sum_{i=1}^{n}\mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w}))\frac{\partial Y_i(\mathbf{w})}{\partial \rho}\right] R(\mathbf{Y}(\mathbf{w})) \tag{A.11}$$

$$+\sum_{1\le i,j\le n}\mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w}))\frac{\partial Y_i(\mathbf{w})}{\partial \tau_j} R_{y_j}(\mathbf{Y}(\mathbf{w})) + \sum_{1\le i,j\le n}\mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w}))\frac{\partial Y_i(\mathbf{w})}{\partial \log p_j}\frac{\partial \log p_j^R(t)}{\partial t}.$$

Combining (A.11) and (15), and aggregating for all types leads to (17) and (18). Note that $\mathscr{B}^R(t)$, $\mathscr{W}^R(t)$ and $\mathscr{L}^R(t)$ inherit the Gateaux differentiability with respect to the direction $R(\cdot)$ of the tax schedule from that of $Y_i^R(\mathbf{w},t)$ and of $U^R(\mathbf{w},t)$.

B.2 Proof of Proposition 1

B.2.a Proof of Lemma 2

Define the aggregate i^{th} income as function of the price $\mathbf{p}$ and of the magnitude t of the tax perturbation $\mathbf{y} \mapsto \widetilde{\mathcal{T}}(\mathbf{y},t)$ as follows:

$$\mathcal{Y}_i^{R,PE}(t,\mathbf{p}) \overset{\text{def}}{\equiv} \int_{\mathcal{W}} Y_i^{R,PE}(\mathbf{w},t,\mathbf{p})\,\mathrm{d}F(\mathbf{w})$$

From the inverse demand equations (6), and since an equilibrium is assumed to exist and to be unique, prices $\mathbf{p}^R(t) = (p_1^R(t),...,p_n^R(t))$ solve:

$$\forall t, \forall i\in\{1,...,n\} \qquad p_i^R(t) = \mathscr{P}_i\left(\frac{\mathcal{Y}_1^{R,PE}(t,\mathbf{p})}{p_1^R(t)},...,\frac{\mathcal{Y}_n^{R,PE}(t,\mathbf{p})}{p_n^R(t)},\boldsymbol{\alpha}\right). \tag{A.12}$$

Log-differentiating the latter equation and using (24a) leads to:

$$\left[\frac{\mathrm{d}p_i^R}{p_i}\right]_i = \Xi\cdot\left[\frac{\mathrm{d}\mathcal{X}_i^R}{\mathcal{X}_i}\right]_i + \sum_{\ell=1}^{L}\left[\frac{\partial \log \mathscr{P}_i}{\partial \alpha_\ell}\right]\mathrm{d}\alpha_\ell, \tag{A.13a}$$

where $\left[\mathrm{d}\mathcal{X}_i^R/\mathcal{X}\right]_i = \left[\mathrm{d}\mathcal{Y}_i^R/\mathcal{Y}\right]_i - \left[\mathrm{d}p_i^R/p_i\right]_i$ is the vector of log variations of aggregate factor. Aggregating (14) over all types and using (24b) and (26) yields:

$$\left[\frac{\mathrm{d}\mathcal{X}_i^R}{\mathcal{X}_i}\right]_i = \Gamma \cdot \left[\frac{\mathrm{d}p_i^R}{p_i}\right]_i + \left[\frac{\partial \log \mathcal{Y}_i^{R,PE}}{\partial t}\right]_i \mathrm{d}t. \tag{A.13b}$$

Combining the latter two equations leads to:

$$\left[\frac{\mathrm{d}p_i^R}{p_i}\right]_i = \Xi \cdot \Gamma \cdot \left[\frac{\mathrm{d}p_i^R}{p_i}\right]_i + \Xi \cdot \left[\frac{\partial \log \mathcal{Y}_i^{R,PE}}{\partial t}\right]_i \mathrm{d}t + \sum_{\ell=1}^{L} \left[\frac{\partial \log \mathscr{P}_i}{\partial \alpha_\ell}\right] \mathrm{d}\alpha_\ell$$

and finally:

$$(I_n - \Xi \cdot \Gamma) \cdot \left[\frac{\mathrm{d}p_i^R}{p_i}\right]_i = \Xi \cdot \left[\frac{\partial \log \mathcal{Y}_i^{R,PE}}{\partial t}\right]_i \mathrm{d}t + \sum_{\ell=1}^{L} \left[\frac{\partial \log \mathscr{P}_i}{\partial \alpha_\ell}\right] \mathrm{d}\alpha_\ell$$

Under Assumption 2, one can apply the implicit function theorem to ensure that the vector of prices is differentiable with respect to t and $\boldsymbol{\alpha}$, and we obtain Equations (25) and (52).

B.2.b Proof of Lemma 3 and of Part *i)* of Proposition 1

The tax reform impact on the Lagrangian induced by price changes is given by:

$$\begin{aligned}
\sum_{j=1}^{n} \frac{\partial \mathscr{L}}{\partial \log p_j} \frac{\partial \log p_j^R}{\partial t} &= \frac{\boldsymbol{\partial} \mathscr{L}}{\boldsymbol{\partial \log p}} \cdot \frac{\boldsymbol{\partial \log p}^R}{\partial t} \\
&= \frac{\boldsymbol{\partial} \mathscr{L}}{\boldsymbol{\partial \log p}} \cdot (I_n - \Xi \cdot \Gamma)^{-1} \cdot \Xi \cdot \frac{\boldsymbol{\partial \log Y}^{R,PE}}{\partial t} \\
&= \sum_{i=1}^{n} \eta_i \frac{1}{\mathcal{Y}_i} \frac{\partial \mathcal{Y}_i^{R,PE}(t)}{\partial t} \\
&= \sum_{i=1}^{n} \mu_i \frac{\partial \mathcal{Y}_i^{R,PE}(t)}{\partial t}
\end{aligned}$$

The first equality rewrites the LHS in matrix terms. The second equality is derived from (25). The third equality uses (29) and expands the obtained expression in scalar terms, and the last equality is a result of the definition $\mu_i \stackrel{\text{def}}{\equiv} \eta_i/\mathcal{Y}_i$. Substituting (27) into (17) yields Equation (30), which concludes the proof of Part *i)* of Proposition 1.

B.2.c Proof of Part *ii)* of Proposition 1

Consider a tax reform direction $R(\cdot)$. Let $r(t)$ be the lump-sum rebate required to ensure a balanced budget for the tax reform $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y}) - t\, R(\mathbf{y}) + r(t)$. For any variable A, we denote $\partial A^{R,BB}/\partial t$ as the partial derivative of A along the budget-balanced tax reform $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y}) - t\, R(\mathbf{y}) + r(t)$. Consequently, we find that $\partial \mathscr{B}^{R,BB}/\partial t = 0$, and thus:

$$\frac{1}{\lambda} \frac{\partial \mathscr{W}^{R,BB}}{\partial t} = \frac{\partial \mathscr{L}^{R,BB}}{\partial t}$$

Let $\partial A^{\rho}/\partial t$ be the partial derivative of A along the lump-sum perturbation $\mathcal{T}(\mathbf{y}) - t$. According to (30), by Gateaux differentiability of $\mathscr{L}$, we get:

$$\frac{\partial \mathscr{L}^{R,BB}}{\partial t} = \frac{\partial \mathscr{L}^{R}}{\partial t} + r'(0)\, \frac{\partial \mathscr{L}^{\rho}}{\partial t}.$$

Equation (31) implies:

$$\frac{\partial \mathscr{L}^{\rho}}{\partial t} = 0.$$

Combining these three equations leads to:

$$\frac{1}{\lambda}\frac{\partial \mathscr{W}^{R,BB}}{\partial t} = \frac{\partial \mathscr{L}^{R}}{\partial t}.$$

Since $\lambda > 0$, the budget-balanced reform improves welfare (i.e. $\partial \mathscr{W}^{R,BB}/\partial t > 0$) if and only if $\partial \mathscr{L}^{R}/\partial t$ is positive.

B.3 GE-replicating reforms and proof of Proposition 2

According to Equations (33) and (A.9a), the PE effect of the j^{th} GE-replicating reform on utility is given by:

$$\frac{\partial U^{\mathcal{R}^j,PE}(\mathbf{w},t)}{\partial t} = \mathscr{U}_c\left(C(\mathbf{w}),\mathbf{X}(\mathbf{w});\mathbf{w}\right)\ \left(1-\mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w}))\right) Y_j(\mathbf{w}) = \frac{\partial U(\mathbf{w})}{\partial \log p_j}$$

The second equality follows from (A.9b). According to (A.8), the PE effect of the j^{th} GE-replicating reform on the supply of the i^{th} factor is given by:

$$\begin{aligned}\frac{\partial X_i^{\mathcal{R}^j,PE}(\mathbf{w},t)}{\partial t} &= \sum_{k=1}^{n}\frac{\partial X_i(\mathbf{w})}{\partial \tau_k}\frac{\partial \mathcal{R}^j(\mathbf{Y}(\mathbf{w}))}{\partial y_k} + \frac{\partial X_i(\mathbf{w})}{\partial \rho}\mathcal{R}^j(\mathbf{Y}(\mathbf{w}))\\ &= \left(1-\mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w}))\right)\left[\frac{\partial X_i(\mathbf{w})}{\partial \tau_j} + Y_j(\mathbf{w})\frac{\partial X_i(\mathbf{w})}{\partial \rho}\right] - Y_j(\mathbf{w})\sum_{k=1}^{n}\mathcal{T}_{y_k y_j}(\mathbf{Y}(\mathbf{w}))\frac{\partial X_i(\mathbf{w})}{\partial \tau_k}\\ &= \left(1-\mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w}))\right)\frac{\partial X_i^u(\mathbf{w})}{\partial \tau_j} - Y_j(\mathbf{w})\sum_{k=1}^{n}\mathcal{T}_{y_k y_j}(\mathbf{Y}(\mathbf{w}))\frac{\partial X_i(\mathbf{w})}{\partial \tau_k}\end{aligned}$$

The second equality uses (33), and the third equality uses (A.7d). According to (A.7f), we therefore get:

$$\frac{\partial X_i^{\mathcal{R}^j,PE}(\mathbf{w},t)}{\partial t} = \frac{\partial X_i(\mathbf{w})}{\partial \log p_j} \quad \Rightarrow \quad \frac{\partial Y_i^{\mathcal{R}^j,PE}(\mathbf{w},t)}{\partial t} = \frac{\partial Y_i(\mathbf{w})}{\partial \log p_j} - \mathbb{1}_{i=j}\,Y_j(\mathbf{w}). \tag{A.14}$$

Finally, applying the chain rule to $C(\mathbf{w}) = \mathscr{C}\left(U(\mathbf{w}),\mathbf{X}(\mathbf{w});\mathbf{w}\right)$ yields:

$$\frac{\partial C^{\mathcal{R}^j,PE}(\mathbf{w},t)}{\partial t} = \frac{\partial C(\mathbf{w})}{\partial \log p_j}.$$

This ends the proof that the j^{th} GE-replicating reform at the PE replicates the effects of a log-change in the j-th price on taxpayers' supplies of production factors, consumption levels and utilities.

We now turn to Proposition 2. For any direction $R(\cdot)$, Let $R^N(\cdot)$ be defined as $R^N(\mathbf{y}) \overset{\text{def}}{\equiv} R(\mathbf{y}) - \sum_{j=1}^{n}\gamma_j^R\,\mathcal{R}^j(\mathbf{y})$, with $\mathcal{R}^j(\cdot)$ defined in (33). To ensure that reforms in the direction R^N has the same effect on factor supplies and utilities at the GE as do reforms in the direction R at the PE, the γ_j^R's must solve $\gamma_j^R = -\partial \log p_j^{R^N}/\partial t$ for $j = 1, ..., n$. This condition arises from the equivalence, for each factor j, between a reform in the j^{th} GE-replicating direction of magnitude dp_j/p_j and a log change in the j^{th} price. Since the matrix Γ of elasticities of factors supplies with respect to price is also the matrix of PE responses of GE-replicating reforms on aggregate income, we thus get:

$$\frac{\partial \log \boldsymbol{\mathcal{Y}}^{R^N,PE}}{\partial t} = \frac{\partial \log \boldsymbol{\mathcal{Y}}^{R,PE}}{\partial t} - \Gamma \cdot \frac{\partial \log \boldsymbol{p}^{R^N}}{\partial t}. \tag{A.15}$$

From (25), we thus get:

$$(I_n - \Xi \cdot \Gamma)\ \frac{\partial \log \boldsymbol{p}^{R^N}}{\partial t} = \Xi \cdot \frac{\partial \log \boldsymbol{\mathcal{Y}}^{R,PE}}{\partial t} - \Xi \cdot \Gamma \cdot \frac{\partial \log \boldsymbol{p}^{R^N}}{\partial t}.$$

We therefore get:

$$\frac{\partial \log \boldsymbol{p}^{R^N}}{\partial t} = \Xi \cdot \frac{\partial \log \boldsymbol{\mathcal{Y}}^{R,PE}}{\partial t}$$

which eventually leads to $\gamma_j^R = \sum_{i=1}^n \Xi_{j,i}\ (\partial \mathcal{Y}_i^{R,PE}/\partial t)/\mathcal{Y}_i$.

Finally, to confirm that $\mathrm{d}\mathcal{X}^{R^N}/\mathcal{X} = \partial \log \mathcal{X}^{R,PE}/\partial t$, we successively notice that:

$$\begin{aligned}
\frac{\partial \log \boldsymbol{\mathcal{X}}^{R^N}}{\partial t} &= \Gamma \cdot \frac{\partial \log \boldsymbol{p}^{R^N}}{\partial t} + \frac{\partial \log \boldsymbol{\mathcal{Y}}^{R^N,PE}}{\partial t} \\
&= \Gamma \cdot \frac{\partial \log \boldsymbol{p}^{R^N}}{\partial t} + \frac{\partial \log \boldsymbol{\mathcal{Y}}^{R,PE}}{\partial t} - \Gamma \cdot \frac{\partial \log \boldsymbol{p}^{R^N}}{\partial t} \\
&= \frac{\partial \log \boldsymbol{\mathcal{Y}}^{R,PE}}{\partial t} = \frac{\partial \log \boldsymbol{\mathcal{X}}^{R,PE}}{\partial t}
\end{aligned}$$

where the first equality follows (A.13b), the second is induced by (A.15) and the last equality holds because $\partial \log \mathcal{X}^{R,PE}/\partial t = \partial \log \mathcal{Y}^{R,PE}/\partial t$ at the PE.

B.4 Proof of Proposition 3

Let us use (14) to rewrite (30) as:

$$\frac{\partial \mathscr{L}^R(t)}{\partial t} = \int_W \left\{ (g(\mathbf{w}) - 1)\ R(\mathbf{Y}(\mathbf{w})) + \sum_{i=1}^n (\mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w})) + \mu_i)\ \frac{\partial Y_i^{R,PE}(\mathbf{w}, t)}{\partial t} \right\} \mathrm{d}F(\mathbf{w}).$$

Plugging (33) and (A.14) in the latter equation, we can write:

$$\begin{aligned}
\frac{\partial \mathscr{L}^{\mathcal{R}^j}(t)}{\partial t} &= \int_{\mathcal{W}} \Bigg\{ (g(\mathbf{w}) - 1)\left(1 - \mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w}))\right)\ Y_j(\mathbf{w}) \\
&+ \sum_{i=1}^n (\mathcal{T}_{y_i}(Y_i(\mathbf{w})) + \mu_i) \left[\frac{\partial Y_i(\mathbf{w})}{\partial \log p_j} - \mathbb{1}_{i=j}\ Y_j(\mathbf{w})\right] \Bigg\} \mathrm{d}F(\mathbf{w}) \\
&= \int_{\mathcal{W}} \left\{ \left[g(\mathbf{w}) \left(1 - \mathcal{T}_{y_j}(\mathbf{Y}(\mathbf{w}))\right) - 1 - \mu_j\right] Y_j(\mathbf{w}) + \sum_{i=1}^n (\mathcal{T}_{y_i}(\mathbf{Y}(\mathbf{w})) + \mu_i)\ \frac{\partial Y_i(\mathbf{w})}{\partial \log p_j} \right\} \mathrm{d}F(\mathbf{w}).
\end{aligned}$$

Using Equations (18) yields:

$$\frac{\partial \mathscr{L}^{\mathcal{R}^j}(t)}{\partial t} = \frac{\partial \mathscr{L}}{\partial \log p_j} - (1 + \mu_j)\mathcal{Y}_j(\mathbf{w}) + \sum_{i=1}^n \mu_i\ \frac{\partial \mathcal{Y}_i}{\partial \log p_j}.$$

From $\partial \mathcal{Y}_j(\mathbf{w})/\partial \log p_j = \partial \mathcal{X}_j/\partial \log p_j + \mathcal{Y}_j$ and (24b), we obtain:

$$\frac{\partial \mathscr{L}}{\partial \log p_j} = \mathcal{Y}_j - \sum_{i=1}^n \mu_i\ \mathcal{Y}_i\ \Gamma_{i,j} + \frac{\partial \mathscr{L}^{\mathcal{R}^j}(t)}{\partial t},$$

where, substituting $\eta_i = \mu_i\ \mathcal{Y}_i$ leads, in matrix terms to:

$$\frac{\boldsymbol{\partial \mathscr{L}}}{\boldsymbol{\partial \log p}} = \boldsymbol{\mathcal{Y}} + \frac{\boldsymbol{\partial \mathscr{L}^{\mathcal{R}}}}{\boldsymbol{\partial t}} - \boldsymbol{\eta} \cdot \boldsymbol{\Gamma}. \tag{A.16}$$

Substituting (29) into (A.16) leads to:

$$\frac{\partial \boldsymbol{\mathscr{L}}}{\partial \log \boldsymbol{p}} \quad = \quad \boldsymbol{\mathcal{Y}} + \frac{\partial \boldsymbol{\mathscr{L}}^{\boldsymbol{\mathcal{R}}}}{\partial \boldsymbol{t}} - \frac{\partial \boldsymbol{\mathscr{L}}}{\partial \log \boldsymbol{p}} \cdot (I_n - \Xi \cdot \Gamma)^{-1} \cdot \Xi \cdot \Gamma. \tag{A.17}$$

We notice that:

$$I_n + (I_n - \Xi \cdot \Gamma)^{-1} \cdot \Xi \cdot \Gamma = (I_n - \Xi \cdot \Gamma)^{-1} (I_n - \Xi \cdot \Gamma + \Xi \cdot \Gamma) = (I_n - \Xi \cdot \Gamma)^{-1}.$$

This equality matrix allows us to simplify Equation (A.17) as follows:

$$\frac{\partial \boldsymbol{\mathscr{L}}}{\partial \log \boldsymbol{p}} \cdot (I_n - \Xi \cdot \Gamma)^{-1} \quad = \quad \boldsymbol{\mathcal{Y}} + \frac{\partial \boldsymbol{\mathscr{L}}^{\boldsymbol{\mathcal{R}}}}{\partial \boldsymbol{t}}. \tag{A.18}$$

Multiplying on the right both sides of (A.18) by matrix Ξ and using (29), we eventually arrive at:

$$\boldsymbol{\eta} = \boldsymbol{\mathcal{Y}} \cdot \Xi + \frac{\partial \boldsymbol{\mathscr{L}}^{\boldsymbol{\mathcal{R}}}}{\partial \boldsymbol{t}} \cdot \Xi. \tag{A.19}$$

To compute the term $\boldsymbol{\mathcal{Y}} \cdot \Xi$, we differentiate both sides of (7) with respect to $\mathcal{X}_i$. We obtain: $p_i + \sum_{j=1}^{n} \mathcal{X}_j \, \partial \mathscr{P}_j / \partial \mathcal{X}_i = \mathcal{F}_{\mathcal{X}_i}$. Rearranging terms leads to: $\sum_{j=1}^{n} \mathcal{Y}_j \, \partial \log \mathscr{P}_j / \partial \mathcal{X}_i = \mathcal{F}_{\mathcal{X}_i} - p_i$. Multiplying both sides by $\mathcal{X}_i$ and using (24a) yields:

$$\sum_{j=1}^{n} \mathcal{Y}_j \, \Xi_{j,i} = (\mathcal{F}_{\mathcal{X}_i} - p_i) \, \mathcal{X}_i. \tag{A.20}$$

Therefore, the i^{th} column of Equation (A.19) writes:

$$\eta_i = \sum_{j=1}^{n} \mathcal{Y}_j \, \Xi_{j,i} + \sum_{j=1}^{n} \frac{\partial \mathscr{L}^{\mathcal{R}^j}}{\partial t} \, \Xi_{j,i} = (\mathcal{F}_{\mathcal{X}_i} - p_i) \, \mathcal{X}_i + \sum_{j=1}^{n} \frac{\partial \mathscr{L}^{\mathcal{R}^j}}{\partial t} \, \Xi_{j,i}$$

where the second equality is obtained using (A.20). Utilizing $\eta_i = \mu_i \, \mathcal{Y}_i$ leads to (38a). Under perfect competition, Equation (8) simplifies (38a) to (38b). If the tax schedule is optimized in the directions $\mathcal{R}^1, ..., \mathcal{R}^n$, yielding $\partial \mathscr{L}^{\mathcal{R}^1}(t)/\partial t = ... = \partial \mathscr{L}^{\mathcal{R}^n}(t)/\partial t = 0$, it modifies (38a) to (38c). Moreover, if one further assumes perfect competition, Equation (8) simplifies (38c) to (38d).

Alternative proof of (38a)

We now propose a more intuitive proof. Equation (34) implies:

$$\frac{\partial \mathscr{L}^{R^N}}{\partial t} = \frac{\partial \mathscr{L}^{R}}{\partial t} - \sum_{j=1}^{n} \gamma_j^R \, \frac{\partial \mathscr{L}^{\mathcal{R}^j}}{\partial t}.$$

Combining with (35) and (36) leads to (37). Comparing with (32) leads to:

$$\sum_{i=1}^{n} \mu_i \, \frac{\partial \mathcal{Y}_i^{R,PE}}{\partial t} = \sum_{1 \leq i,j \leq n} \left(\mathcal{Y}_j + \frac{\partial \mathscr{L}^{\mathcal{R}^j}}{\partial t} \right) \frac{\Xi_{j,i}}{\mathcal{Y}_i} \, \frac{\partial \mathcal{Y}_i^{R,PE}}{\partial t}.$$

The latter equality having to hold for any direction $R(\cdot)$,[39] the preceding equality leads to:

$$\forall i = 1, ..., n: \qquad \mu_i = \sum_{i=1}^{n} \left(\mathcal{Y}_j + \frac{\partial \mathscr{L}^{\mathcal{R}^j}}{\partial t} \right) \frac{\Xi_{j,i}}{\mathcal{Y}_i}.$$

Differentiating (7) with respect to the i^{th} factor, we obtain $\mathcal{F}_{\mathcal{X}_i} - p_i = \sum_j \left(\partial \mathscr{P}_j / \partial \mathcal{X}_i \right) \, \mathcal{X}_j$. Using (24a) leads to: $\mathcal{F}_{\mathcal{X}_i} - p_i = (1/\mathcal{X}_i) \sum_j \Xi_{ji} \, \mathcal{Y}_j$. Dividing both sides by p_i yields $\sum_{j=1}^{n} \mathcal{Y}_j \, \Xi_{j,i} / \mathcal{Y}_i = \left(\mathcal{F}_{\mathcal{X}_i} - p_i \right) / p_i$, thereby leading to (38a).

[39] This requires that the set of vectors $\left(\partial \mathcal{Y}_1^{R,PE} / \partial t, ..., \partial \mathcal{Y}_n^{R,PE} / \partial t \right)$ for all directions $R(\cdot)$ is $n-$dimensional, an assumption that is not necessary in the previous proof.

B.5 Proof of Proposition 4

Rewriting Equation (30) in terms of income $\mathbf{y}$ rather than type $\mathbf{w}$ yields:

$$\begin{aligned}\frac{\partial \mathscr{L}^R(t)}{\partial t} &= \int_{\mathcal{W}_Y} \left\{ - \left[1 - \widehat{g}(\mathbf{y}) - \sum_{i=1}^{n} \left(\mathcal{T}_{y_i}(\mathbf{y}) + \mu_i \right) \frac{\partial \widehat{Y}_i(\mathbf{y})}{\partial \rho} \right] R(\mathbf{y}) \right. \\ &+ \left. \sum_{1 \le i,j \le n} \left(\mathcal{T}_{y_i}(\mathbf{y}) + \mu_i \right) \frac{\partial \widehat{Y}_i(\mathbf{y})}{\partial \tau_j} R_{y_j}(\mathbf{y})) \right\} h(\mathbf{y}) \mathrm{d}\mathbf{y}.\end{aligned}$$

Using the divergence theorem on the term of the second line and rearranging, we obtain:

$$\begin{aligned}\frac{\partial \mathscr{L}^R(t)}{\partial t} &= \oint_{\partial W_Y} \sum_{1 \le i,j \le n} \left(\mathcal{T}_{y_i}(\mathbf{y}) + \mu_i \right) \frac{\partial \widehat{Y}_i(\mathbf{y})}{\partial \tau_j} h(\mathbf{y}) \phi_j(\mathbf{y}) R(\mathbf{y}) d\sigma(\mathbf{y}) \\ &- \int_{\mathcal{W}_Y} \left\{ \left[1 - \widehat{g}(\mathbf{y}) - \sum_{i=1}^{n} \left(\mathcal{T}_{y_i}(\mathbf{y}) + \mu_i \right) \frac{\partial \widehat{Y}_i(\mathbf{y})}{\partial \rho} \right] h(\mathbf{y}) \right. \\ &\left. + \sum_{j=1}^{n} \frac{\partial \left[\sum_{i=1}^{n} \left(\mathcal{T}_{y_i}(\mathbf{y}) + \mu_i \right) h(\mathbf{y}) \right]}{\partial y_j} \right\} R(\mathbf{y})\, d\mathbf{y}.\end{aligned}$$

where $d\sigma(\mathbf{y})$ is the corresponding measure of a surface integral (denoted by $\oint$). If the tax system $\mathbf{y} \mapsto \mathcal{T}(\mathbf{y})$ is optimal, the latest equation has to be equal to zero for all possible directions $R(\cdot)$. This is only possible if both equations given in Proposition 4 are satisfied.

At this optimum, one must have $\partial \mathscr{L}^{\mathcal{R}^j}/\partial t = 0$ for all $j \in \{1, ..., n\}$. This implies that Equation (38a) reduces to $\mu_1, ..., \mu_n = 0$ under perfect competition. Revealed welfare weights $\widetilde{g}(\mathbf{y})$ solve Equation (40) with $\mu_1, ..., \mu_n = 0$ for $\widehat{g}(\mathbf{y})$ for the current tax schedule, which leads to (42).

B.6 Proof of Proposition 5

From the definition of revealed welfare weights, we get that for any direction $R(\cdot)$: $\partial \mathscr{L}^R(t)/\partial t = 0$. Moreover, since $\mu_1 = ... = \mu_n = 0$, we have that for any direction $R(\cdot)$: $\partial \mathscr{L}^R(t)/\partial t = \partial \mathscr{L}^{R,PE}(t)/\partial t = 0$ from (32). Therefore, using $\partial \mathscr{L}^{R,PE}(t)/\partial t = \partial \mathscr{B}^{R,PE}(t)/\partial t + (1/\lambda) \partial \mathscr{W}^{R,PE}(t)/\partial t$ and Equation (15), it yields:

$$\frac{\partial \mathscr{B}^{R,PE}(t)}{\partial t} = - \int_{\mathcal{W}_Y} \widehat{g}(\mathbf{y})\ R(\mathbf{y})\ h(\mathbf{y}) \mathrm{d}\mathbf{y}. \tag{A.21}$$

Therefore, a tax reform with a small positive magnitude t and a direction $R(\cdot)$ that verifies (43) increases tax revenue at the PE. According to (15), such a reform also increases at the PE the welfare of taxpayers for which $R(\mathbf{Y}(\mathbf{w})) > 0$ and leave the welfare of the others unchanged. It is therefore a Pareto-improving tax reform at the PE.

According to Proposition 2, a reform with a small positive t in the direction $R^N(\cdot)$ defined in (34) has the same effects at the GE on taxpayers' utility $U(\mathbf{w})$ and factor supplies $\mathbf{X}(\mathbf{w})$ as a reform in the direction $R(\cdot)$ and the same magnitude t at the PE. Since tax revenues are equal to: $\sum_{j=1}^{n} p_j\ \mathcal{X}_j - \int_{\mathcal{W}} \mathscr{C}\left(U(\mathbf{w}), \mathbf{X}(\mathbf{w}); \mathbf{w}\right)\ \mathrm{d}F(\mathbf{w})$, if a tax reform with a small positive magnitude t and a direction $R(\cdot)$ is Pareto-improving at the PE, which is the case when some revealed welfare are negative and the direction $R(\cdot)$ verifies (43), a reform with a small positive magnitude t and the direction $R^N(\cdot)$ defined by (34) and (35) is Pareto-improving if $\sum_j \mathcal{X}_j\ \partial p^{R^N}/\partial t \ge 0$. From (7) we get:

$$\mathcal{F}\left(\mathcal{X}_1^{R,PE}(t), ..., \mathcal{X}_n^{R,PE}(t) \right) = \sum_{j=1}^{n} p_j^R(t)\ \mathcal{X}_j^{R,PE}(t)$$

Differentiating both sides with respect to t and using (8) leads to: $\sum_j \mathcal{X}_j\ \partial p^{R^N}/\partial t = 0$. Hence, If a reform with a small positive magnitude t and a direction $R(\cdot)$ is Pareto-improving at the PE, then, under perfect competition, a reform with a small positive magnitude t and the direction $R^N(\cdot)$ defined by (34) and (35) is Pareto-improving at the GE.

B.7 Proof of Proposition 6

We consider the case where revealed welfare weights $\widehat{g}(\mathbf{y}) > 0$ are almost everywhere positive.

We first notice that, according to Proposition 2, under perfect competition, there exists a direction $R^N(\cdot)$ such that reforms with positive t in the direction $R^N(\cdot)$ are Pareto-improving at the GE *if and only* if there exists a direction $R(\cdot)$ such that reforms with positive t in the direction $R^N(\cdot)$ are Pareto-improving at the PE, where $R(\cdot)$ and $R(\cdot)$ are related by (34) and (35)

Assuming, by contradiction, that there exists a direction of tax reform denoted $R^N(\cdot)$ such that a reform in the direction $R^N(\cdot)$ and a small positive magnitude t is Pareto-improving at the GE. According to Proposition 2, this implies the existence of a direction of tax reform denoted $R(\cdot)$, such that a reform with this direction and a positive t is Pareto-improving at the PE. According to (15), since a reform in the direction $R(\cdot)$ improves taxpayer's welfare at the PE, one must have $R(\mathbf{Y}(\mathbf{w})) \geq 0$ for all $\mathbf{w} \in \mathcal{W}$ with a strict inequality for some types. However, according to (A.21), such a reform decreases tax revenues at the PE, leading to a contradiction for a Pareto-improving direction of tax reforms at the PE.

B.8 Proof of Proposition 7

When the tax system is schedular and linear for $i = n' + 1, ..., n$, we get that:

$$\mathcal{T}(\mathbf{y}) = \sum_{i=1}^{n'} T_i(y_i) + \sum_{i=n'+1}^{n} t_i\, y_i \tag{A.22}$$

the admissible directions of tax reforms must also be schedular, i.e. they must depend only on one type of income and take the form $\mathbf{y} \mapsto R_i(y_i)$. Moreover for $i = n' + 1, ..., n$ the directions specific to the i^{th} income must be linear.

Under Equation (A.22), according to (33) the GE-replicating directions are given by $\mathcal{R}^j(\mathbf{y}) = (1 - T'(y_j))y_j$ for $j = 1, ..., n'$ and by $R^j(\mathbf{y}) = (1 - t_j)y_j$ for $j = n' + 1, ..., n$. Perturbing the tax system along the GE-replicating directions thus keeps the tax system being schedular and also linear for $i = n' + 1, ..., n$. Therefore, one has $\partial\mathscr{L}^{\mathcal{R}^j}/\partial t = 0$ for all $j = 1, ..., n$, so, according to Proposition 3, the GE multipliers are given by (38c).

Let $R_i(y_i)$ be any direction of a tax reform specific to the i^{th} income. Because the tax schedule is schedular, Equation (30), stating the impact on the Lagrangian of a tax reform at the GE, simplifies to:

$$\begin{aligned}\frac{\partial\mathscr{L}^{R_i}(t)}{\partial t} &= \int_{\mathcal{W}} \Bigg\{ -\left[1 - g(\mathbf{w}) - \sum_{k=1}^{n} \left(T_k'(Y_k(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k(\mathbf{w})}{\partial \rho}\right] R_i(Y_i(\mathbf{w})) \\ &+ \sum_{1=k}^{n} \left(T_k'(Y_k(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k(\mathbf{w})}{\partial \tau_i}\, R_i'(Y_i(\mathbf{w}))\Bigg\} \mathrm{d}F(\mathbf{w}).\end{aligned} \tag{A.23}$$

since $\partial R_i(y_i)/\partial y_j = 0$ whenever $j \neq i$ under a schedular direction of tax reform. Rewritten in terms of the distribution of the i^{th} income leads to:

$$\begin{aligned}\frac{\partial\mathscr{L}^{R_i}(t)}{\partial t} &= \int_{\mathbb{R}_+} \Bigg\{ -\left[1 - \overline{g(\mathbf{w})}\Big|_{Y_i(\mathbf{w})=y_i} - \sum_{k=1}^{n} \overline{\left(T_k'(Y_k(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k(\mathbf{w})}{\partial \rho}}\Bigg|_{Y_i(\mathbf{w})=y_i}\right] R_i(y_i) \\ &+ \sum_{1=k}^{n} \overline{\left(T_k'(Y_k(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k(\mathbf{w})}{\partial \tau_i}}\Bigg|_{Y_i(\mathbf{w})=y_i} R_i'(y_i)\Bigg\} h_i(y_i)\mathrm{d}y_i.\end{aligned}$$

Integrating by parts the first term and rearranging terms using (44) leads to:

$$\frac{\partial \mathscr{L}^{R_i}(t)}{\partial t} = \int_{\mathbb{R}_+} \left\{ - \int_{z=y_i}^{\infty} \left[1 - \overline{g(\mathbf{w})}|_{Y_i(\mathbf{w})=z} - \sum_{k=1}^{n} \overline{\left(T_k'(Y_k(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k(\mathbf{w})}{\partial \rho}}\Big|_{Y_i(\mathbf{w})=z} \right] \mathrm{d}H_i(z) \right.$$
$$\left. + \frac{T_i'(y_i) + \mu_i}{1 - T_i'(y_i)} \, \varepsilon_i(y_i) \, y_i \, h_i(y_i) + \sum_{1 \le k \le n, k \ne i} \overline{\left(T_k'(Y_k(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k(\mathbf{w})}{\partial \tau_i}}\Big|_{Y_i(\mathbf{w})=y_i} h_i(y_i) \right\} R'(y_i) \mathrm{d}y_i.$$
$$- \lim_{y_i \mapsto \infty} \left\{ \int_{z=y_i}^{\infty} \left[1 - \overline{g(\mathbf{w})}|_{Y_i(\mathbf{w})=z} - \sum_{k=1}^{n} \overline{\left(T_k'(Y_k(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k(\mathbf{w})}{\partial \rho}}\Big|_{Y_i(\mathbf{w})=z} \right] \mathrm{d}H_i(z) \; R_i(y_i) \right\}$$
$$+ \lim_{y_i \mapsto 0} \left\{ \int_{z=y_i}^{\infty} \left[1 - \overline{g(\mathbf{w})}|_{Y_i(\mathbf{w})=z} - \sum_{k=1}^{n} \overline{\left(T_k'(Y_k(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k(\mathbf{w})}{\partial \rho}}\Big|_{Y_i(\mathbf{w})=z} \right] \mathrm{d}H_i(z) \; R_i(y_i) \right\}$$

For $i = 1, ..., n'$, the income specific tax schedule $T_i(\cdot)$ being nonlinear, the above equation must be equal to zero for any non linear direction R_i, which implies (45a).

For $i = n' + 1, ..., n$, the i^{th} income specific tax schedule has to be linear, so the only admissible directions of tax reforms specific to the i^{th} income are proportional to $R_i(y_i) = y_i$. Equation (A.23) then simplifies to:

$$\begin{aligned} \frac{\partial \mathscr{L}^{R_i}(t)}{\partial t} &= \int_{\mathcal{W}} \left\{ - \left[1 - g(\mathbf{w}) - \sum_{k=1}^{n} \left(T_k'(Y_k(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k(\mathbf{w})}{\partial \rho} \right] Y_i(\mathbf{w}) \right. \\ &\quad \left. + \sum_{1=k}^{n} \left(T_k'(Y_k(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k(\mathbf{w})}{\partial \tau_i} \right\} \mathrm{d}F(\mathbf{w}). \end{aligned}$$

Using (20), the preceding equation simplifies to:

$$\frac{\partial \mathscr{L}^{R_i}(t)}{\partial t} = \int_{\mathcal{W}} \left\{ - \left[1 - g(\mathbf{w})\right] Y_i(\mathbf{w}) + \sum_{1=k}^{n} \left(T_k'(Y_k(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k^u(\mathbf{w})}{\partial \tau_i} \right\} \mathrm{d}F(\mathbf{w}).$$

Using (44), the condition $\partial \mathscr{L}^{y_i} / \partial t = 0$ leads to (45b).

B.9 Proof of Proposition 8

When the tax schedule is comprehensive, admissible directions of tax reforms take the form $\mathbf{y} \mapsto R(y_1 + ... + y_n)$. Consequently, Equation (30) simplifies to:

$$\begin{aligned} \frac{\partial \mathscr{L}^{R}(t)}{\partial t} &= \int_{\mathcal{W}} \left\{ - \left[1 - g(\mathbf{w}) - \sum_{k=1}^{n} \left(T_0'(Y_0(\mathbf{w})) + \mu_k\right) \frac{\partial Y_i(\mathbf{w})}{\partial \rho} \right] R(Y_0(\mathbf{w})) \right. \\ &\quad \left. + \sum_{1 \le j,k \le n} \left(T_0'(Y_0(\mathbf{w})) + \mu_k\right) \frac{\partial Y_k(\mathbf{w})}{\partial \tau_j} \, R'(Y_0(\mathbf{w})) \right\} \mathrm{d}F(\mathbf{w}). \end{aligned}$$

Rewriting this expression in terms of the density $h_0(\cdot)$ and CDF $H_0(\cdot)$ of the taxable income, the last equation becomes:

$$\begin{aligned} \frac{\partial \mathscr{L}^{R}(t)}{\partial t} &= \int_{\mathbb{R}_+} \left\{ - \left[1 - \overline{g(\mathbf{w})}|_{Y_0(\mathbf{w})=y_0} - \sum_{k=1}^{n} \left(T_0'(y_0) + \mu_k\right) \overline{\frac{\partial Y_k(\mathbf{w})}{\partial \rho}}\Big|_{Y_0(\mathbf{w})=y_0} \right] R(y_0) \right. \\ &\quad \left. + \sum_{1 \le j,k \le n} \left(T_0'(y_0) + \mu_k\right) \overline{\frac{\partial Y_k(\mathbf{w})}{\partial \tau_j}}\Big|_{Y_0(\mathbf{w})=y_0} R'(y_0) \right\} h_0(y_0) \mathrm{d}y_0. \end{aligned}$$

Using (46)-(48) leads to:

$$\begin{aligned}\frac{\partial \mathscr{L}^R(t)}{\partial t} &= \int_{\mathbb{R}_+}\left\{-\left[1-\overline{g(\mathbf{w})}|_{Y_0(\mathbf{w})=y_0} - T_0'(y_0)\,\frac{\partial Y_0(y_0)}{\partial \rho} - \sum_{k=1}^n \mu_k\,\overline{\frac{\partial Y_k(\mathbf{w})}{\partial \rho}}\Big|_{Y_0(\mathbf{w})=y_0}\right] R(y_0)\right.\\ &+ \left.\frac{T_0'(y_0)}{1-T_0'(y_0)}\,\varepsilon_0(y_0)\,y_0\,R'(y_0) + \sum_{k=1}^n \mu_k\,\overline{\frac{\partial Y_k(\mathbf{w})}{\partial \tau_0}}\Big|_{Y_0(\mathbf{w})=y_0} R'(y_0)\right\} h_0(y_0)\mathrm{d}y_0.\end{aligned}$$

Integrating by parts the first line yields:

$$\begin{aligned}&\frac{\partial \mathscr{L}^R(t)}{\partial t}\\ =&\int_{\mathbb{R}_+}\left\{-\int_{z=y_0}^{\infty}\left[1-\overline{g(\mathbf{w})}|_{Y_0(\mathbf{w})=z} - T_0'(y_0)\frac{\partial Y_0(y_0)}{\partial \rho} - \sum_{k=1}^n \mu_k\,\overline{\frac{\partial Y_k(\mathbf{w})}{\partial \rho}}\Big|_{Y_0(\mathbf{w})=z}\right]\mathrm{d}H_0(y_0)\right.\\ +&\left.\frac{T_0'(y_0)}{1-T_0'(y_0)}\,\varepsilon_0(y_0)\,y_0\,h_0(y_0) + \sum_{k=1}^n \mu_k\,\overline{\frac{\partial Y_k(\mathbf{w})}{\partial \tau_0}}\Big|_{Y_0(\mathbf{w})=y_0} h_0(y_0)\right\} R'(y_0)\mathrm{d}y_0\\ -&\lim_{y\mapsto\infty}\int_{z=y_0}^{\infty}\left[1-\overline{g(\mathbf{w})}|_{Y_0(\mathbf{w})=z} - T_0'(y_0)\frac{\partial Y_0(y_0)}{\partial \rho} - \sum_{k=1}^n \mu_k\,\overline{\frac{\partial Y_k(\mathbf{w})}{\partial \rho}}\Big|_{Y_0(\mathbf{w})=z}\right]\mathrm{d}H_0(y_0)\;R(y_0)\\ +&\lim_{y\mapsto 0}\int_{z=y_0}^{\infty}\left[1-\overline{g(\mathbf{w})}|_{Y_0(\mathbf{w})=z} - T_0'(y_0)\frac{\partial Y_0(y_0)}{\partial \rho} - \sum_{k=1}^n \mu_k\,\overline{\frac{\partial Y_k(\mathbf{w})}{\partial \rho}}\Big|_{Y_0(\mathbf{w})=z}\right]\mathrm{d}H_0(y_0)\;R(y_0)\end{aligned}$$

At the optimal comprehensive tax schedule, one mus have $\partial\mathscr{L}^R/\partial t = 0$ for all directions, which implies Equation (49).

If there are only two production factors and if the elasticity of substitution between these two factors is denoted σ, one gets:

$$\frac{\mathrm{d}p_1}{p_1} - \frac{\mathrm{d}p_2}{p_2} = \frac{1}{\sigma}\left(\frac{\mathrm{d}\mathcal{X}_2}{\mathcal{X}_2} - \frac{\mathrm{d}\mathcal{X}_1}{\mathcal{X}_1}\right)$$

Under perfect competition, and denoting $\alpha_i = \mathcal{Y}_i/\left(\mathcal{Y}_1+\mathcal{Y}_2\right)$ the i^{th} income share, the differentiation of both sides of (7) lead to:

$$0 = \alpha_1\,\frac{\mathrm{d}p_1}{p_1} + \alpha_2\,\frac{\mathrm{d}p_2}{p_2} \qquad \Rightarrow \qquad \frac{\mathrm{d}p_1}{p_1} - \frac{\mathrm{d}p_2}{p_2} = \frac{1}{\alpha_2}\,\frac{\mathrm{d}p_1}{p_1} = -\frac{1}{\alpha_1}\,\frac{\mathrm{d}p_2}{p_2}$$

Combining the two latter equations leads to:

$$\Xi = \begin{pmatrix} -\frac{\alpha_2}{\sigma} & \frac{\alpha_2}{\sigma} \\ \frac{\alpha_1}{\sigma} & -\frac{\alpha_1}{\sigma}\end{pmatrix}$$

Under perfect competition, the GE multipliers are given by Equation (38b), which leads to:

$$\mu_1 = \frac{-\dfrac{\partial \mathscr{L}^{\mathcal{R}^1}}{\partial t}\,\alpha_2 + \dfrac{\partial \mathscr{L}^{\mathcal{R}^2}}{\partial t}\,\alpha_1}{\sigma\,\mathcal{Y}_1} \qquad \text{and :} \qquad \mu_1 = \frac{\dfrac{\partial \mathscr{L}^{\mathcal{R}^1}}{\partial t}\,\alpha_2 - \dfrac{\partial \mathscr{L}^{\mathcal{R}^2}}{\partial t}\,\alpha_1}{\sigma\,\mathcal{Y}_2}$$

Using $\partial\mathscr{L}^{\mathcal{R}^1}/\partial t + \partial\mathscr{L}^{\mathcal{R}^2}/\partial t = 0$ eventually yields (50).

C Appendix related to Section IV

C.1 Proof of Proposition 9

Equation (51c) can be rewritten in matrix form as:

$$\frac{\partial \mathscr{L}}{\partial \alpha_\ell} \mathrm{d}\alpha_\ell = \frac{\boldsymbol{\partial \mathscr{L}}}{\boldsymbol{\partial \log p}} \cdot \frac{\mathbf{d}\boldsymbol{p}}{\boldsymbol{p}}$$

Using (52) leads to:

$$\frac{\partial \mathscr{L}}{\partial \alpha_\ell} = \frac{\boldsymbol{\partial \mathscr{L}}}{\boldsymbol{\partial \log p}} \cdot (\boldsymbol{I_n} - \boldsymbol{\Xi} \cdot \boldsymbol{\Gamma})^{-1} \cdot \frac{\boldsymbol{\partial \log \mathscr{P}}}{\boldsymbol{\partial \alpha_\ell}}$$

. Plugging (A.18) in the latter equation leads to:

$$\frac{\partial \mathscr{L}}{\partial \alpha_\ell} = \left(\boldsymbol{\mathcal{Y}} + \frac{\boldsymbol{\partial \mathscr{L}^{\mathcal{R}}}}{\boldsymbol{\partial t}}\right) \cdot \frac{\boldsymbol{\partial \log \mathscr{P}}}{\boldsymbol{\partial \alpha_\ell}} = \sum_{j=1}^{n} \mathcal{Y}_j \frac{\partial \log \mathscr{P}_j}{\partial \alpha_\ell} + \sum_{j=1}^{n} \frac{\partial \mathscr{L}^{\mathcal{R}^j}}{\partial t} \frac{\partial \log \mathscr{P}_j}{\partial \alpha_\ell}. \tag{A.24}$$

Differentiating both sides of (7) with respect to α_ℓ implies that:

$$\mathcal{F}_{\alpha_\ell} = \sum_{j=1}^{n} \mathcal{X}_j \frac{\partial \mathscr{P}_j}{\partial \alpha_\ell} = \sum_{j=1}^{n} \mathcal{Y}_j \frac{\partial \log \mathscr{P}_j}{\partial \alpha_\ell}. \tag{A.25}$$

Plugging (A.25) into (A.24) ends the proof of Proposition 9.

C.2 Proof of Proposition 10

Since matrix Γ also provides the effects at the PE of the GE-replicating reforms on factor supplies and using (55) we get that:

$$\frac{\boldsymbol{\partial \log \mathcal{Y}^{R^N,PE}}}{\boldsymbol{\partial t}} = -\Gamma \cdot \sum_{\ell=1}^{L} \frac{\boldsymbol{\partial \log \mathscr{P}}}{\boldsymbol{\partial \alpha_\ell}} \alpha'_\ell(t).$$

Using (52) yields:

$$\begin{aligned} (\boldsymbol{I_n} - \boldsymbol{\Xi} \cdot \boldsymbol{\Gamma}) \frac{\mathbf{d}\boldsymbol{p}^{\boldsymbol{R^N}}}{\boldsymbol{p}} &= \sum_{\ell=1}^{L} \frac{\boldsymbol{\partial \log \mathscr{P}}}{\boldsymbol{\partial \alpha_\ell}} \alpha'_\ell(t)\, \mathrm{d}t - \Xi \cdot \Gamma \cdot \sum_{\ell=1}^{L} \frac{\boldsymbol{\partial \log \mathscr{P}}}{\boldsymbol{\partial \alpha_\ell}} \alpha'_\ell(t)\, \mathrm{d}t \\ \frac{\mathbf{d}\boldsymbol{p}^{\boldsymbol{R^N}}}{\boldsymbol{p}} &= \sum_{\ell=1}^{L} \frac{\boldsymbol{\partial \log \mathscr{P}}}{\boldsymbol{\partial \alpha_\ell}} \alpha'_\ell(t)\, \alpha'_\ell(t)\, \mathrm{d}t \end{aligned} \tag{A.26}$$

Using (A.13b) leads to:

$$\frac{\mathbf{d}\boldsymbol{\mathcal{X}}^{\boldsymbol{R^N}}}{\boldsymbol{\mathcal{X}}} = \Gamma \cdot \sum_{\ell=1}^{L} \frac{\boldsymbol{\partial \log \mathscr{P}}}{\boldsymbol{\partial \alpha_\ell}} \alpha'_\ell(t)\, \mathrm{d}t - \Gamma \cdot \sum_{\ell=1}^{L} \frac{\boldsymbol{\partial \log \mathscr{P}}}{\boldsymbol{\partial \alpha_\ell}} \mathrm{d}\alpha_\ell = 0$$

Hence, combining the production policies reforms $t \mapsto \alpha_1(t), ..., \alpha_L(t)$ with the tax reform $-\sum_{j=1}^{n} \gamma_j \mathcal{R}^j$ has no impact on factor supplies. According to (A.9a), (A.9b) and (A.26), the impacts on taxpayers' utility are also nil and so are the impact on taxpayer's consumption. Hence, the impact on government's revenue $\sum_{j=1}^{n} p_j\, \mathcal{X}_j - \int_{\mathcal{W}} \mathscr{C}(U(\mathbf{w}), \mathbf{X}(\mathbf{w}); \mathbf{w}) \mathrm{d}F(\mathbf{w})$ is given by:

$$\begin{aligned} \sum_{j=1}^{n} \mathcal{X}_j\, \mathrm{d}p_j^{R^N} &= \sum_{j=1}^{n} \mathcal{Y}_j \frac{\mathrm{d}p_j^{R^N}}{p_j} = \boldsymbol{\mathcal{Y}} \cdot \frac{\mathbf{d}\boldsymbol{p}^{\boldsymbol{R^N}}}{\boldsymbol{p}} \\ &= \sum_{\ell=1}^{L} \sum_{j=1}^{n} \mathcal{Y}_j \frac{\partial \log \mathscr{P}_j}{\partial \alpha_\ell} \alpha'_\ell(t)\, \mathrm{d}t \\ &= \sum_{\ell=1}^{L} \mathcal{F}_{\alpha_\ell}\, \alpha'_\ell(t)\, \mathrm{d}t \end{aligned}$$

which is positive, ending the proof of Proposition 10.

C.3 Appendix related to Section IV.2

In this appendix, we show that the constrained profit maximization of firms (Equation (59)) is equivalent to the constrained maximization of a hypothetical "production coordinator". This coordinator maximizes the total production of the final good minus the consumption of the final good required for the production of intermediate goods.

The Lagrangian of the production coordinator's program (61) writes:

$$\sum_{s=0}^{S} q_s \left[\sum_{\varphi=1}^{N_s}(1-\alpha_s)\mathcal{F}^{\varphi,s}(\boldsymbol{\mathcal{X}}^{\varphi,s}, \boldsymbol{z}^{\varphi,s}) - \sum_{s'=0 s'\neq s}^{S}\sum_{\varphi=1}^{N_{s'}} z_s^{\varphi,s'} + \overline{Z}_s\right] + \sum_{i=1}^{n} p_i \left[\mathcal{X}_i - \sum_{s=0}^{S}\sum_{\varphi=1}^{N_s}\mathcal{X}_i^{\varphi,s}\right]$$

where we denote p_i the Lagrange multiplier associated to the i^{th} constraint (61b) and where we denote q_s the Lagrange multiplier associated with the s^{th} constraint (61c). We normalize $\overline{Z}_0 = \alpha_0 = 0$ and $q_0 = 1$. The first-order conditions with respect to $\mathcal{X}_i^{\varphi,s}$ and $z_{s'}^{\varphi,s}$ coincide with Equations (60). Since plugging (61d) into (61c) leads to (56) and equation (61b) coincides with (58), the competitive allocation of the production resources coincides with the solution of the production coordinator's Program (61).

We now verify that the sum of income factors $\sum_{i=1}^{n} p_i\ \mathcal{X}_i$, profits $\sum_{s=0}^{S}\sum_{\varphi=1}^{N_s}\pi^{\varphi,s}$ and revenues from intermediate good taxation $\sum_{s=1}^{S}\sum_{\varphi=1}^{N_s}\alpha_s q_s \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s}\right)$ is equal to the total production of the final good $\sum_{\varphi=1}^{N_0}\mathcal{F}^{\varphi,0}\left(\boldsymbol{\mathcal{X}}^{\varphi,0}, \mathbf{z}^{\varphi,0}\right)$ net of the consumption of the final good by intermediate good producers, i.e. $\sum_{s=1}^{S}\sum_{\varphi=1}^{N_s} z_0^{\varphi,s}$. According to (59), the sum of income factors, profits and tax liabilities on intermediate goods from firm $\varphi \in \{1,...,N_s\}$ in sector $s \in \{0,...,S\}$ is equal to:

$$\sum_{i=1}^{n} p_i\ \mathcal{X}_i^{\varphi,s} + \pi^{\varphi,s} + \alpha_s\ q_s\ \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s}\right) = q_s\ \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s}\right) - \sum_{\substack{s'=0\\ s'\neq s}}^{S} q_{s'}\ z_{s'}^{\varphi,s}.$$

In other words, the value added of firm φ in sector s in the right-hand side is equal to the sum of factor incomes, profits and tax revenue from intermediate goods taxation in the left-hand side. Adding the later equality for all firms in all sectors and using (58), $\alpha_0 = 0$ and $q_0 = 1$ yield:

$$\begin{aligned}
&\sum_{i=1}^{n} p_i\ \mathcal{X}_i + \sum_{s=0}^{S}\sum_{\varphi=1}^{N_s}\pi^{\varphi,s} + \sum_{s=0}^{S}\sum_{\varphi=1}^{N_s}\alpha_s\ q_s\ \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s}\right)\\
=\ &\sum_{s=0}^{S}\sum_{\varphi=1}^{N_s} q_s\ \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s}\right) - \sum_{\substack{0\leq s,s'\leq S\\ s'\neq s}}\sum_{\varphi=1}^{N_s} q_{s'}\ z_{s'}^{\varphi,s}\\
=\ &\sum_{\varphi=1}^{N_0}\mathcal{F}^{\varphi,0}\left(\boldsymbol{\mathcal{X}}^{\varphi,0}, \mathbf{z}^{\varphi,0}\right) + \sum_{s=1}^{S}\sum_{\varphi=1}^{N_s} q_s\ \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s}\right) - \sum_{\substack{0\leq s,s'\leq S\\ s'\neq s}}\sum_{\varphi=1}^{N_s} q_{s'}\ z_{s'}^{\varphi,s}\\
=\ &\sum_{\varphi=1}^{N_0}\mathcal{F}^{\varphi,0}\left(\boldsymbol{\mathcal{X}}^{\varphi,0}, \mathbf{z}^{\varphi,0}\right) + \sum_{s=1}^{S}\sum_{\varphi=1}^{N_s} q_s\ \mathcal{F}^{\varphi,s}\left(\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s}\right) - \sum_{\substack{0\leq s,s'\leq S\\ s'\neq s}}\sum_{\varphi=1}^{N_{s'}} q_s\ z_s^{\varphi,s'}
\end{aligned}$$

where the last equality has been obtained by inverting indexes s and s' in the last term. Combining the latter equality with (56) leads to:

$$\sum_{i=1}^{n} p_i \ \mathcal{X}_i + \sum_{s=0}^{S} \sum_{\varphi=1}^{N_s} \pi^{\varphi,s} + \sum_{s=0}^{S} \sum_{\varphi=1}^{N_s} \alpha_s \ q_s \ \mathcal{F}^{\varphi,s} \left(\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s} \right)$$

$$= \sum_{\varphi=1}^{N_0} \mathcal{F}^{\varphi,0} \left(\boldsymbol{\mathcal{X}}^{\varphi,0}, \mathbf{z}^{\varphi,0} \right) + \sum_{\substack{0 \le s,s' \le S \\ s' \neq s, s \neq 0}} \sum_{\varphi=1}^{N_{s'}} q_s \ z_s^{\varphi,s'} - \sum_{\substack{0 \le s,s' \le S \\ s' \neq s}} \sum_{\varphi=1}^{N_{s'}} q_s \ z_s^{\varphi,s'}.$$

Simplifying eventually leads to:

$$\sum_{i=1}^{n} p_i \ \mathcal{X}_i + \sum_{s=0}^{S} \sum_{\varphi=1}^{N_s} \pi^{\varphi,s} + \sum_{s=1}^{S} \sum_{\varphi=1}^{N_s} \alpha_s \ q_s \ \mathcal{F}^{\varphi,s} \left(\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s} \right) = \sum_{\varphi=1}^{N_0} \mathcal{F}^{\varphi,0} \left(\boldsymbol{\mathcal{X}}^{\varphi,0}, \mathbf{z}^{\varphi,0} \right) - \sum_{s'=1}^{S} \sum_{\varphi=1}^{N_{s'}} z_0^{\varphi,s'},$$

i.e. the sum of income factors $\sum_{i=1}^{n} p_i \ \mathcal{X}_i$, profits $\sum_{s=0}^{S} \sum_{\varphi=1}^{N_s} \pi^{\varphi,s}$ and tax revenue from intermediate good taxation $\sum_{s=1}^{S} \sum_{\varphi=1}^{N_s} \alpha_s q_s \mathcal{F}^{\varphi,s} \left(\boldsymbol{\mathcal{X}}^{\varphi,s}, \mathbf{z}^{\varphi,s} \right)$ is equal to the total production of the final good $\sum_{\varphi=1}^{N_0} \mathcal{F}^{\varphi,0} \left(\boldsymbol{\mathcal{X}}^{\varphi,0}, \mathbf{z}^{\varphi,0} \right)$ net of the consumption of the final good by intermediate goods producers, i.e. $\sum_{s'=1}^{S} \sum_{\varphi=1}^{N_{s'}} z_0^{\varphi,s}$.