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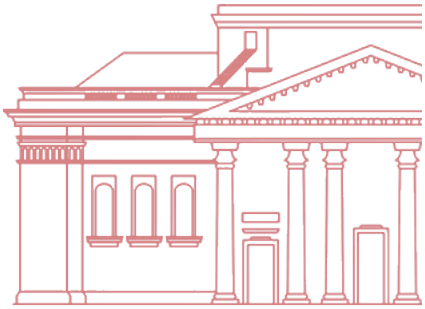

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Non-Deceptive Counterfeiting and Consumer Welfare: A Differential Game Approach

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Abstract

Counterfeiting, which is defined as illegally copying genuine goods with a brand name, is a widespread phenomenon and is imposing a huge cost on owners of trademarks. As a consequence, to deter counterfeiting authorities fine anyone producing or trading (in) fake goods. Yet, consuming counterfeit products is not prosecuted in certain countries, like the US or the UK, whereas it is in France or Italy. Why is it so? To tackle this issue we study how the entry of a counterfeiter affects the legal firm's pricing and advertising strategies and profits when there is no public nor private enforcement of property rights. The rationale for focusing on price and advertising is in fact straightforward. First, it is probably the high margin, that is, the difference between the price and the (comparatively very low) production cost that makes counterfeiting financially attractive. Second, the high willingness-to-pay by consumers is driven by the brand image or reputation, and this asset is notably built through advertising. Third, public enforcement of property rights is often lax or imperfect and not all legal firms can afford private enforcement policies.

Our results are as follows: First, we obtain that counterfeiting affects negatively pricing and advertising strategies before and after entry occurs. Thus counterfeiting nevers stimulates the building of brand reputation. Second, we show that while counterfeiting always reduces the profits of the owner of the genuine product, there are circumstances under which consumers benefit from this illegal trade (in this case the decrease in the price of the genuine good compensates the decrease in the brand reputation of this good). Such a result can rationalize the observation that fining producing and trading in fake goods can go in pairs with non fining consumers of fake products.

JEL codes: K42, L15, O34, P14, P37.

Keywords: OR in Marketing; Counterfeiting; Dynamic Games; Pricing; Advertising;

1 Introduction

Grossman and Shapiro (1988a, 1988b) define counterfeiting as illegally copying genuine goods with a brand name, whereas Cordell et al. (1996) state that "Any unauthorized manufacturing of goods whose special characteristics are protected as intellectual property rights (trademarks, patents and copyrights) constitutes product counterfeiting." As clearly shown by the numbers to follow, the worldwide magnitude of this illegal activity is simply astonishing. According to Levin (2009), American businesses and industries lose approximately $200 billion in revenues annually due to counterfeits, and on a broader scale, counterfeit goods account for more than half a trillion dollars each year.[1] Research analysts estimate that the number of jobs lost worldwide to counterfeit black markets is approximately 2.5 million with 750,000 of them being located in the United States (Levin, ibid) and 300,000 in Europe (Eisend and Schichert-Guler (2006)). Even though they are already impressive, these figures probably do not tell the whole story. For instance, it may well be that by violating property rights, counterfeiting discourages the owners from investing in improving the quality of their products, which undoubtedly has a private and a social cost.[2]

[1] See also A. Sowder, "The Harmful Effects of Counterfeit Goods", Athens State University, http://www.athens.edu/business-journal/spring-2013/asowder-couterfeit/.

[2] Staake et al. (2009) provides a comprehensive literature review and discusses the existing body of research on the structures and mechanisms of counterfeit trade before 2010

It is natural to wonder how to efficiently combat and deter counterfeiting, and one can distinguish between private and public efforts. Public enforcement of property rights has often relied on the seizure of counterfeit goods, which is prescribed in the commercial laws of many countries. For instance, more than 40 million counterfeit products were seized at the European Union's external border in 2012: their equivalent value in genuine products is nearly €1 billion.[3,4] In addition to confiscation, authorities can fine anyone producing or trading (in) fake goods. [5] Designing fines involves two decisions. The first pertains to determining of the fines' values, and the second relates to how the proceeds of the fines are used. As regards the first issue, the penalty for counterfeiting is often set as a function of the price charged by the intellectual property right (henceforth IPR) holder. To illustrate, in the U.S., the Anticounterfeiting Consumer Protection Act of 1996, S. 1136, provides civil fines pegged to the value of genuine goods. The fines are often rebated to the producers of the genuine goods. For instance, in June 2008, a French Court "ordered e-Bay to pay $63 million in damages to units of the Paris-based luxury goods mammoth LVMH, after agreeing that the site had facilitated the sale of counterfeit versions of its high-end products, particularly Louis Vuitton luggage..."). [6]

Another important issue when it comes to deterring counterfeiting is whether consumers of fake products should be fined as well (in addition to being exposed to seizure). This depends on whether consumers are victims of counterfeiting or whether know perfectly well that the products they are buying are imitations. One can argue that punishing the purchase of counterfeit products would deter the illegal trade of such goods. For example, in Italy, purchasing counterfeit products is considered a crime. Buyers of counterfeit goods are given on-the-spot fines of up to 10,000 euros. In France, the maximum fine for buying fake goods is 300,000 euros or three years in jail.[7] In other countries, like the U.S.A or the UK, authorities target those who trade in fake goods, but refrain from criminalizing consumers who buy them. A possible drawback of prosecuting consumers of fake products is reducing the incentive of consumers to buy genuine products when they cannot distinguish between fake items and the genuine product (Yao (2015)).

Private enforcement of property rights can essentially take two forms, namely, policing and policies by their owners. Qian (2014) notes that the luxury house LVMH assigns approximately 60 full-time employees to anti-counterfeiting, working in collaboration with a wide network of outside investigators and a team of lawyers, and that it spent more than 16 million dollars on investigations and legal fees in 2004 alone. In terms of policies, a number of anti-counterfeiting strategies have been recommended by numerous researchers. For instance, Chaudhry and Zimmerman (2009) suggest aggressively cutting prices, providing financial incentives to distributors so they will reject

[3]See T. Bashir: http://brandandcommercial.com/articles/show/brand-building/214/counterfeiting-the-challenge-to-brand-owners-and-manufacturers1.

[4]Interestingly, the law can even specify what to do with the confiscated products. In the US case, the law gives the Customs Service four options regarding the uses of the seized goods at the border, namely: reexportation of the goods, donation to charity, destruction, or turning them to the General Services Administration for relabeling and sale (see Grossman and Shapiro, p.72 (1988a)).

[5]There can be either monetary or non-monetary sanctions. There are other policies that prevent counterfeiting. For instance, a tariff on copying devices may prevent copyright infringement when the copying cost is relatively low and the tariff raises the effective copying cost. The Copyright Board of Canada has the power to impose tariffs on copying devices (subject to the approval of the Supreme Court of Canada).

[6]Pocketing, i.e., rebating fines to the producers of the genuine goods, affects their production decisions. When fines imposed on counterfeiters are pegged to the price of the genuine items, a luxury monopolist can find counterfeiting profitable (in comparison to the case where IPRs are completely enforced) by raising its selling price (Yao (2005)). This result is also obtained by Di Liddo (2018) in the case where the genuine firm can pocket fines not necessarily pegged to the price of its product.

[7]Cox and Collins (2014), which focuses on music and movie piracies in Finland, derives a demand function for pirated products that takes into account the expected cost of punishment.

counterfeits, and educating consumers about the harmful effects of fake goods. Shultz and Saporito (1996) propose ten anti-counterfeiting strategies, among them, advertising as a tool to differentiate real products from phony ones, pricing to influence demand; and finally, involvement in coalitions with organizations that have similar intellectual property right (IPR) interests.

This paper looks at how the entry of a counterfeiter on the market affects the legal firm's pricing and advertising strategies and profits when there is no public nor private enforcement of property rights. The rationale for focusing on price and advertising is straightforward. First, it is probably the high margin, that is, the difference between the price and the (comparatively very low) production cost that makes counterfeiting financially attractive. Second, the high willingness-to-pay by consumers is driven by the brand image or reputation, and this asset is built through advertising, and of course, through other features such as design, quality, etc. Third, public enforcement of property rights is often lax or imperfect and not all legal firms can afford private enforcement policies. In such a setting, fining the consumption of fake products would be especially relevant if counterfeiting were actually detrimental to both the legal firm and the consumers.

To the best of our knowledge, excepting Buratto et al. (2016) and Crettez et al. (2018) there are no papers analyzing brand quality dynamics in the presence of counterfeiting. To be sure, the impact of counterfeiting and piracy on brand reputation (and quality) has already been analyzed—see for instance Banerjee (2013), Qian (2014), Qian et al. (2014), Zhang (2012), and the review by Di Liddo (2017). But in these contributions, the analysis is restricted to a two-period setup (or a static setting). By contrast, the present paper, like Buratto et al. (ibid), and Crettez et al. (ibid), considers a continuous time framework, which allows us to study how the genuine firm's strategic decisions regarding pricing and advertising change with the date of the counterfeiter's arrival and the parameters describing the dynamics of its brand reputation. Moreover, our framework allows us to study the dynamics of brand reputation before as well as after the counterfeiter's entry.[8] We will later highlight the differences between Buratto et al. and Crettez et al's papers and ours. We shall answer the following research questions:

1. How does the counterfeiter's entry affect the legal firm's pricing and advertising decisions?
2. Are there conditions under which the legal firm benefits from counterfeiting?
3. Does the consumer benefit from counterfeiting?

In a nutshell, our results are as follows: First, we obtain that counterfeiting influences pricing and advertising strategies before and after entry occurs. The legal firm decreases its price and advertising investments in the counterfeiting scenarios. This leads to a loss in a long-term brand equity, that is, counterfeiting has a long-lasting effect on the legal firm even when the counterfeiter stops. This result contradicts some findings in the literature, according to which counterfeiting may stimulate innovation or the quality of the genuine good through product differentiation (e.g., Banerjee (2013), Qian (2014), Qian et al. (2014), Zhang et al. (2012)). A common feature of these results is that the legal firm is able to sustainably differentiate the quality of its product from that of the counterfeiters. This, however, possibly overlooks the case where the counterfeiters interact *repeatedly* with the legal firm. In such a case, it makes sense for counterfeiters to react to the differentiation efforts of the legal firm by adapting their own products. By construction, our analysis captures the repeated interactions between the genuine firm and the counterfeiter and illustrates the

[8] Our approach also differs from that of dynamic general equilibrium models, which study innovation in the case where intellectual property rights are poorly protected (see, e.g. Suzuki, (2015)). An important difference between these models and the present paper is that we pay more attention to the brand reputation and to the nature of the imperfect competition between the genuine firm and the counterfeiter.

relevance of a differentiable game approach to counterfeiting.
Second, we show that while under no circumstances will counterfeiting be welcomed by a legal firm, there are indeed circumstances under which the consumer benefits from this illegal trade (the decrease in the price of the genuine good compensates for the decrease in the brand reputation of this good). This result can serve as a rationale for not fining consumers of fake products.

The rest of the paper is organized as follows: In Section 2, we introduce the model and present the two considered scenarios. In Section 3, the optimal strategies and outcomes are determined in the no-counterfeiting scenario, which is our benchmark. In Section 4, we characterize the equilibrium strategies and payoffs in the counterfeiting scenario; and in Section 5, we compare the results of the two scenarios. Section 6 briefly concludes.

2 Model

We consider a planning horizon $[0,T]$, with time t running continuously. The initial date corresponds to the launch of a new product by an established legal manufacturer, player l, and T to the end of the selling season. After T, the product loses its appeal because of, e.g., a change of season for fashion apparel, or the arrival of a new version of software. At an intermediate date $\mathcal{E} \in (0,T]$ a counterfeiter, player c, enters the market and offers a fake product, which performs the same functions as the legal product, e.g., typing a scientific paper in the case of software. Denote by $p_l(t)$ the price of the manufacturer's product at time $t \in [0,T]$ and by $p_c(t)$ the price of the copied product at $t \in [\mathcal{E},T]$.

Denote by $R(t)$ the manufacturer's brand reputation, to which we can also refer as goodwill or brand equity. In the absence of counterfeiting, the demand for the legal firm is given by

$$q_l(t) = \max\left\{0, \tilde{\delta}_l\sqrt{R(t)} - \tilde{\beta}_l p_l(t)\right\}, \quad t \in [0,T],$$

and in the scenario with counterfeiting by

$$q_{l1}(t) = \max\left\{0, \tilde{\delta}_l\sqrt{R(t)} - \tilde{\beta}_l p_{l1}(t)\right\}, \qquad t \in [0,\mathcal{E}), \tag{1}$$

$$q_{l2}(t) = \max\left\{0, \delta_l\sqrt{R(t)} - \beta_l p_{l2}(t) + \gamma p_c(t)\right\}, \quad t \in [\mathcal{E},T], \tag{2}$$

$$q_c(t) = \max\left\{0, \delta_c\sqrt{R(t)} - \beta_c p_c(t) + \gamma p_{l2}(t)\right\}, \quad t \in [\mathcal{E},T], \tag{3}$$

where $\tilde{\delta}_l, \delta_l, \delta_c$ and $\beta_j, j \in \{l,c\}$ are positive parameters and $\gamma \geq 0$ with $\beta_j > \gamma$, that is, the direct-price effect is larger than the cross-price effect.[9] The subscripts 1 and 2 are used to distinguish between the two periods, that is, before and after the counterfeiter's entry.

Remark 1. *The fake product is non-deceptive, meaning that the buyer knows perfectly well that the product is not genuine. To illustrate, think of a consumer purchasing an illegal copy of software on the Internet, or a tourist buying a Lancel bag from a street seller in Paris.*

We make the following comments on the above demand functions:

1. We show in Appendix A that these demand functions are obtained at each date by maximizing the following consumer's utility function:

$$U(q_l, q_c, y) = \sigma_l\sqrt{R}q_l + \sigma_c\sqrt{R}q_c - \frac{\kappa_l q_l^2}{2} - \frac{\kappa_c q_c^2}{2} - \psi q_l q_c + y,$$

[9] To study the interactions between firms in a dynamic setting it is most convenient to use linear demand function (see, *i.e.*, Cellini and Lambertini (2007)).

subject to the budget constraint given by

$$p_l q_l + p_c q_c + y = I,$$

where: q_l (resp. q_c) is the quantity of legal (resp. fake) product; y is a composite good; I the consumer's income; and $\sigma_l, \sigma_c, \psi, \kappa_l$ and κ_c are positive parameters. The derivation of demand functions from utility maximization provides a micro foundation for the specifications in (1)-(3).[10] Assuming that the set of consumers can be represented by a single consumer at each date is probably the simplest setting that allows to the study the welfare effects of counterfeiting.[11]

2. The demands for the genuine product, with and without the presence of a fake good, are structurally different, that is, $\tilde{\delta}_l \neq \delta_l$ and $\tilde{\beta}_l \neq \beta_l$, with $\tilde{\delta}_l > \delta_l$ and $\tilde{\beta}_l < \beta_l$. Put differently, setting $p_c(t) = 0$ in the duopoly market does not yield the demand in the monopoly market.

3. The demand functions have the familiar affine shape, with, however, the additional feature that the market potential is not a given constant but depends positively on the brand reputation. The square root function is to account for marginal decreasing returns in reputation.

4. As expected, each demand is decreasing in own price and increasing in competitor's price.

The manufacturer can increase the brand reputation by investing in advertising. The evolution of the brand's reputation is described by the following linear differential equation:

$$\dot{R}(t) = ka(t) - \sigma R(t), \quad R(0) = R_0 > 0, \tag{4}$$

where $a(t)$ is the advertising effort of the legal producer at time $t, k > 0$ is an efficiency parameter, and σ is the decay rate.[12] Following a substantial literature in both optimal control and differential games (see, e.g., the book by Jørgensen and Zaccour (2004) and the survey by Huang et al. (2012)), we suppose that the advertising cost is convex increasing and given by the quadratic function

$$C_l(a) = \frac{\omega}{2} a^2(t),$$

where ω is a positive parameter. Further, we suppose that the marginal production costs of both players are constant and we set them equal to zero. This is not a severe assumption as adding costs will have only a quantitative impact on the results without altering the qualitative insights.

[10] A similar approach can be founded in Lai and Chang (2012).

[11] By contrast with the vertical product differentiation model used in several papers in the literature (see *inter alia* Banerjee (2003), Di Liddo (2018), Zhang *et al.* (2012)), in our approach the "representative consumer" buys both the genuine and the fake product. It is possible, however, to give an alternative derivation of the linear demand functions and the quadratic objective under which some consumers do not buy any product, some consumers buy the two kinds of products and some other consumers buy one kind of good only (see Martin (2009)). A general discussion of demand functions can be found in Huang et al. (2013) (see especially subsection 2.2). The fact that some consumers buy both genuine goods and counterfeits, is documented *e.g.*, in Kapferer and Michaut (2014) or Stöttinger and Penz (2015). Thus, it seems acceptable to assume that the *representative* consumer buys both the genuine good and the counterfeit.

[12] We do not take into account word-of-mouth communication effects (see remark 2 below). For instance Givon et al. (1995) studies on an innovation diffusion model where pirates play an important role in converting potential users into users and even buyers of the software (they show that this effect was at work in the diffusion on spreadsheets and word processors during the 1990's in the United Kingdom). Peres et al. (2010) reviews the literature on innovation diffusion that, in addition to word-of-mouth communications, incorporate network externalities and social signals.

The legal producer maximizes its stream of profit over the planning horizon.[13] Its optimization problem is defined as follows:

$$\begin{aligned}\max_{p_{l1}(t),\,p_{l2}(t),\,a_1(t),\,a_2(t)} \Pi_l \;=\; & \left[\int_0^{\mathcal{E}} \left(p_{l1}(t)\left(\tilde{\delta}_l\sqrt{R(t)}-\tilde{\beta}_l p_{l1}(t)\right)-\frac{\omega}{2}a_1^2(t)\right)dt\,+\right.\\ & \left.\int_{\mathcal{E}}^{T}\left(p_{l2}(t)\left(\delta_l\sqrt{R(t)}-\beta_l p_{l2}(t)+\gamma p_c(t)\right)-\frac{\omega}{2}a_2^2(t)\right)dt\right] \qquad (5)\\ & +S(R(T)),\\ & \text{subject to (4)},\end{aligned}$$

where $S(R(T))$ is the salvage value of the brand at T, which captures the potential future payoffs that the manufacturer can derive from other products having the same brand name. We suppose that the salvage value can be well approximated by a linear function, that is, $S(R(T)) = sR(T)$. Clearly, this is a simplifying assumption and there is no conceptual difficulty in adopting a non-linear salvage value. However, retaining a non-linear function would come at the cost of complicating considerably the computations, without adding any qualitative gain in terms of our research questions.

The counterfeiter's optimization problem is given by

$$\max_{p_c(t)} \Pi_c = \int_{\mathcal{E}}^{T} p_c(t)\left(\delta_c\sqrt{R(t)}-\beta_c p_c(t)+\gamma p_{l2}(t)\right)dt, \qquad t\in[\mathcal{E},T]. \qquad (6)$$

As the counterfeiter's decision does not affect the dynamics, its optimization problem is equivalent to solving the following static one:

$$\max_{p_c(t)} \pi_c = \max_{p_c(t)} p_c(t)\left(\delta_c\sqrt{R(t)}-\beta_c p_c(t)+\gamma p_{l2}(t)\right), \qquad \forall t\in[\mathcal{E},T].$$

To address our research questions, we shall characterize and compare the solutions in the following two scenarios:

No Counterfeiting. The product cannot be copied and the only demand is legal. The manufacturer then solves the following optimal control problem:

$$\begin{aligned}\max_{p_l(t),\,a(t)} \Pi_l^{\mathcal{N}} \;=\; & \max_{p_l(t),\,a(t)} \int_0^{T}\left(p_l(t)\left(\tilde{\delta}_l\sqrt{R(t)}-\tilde{\beta}_l p_l(t)\right)-\frac{\omega}{2}a^2(t)\right)dt + sR(T), \qquad (7)\\ & \dot{R}(t) = ka(t)-\sigma R(t),\; R(0)=R_0,\end{aligned}$$

where the superscript $\mathcal{N}$ refers to no counterfeiting. This is our benchmark scenario, which corresponds either to a situation where the product life cycle is so short that illegal producers do not have enough time to enter the market or to a case where the institutions acting against counterfeiting are highly efficient.

Counterfeiting: Entry of the illegal producer occurs at time $\mathcal{E} \le T$. The counterfeiter and the manufacturer play a finite-horizon differential game during the time interval $[\mathcal{E}, T]$. The manufacturer maximizes

$$\begin{aligned}\Pi_{l2}^{\mathcal{C}} \;=\; & \int_{\mathcal{E}}^{T}\left(p_{l2}(t)\left(\delta_l\sqrt{R(t)}-\beta_l p_{l2}(t)+\gamma p_c(t)\right)-\frac{\omega}{2}a_2^2(t)\right)dt + sR(T),\\ & \text{subject to (4) and } R(\mathcal{E}),\end{aligned}$$

[13] As the producer's problem is defined on a short horizon, we do not include a discount factor in the objective functional.

and the counterfeiter maximizes (6). A Nash equilibrium will be sought and the equilibrium state and strategy will be superscripted with $\mathcal{C}$ (for counterfeiting). To this Nash equilibrium we will associate a value function W_l to the manufacturer problem over the horizon $[\mathcal{E}, T]$. Next, we solve the following maximization problem over the horizon $[0, \mathcal{E}]$:

$$\Pi_{l1}^{\mathcal{C}} = \int_0^{\mathcal{E}} \left(p_{l1}(t) \left(\tilde{\delta}_l \sqrt{R(t)} - \tilde{\beta}_l p_{l1}(t) \right) - \frac{\omega}{2} a_1^2(t) \right) dt + W_l(\mathcal{E}, R(\mathcal{E})).$$

By comparing the outcomes of the two scenarios, we will be able to measure the impact of counterfeiting on the manufacturer's profit and on the consumer. We henceforth omit the time argument when no ambiguity may arise.

Remark 2. *The closest papers to ours are Buratto et al. (2016) and Crettez et al. (2018), and we wish to point out the following important differences between these two contributions. With respect to Buratto et al. (ibid): (i) The demand functions are different. In particular, in Buratto et al. (2016) the demand functions are structurally the same with and without counterfeiting. (ii) The demand functions adopted here are micro founded. (iii) The dynamics are different in two respects. First, in Buratto et al., the illegal firm also advertises the product, which increases the reputation of the legal brand. Here, the counterfeiter does not engage in such activities, which is probably more in line with what is observed empirically. Second, our dynamics include a decay rate to account for consumer forgetting. (iv) The strategies in the counterfeiting scenario are feedback, which is conceptually more attractive than open-loop strategies. (v) And lastly, here, all results are analytical. With respect to Crettez et al. (ibid): (i) The demand function is slightly different. (ii) the present paper deals with counterfeiting whereas Crettez et al. (ibid) also consider imitation more broadly conceived (e.g., knockoffs). (iii) The present paper addresses the effect of counterfeiting on consumer's welfare, whereas Crettez et al focuse on the comparaison of before and after entry strategies. (iv) Crettez et al. assume that the evolution of the incumbent's brand reputation also depends on the entrant's sales during the duopoly period. Here, as was mentionned above, we do not consider dilution or promotion effects. This is because we are interested in the welfare effects of counterfeiting. For instance, it is clear that in the presence of dilution or promotion effects, counterfeiting can be either welfare decreasing or welfare increasing. To better understand the welfare effects of counterfeiting, we concentrate on the case where counterfeiting has neutral effects on brand reputation.* [14]

3 No counterfeiting

In this section, we characterize the optimal solution in the absence of counterfeiting and derive some properties.

Denote by $V_l(t, R(t)) : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$ the value function of the legal firm.[15] The following proposition provides the optimal solution.

Proposition 1. *In the absence of counterfeiting, the optimal pricing and advertising policies are*

[14] According to Qian (2014) "counterfeits have both advertising effects for a brand and substitution effects for authentic products, additionally the effects linger for some years. The advertising effect dominates the substitution effect for high-end authentic-product sales, and the substitution effect the advertising effect for low-end product sales. Our model refers to the case where these two effects are small. "

[15] As a reminder, the value function gives the optimal payoff that can be obtained from $(t, R(t))$, assuming that optimal policies are followed.

given by

$$p_l^{\mathcal{N}}(t, R(t)) = p_l^{\mathcal{N}}(R(t)) = \frac{\tilde{\delta}_l}{2\tilde{\beta}_l}\sqrt{R(t)}, \tag{8}$$

$$a^{\mathcal{N}}(t, R(t)) = a^{\mathcal{N}}(t) = \frac{k}{4\sigma\tilde{\beta}_l\omega}(\tilde{\delta}_l^2 + (4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2)e^{\sigma(t-T)}), \tag{9}$$

and the brand's reputation trajectory by

$$R^{\mathcal{N}}(t) = R_0 e^{-\sigma t} + \frac{k^2}{\omega}\frac{4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2}{8\sigma^2\tilde{\beta}_l}\left(e^{\sigma(t-T)} - e^{-\sigma(T+t)}\right) + \frac{k^2}{\omega}\frac{\tilde{\delta}_l^2}{4\sigma^2\tilde{\beta}_l}\left(1 - e^{-\sigma t}\right). \tag{10}$$

Proof. See Appendix B. ■

The above proposition calls for the following remarks. First, it is easy to see that the advertising level is strictly positive at each instant of time, which, along with the assumption that $R_0 > 0$, implies that $R^{\mathcal{N}}(t)$ is strictly positive for all $t \in [0, T]$. Consequently, the price is also strictly positive, and hence, the solution is indeed interior. Second, from the proof in Appendix B, we see that the optimal advertising effort is dictated by the familiar rule of marginal cost (given by wa) equals marginal revenue, which is measured by $k\frac{\partial V_l}{\partial R}$, that is, the marginal efficiency of advertising in raising reputation times the shadow price of the brand's reputation, measured by the derivative of the value function with respect to reputation. Third, the firm adopts a pricing policy that follows reputation: the higher the reputation, the higher the price. This is observed empirically and is due to the fact that the market potential is increasing in the brand's reputation. Finally, the strategies vary as follows with the different parameter values:

	$\tilde{\delta}_l$	$\tilde{\beta}_l$	k	σ	ω	s
$p_l^{\mathcal{N}}$	$+$	$-$				
$a^{\mathcal{N}}$	$+$	$-$	$+$	$-$	$-$	$+$

We note that the price only depends on the demand function parameters, namely, $\tilde{\delta}_l$ and $\tilde{\beta}_l$, and is increasing in market size parameter $\tilde{\delta}_l$ and decreasing in consumer's sensitivity to price $\tilde{\beta}_l$. Advertising expenditures increase with $\tilde{\delta}_l$, with advertising efficiency k, and with the marginal salvage value of reputation s, and they decrease with advertising cost ω, with the decay rate σ and the consumer's sensitivity to price $\tilde{\beta}_l$. These results are fairly intuitive.

Proposition 2. *The optimal advertising policy is monotonically decreasing over time if, and only if,* $s \leq \frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}$.

Proof. It suffices to compute

$$\dot{a}^{\mathcal{N}}(t) = \frac{ke^{\sigma(t-T)}}{4\tilde{\beta}_l\omega}\left(-\tilde{\delta}_l^2 + 4\sigma\tilde{\beta}_l s\right),$$

to get the result. ■

The intuition behind this result is as follows: if the marginal value of the brand reputation at the end of the planning horizon is sufficiently low, then the firm should start by advertising at a relatively high level and decrease it over time. Early investments in advertising allow the firm to benefit from a high reputation for a longer period of time. In particular, if the salvage value is zero, then the condition in the above proposition will always be satisfied.

The evolution of the price over time follows the evolution of reputation. Indeed,

$$\dot{p}_l^{\mathcal{N}}\left(R\left(t\right)\right)=\frac{\tilde{\delta}_l\dot{R}\left(t\right)}{4\tilde{\beta}_l\sqrt{R\left(t\right)}}.$$

It can be easily verified that

$$\dot{R}^{\mathcal{N}}\left(t\right)\geq 0\Leftrightarrow s\geq\frac{8\sigma^2\tilde{\beta}_l\omega R_0e^{-\sigma t}-k^2\tilde{\delta}_l^2\left(2e^{-\sigma t}-e^{\sigma(t-T)}-e^{-\sigma(T+t)}\right)}{4\sigma\tilde{\beta}_lk^2\left(e^{\sigma(t-T)}+e^{-\sigma(T+t)}\right)}.$$

The above inequality, which involves all the model's parameters, states that, for the reputation to be increasing over time, the marginal salvage value must be high enough. Note that if the brand enjoys a large initial reputation value R_0 or if the advertising cost ω is high, then the condition becomes harder to satisfy. On the other hand, the condition is easier to satisfy when the advertising efficiency k is high.

It is shown in Appendix B that the value function is linear and given by

$$V_l\left(t,R\left(t\right)\right)=z\left(t\right)R\left(t\right)+y\left(t\right),$$

where

$$\begin{aligned}
z\left(t\right) &= \frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}+\frac{4\sigma\tilde{\beta}_ls-\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{\sigma(t-T)},\\
y\left(t\right) &= \frac{k^2}{16\sigma^3\omega\tilde{\beta}_l^2}\left(\frac{\tilde{\sigma\delta}_l^4}{2}(T-t)+\tilde{\delta}_l^2(4\sigma\tilde{\beta}_ls-\tilde{\delta}_l^2)(1-e^{\sigma(t-T)})+\frac{(4\sigma\tilde{\beta}_ls-\tilde{\delta}_l^2)^2}{4}(1-e^{2\sigma(t-T)})\right).
\end{aligned}$$

Proposition 3. *The coefficients $z\left(t\right)$ and $y(t)$ are nonnegative for all $t\in[0,T]$.*

Proof. The coefficient $z\left(t\right)$ is clearly strictly positive for all $t\in[0,T]$. To show that $y\left(t\right)\geq 0$ for all t, it suffices to note that its derivative over time

$$\dot{y}(t)=-\frac{k^2}{32\sigma^2\omega\tilde{\beta}_l^2}\left(\tilde{\delta}_l^2+\left(4\sigma\tilde{\beta}_ls-\tilde{\delta}_l^2\right)e^{\sigma(t-T)}\right)^2,$$

is strictly negative and that $y(T)=0$. ■

The implications of the above proposition are as follows: (i) the value function is strictly increasing in reputation; and (ii) even if the firm is new, that is, if its reputation at initial instant of time is zero, it can still secure a nonnegative profit.

In the absence of counterfeiting, the legal firm's payoff over the whole planning horizon is given by

$$\begin{aligned}
V_l\left(0,R_0\right) &= \left(\frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}+\frac{4\sigma\tilde{\beta}_ls-\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{-\sigma T}\right)R_0+ \qquad (11)\\
&\quad \frac{k^2}{16\sigma^3\omega\tilde{\beta}_l^2}\left(\frac{\tilde{\sigma\delta}_l^4}{2}T+\tilde{\delta}_l^2(4\sigma\tilde{\beta}_ls-\tilde{\delta}_l^2)(1-e^{-\sigma T})+\frac{(4\sigma\tilde{\beta}_ls-\tilde{\delta}_l^2)^2}{4}(1-e^{-2\sigma T})\right).
\end{aligned}$$

This value will be compared to the total profit that the legal firm obtains in the presence of counterfeiting. Finally, the reputation of the legal firm by the terminal planning date is

$$R^{\mathcal{N}}\left(T\right)=R_0e^{-\sigma T}+\frac{k^2}{\omega}\frac{4\sigma\tilde{\beta}_ls-\tilde{\delta}_l^2}{8\sigma^2\tilde{\beta}_l}\left(1-e^{-2\sigma T)}\right)+\frac{k^2}{\omega}\frac{\tilde{\delta}_l^2}{4\sigma^2\tilde{\beta}_l}\left(1-e^{-\sigma T}\right).$$

4 Counterfeiting

The manufacturer's optimization problem is in two stages: between 0 and $\mathcal{E}$, it is a dynamic optimization problem with the solution being (qualitatively) similar to the problem without counterfeiting; between $\mathcal{E}$ and T, the two agents play a noncooperative game and a Nash equilibrium is sought. To obtain a subgame-perfect Nash equilibrium (SPNE) in the two-stage problem, we first solve the second stage with $R^{\mathcal{C}}(\mathcal{E})$ as the initial value of the brand's reputation.

4.1 The duopoly equilibrium

In this second-stage game, the counterfeiter solves the following static optimization problem:

$$\max_{p_c(t)} p_c\left(t\right)\left(\delta_c\sqrt{R\left(t\right)}-\beta_c p_c\left(t\right)+\gamma p_{l2}\left(t\right)\right), \quad \forall t\in\left[\mathcal{E},T\right],$$

while the legal firm solves

$$\begin{aligned}\Pi_{l2}^{\mathcal{C}}= \quad & \max_{p_{l2}(t),\,a_2(t)}\int_{\mathcal{E}}^{T}\left(p_{l2}\left(t\right)\left(\delta_l\sqrt{R\left(t\right)}-\beta_l p_{l2}\left(t\right)+\gamma p_c\left(t\right)\right)-\frac{\omega}{2}a_2^2\left(t\right)\right)dt\\ & +sR\left(T\right),\\ & \text{subject to (4) and } R^{\mathcal{C}}(\mathcal{E}).\end{aligned}$$

Denote by φ_i the strategy of player $i=l,c$. We assume that each player implements a feedback strategy that selects the control action according to the rule $u_i(t)=\varphi_i(t,R(t))$, where

$$u_l(t)=\left(p_{l2}\left(t\right),a_2\left(t\right)\right)\in\mathbb{R}_+^2 \quad \text{and} \quad u_c(t)=\left(p_c\left(t\right)\right)\in\mathbb{R}_+.$$

This means that firm $i=l,c$ observes the state $(t,R(t))$ of the system and then chooses its action as prescribed by the decision rule φ_i.

Definition 1. *A pair (φ_l,φ_c) of functions $\varphi_i:[\mathcal{E},T]\times\mathbb{R}_+\longrightarrow R^{m_i}$, $i=l,c$, is a feedback-Nash equilibrium if*

$$\begin{aligned}\Pi_{l2}^{\mathcal{C}}(\varphi_l,\varphi_c) &\geq \Pi_{l2}^{\mathcal{C}}(u_1,\varphi_c), \quad \forall u_l\in\mathbb{R}_+^2,\\ \Pi_c(\varphi_l,\varphi_c) &\geq \Pi_c(\varphi_l,u_c), \quad \forall u_c\in\mathbb{R}_+.\end{aligned}$$

To characterize a feedback-Nash equilibrium, denote by $W_l\left(t,R\left(t\right)\right):[\mathcal{E},T]\times\mathbb{R}_+\to\mathbb{R}$ the legal firm's value function. The following proposition gives the equilibrium solution of the duopoly game.[16]

Proposition 4. *Assuming that the counterfeiter enters the market at date $\mathcal{E}\leq T$, then the feedback-Nash pricing and advertising strategies are given by*

$$p_{l2}^{\mathcal{C}}\left(t,R\left(t\right)\right) = p_{l2}^{\mathcal{C}}\left(R\left(t\right)\right)=\frac{2\beta_c\delta_l+\delta_c\gamma}{4\beta_c\beta_l-\gamma^2}\sqrt{R\left(t\right)}, \tag{12}$$

$$p_c^{\mathcal{C}}\left(t,R\left(t\right)\right) = p_c^{\mathcal{C}}\left(R\left(t\right)\right)=\frac{2\beta_l\delta_c+\delta_l\gamma}{4\beta_c\beta_l-\gamma^2}\sqrt{R\left(t\right)}, \tag{13}$$

$$a_2^{\mathcal{C}}\left(t,R\left(t\right)\right) = a_2^{\mathcal{C}}\left(t\right)=\frac{k}{\omega}\left(\Gamma+\left(s-\Gamma\right)e^{-\sigma(T-t)}\right), \tag{14}$$

[16] See Haurie et al. (2012) for details on determining a feedback-Nash equilibrium in differential games.

where

$$\Gamma = \frac{\beta_l}{\sigma}\left(\frac{2\beta_c\delta_l + \delta_c\gamma}{4\beta_c\beta_l - \gamma^2}\right)^2 > 0.$$

The reputation trajectory is given by

$$R_2^{\mathcal{C}}(t) = R(\mathcal{E})\, e^{-\sigma(t-\mathcal{E})} + \frac{k^2\Gamma}{\sigma\omega}\left(1 - e^{-\sigma(t-\mathcal{E})}\right) + \frac{k^2(s-\Gamma)}{2\sigma\omega}\left(1 - e^{-2\sigma(t-\mathcal{E})}\right) e^{-\sigma(T-t)}. \tag{15}$$

Proof. See Appendix B ■

The results in the above proposition deserve the following comments. First, by the same arguments provided after Proposition 1, it is easy to verify that the equilibrium solution is indeed interior.

Second, the pricing policies are increasing in the legal firm's reputation and are invariant over time, that is, the time dependency is only through the reputation value. Interestingly, the ratio of the two prices is constant, that is, independent of the state R and of time. Indeed,

$$\frac{p_{l2}^{\mathcal{C}}(R(t))}{p_c^{\mathcal{C}}(R(t))} = \frac{2\beta_c\delta_l + \delta_c\gamma}{2\beta_l\delta_c + \delta_l\gamma}.$$

It is shown in Appendix A that the assumptions made on the utility function imply that the above ratio is always larger than one, which means that the price of the genuine product is always higher than the price of the fake one. Clearly, this is in line with what is observed in the market.

Third, the advertising policy is again determined by equating the marginal cost ωa to the marginal revenue given by $k\frac{\partial W_l}{\partial R}$, and is monotonically decreasing over time if $s \leq \Gamma$. Further, because the advertising policy is independent of $R(t)$ and of the counterfeiter's entry date, it may appear at first glance that the legal firm's advertising policy is not affected by entry. This is clearly not the case since advertising depends on Γ, which involves the counterfeiter's parameters, i.e., β_c and γ.

Finally, we show in Appendix B that the value function of the second-stage problem is linear and given by

$$W_l(t, R(t)) = x(t)\, R(t) + v(t),$$

where

$$x(t) = \Gamma + (s-\Gamma)\, e^{-\sigma(T-t)}, \tag{16}$$

$$v(t) = \frac{k^2}{2\omega}\left(\Gamma^2(T-t) + \frac{(s-\Gamma)^2}{2\sigma}(1 - e^{2\sigma(t-T)}) + \frac{2\Gamma(s-\Gamma)}{\sigma}(1 - e^{\sigma(t-T)})\right). \tag{17}$$

4.2 The first-stage optimal solution

Inserting the equilibrium strategies $p_c^{\mathcal{C}}, p_l^{\mathcal{C}}$ and $a^{\mathcal{C}}$ in the legal firm's second-stage profit ultimately yields a function that depends on the reputation value at counterfeiter's entry time $\mathcal{E}$, which we denote by $W_l(\mathcal{E}, R(\mathcal{E}))$. This function is the salvage value in the first-stage optimization problem of the legal firm, which is,

$$\max_{p_{l1}(t),\, a_1(t)} \Pi_l^{\mathcal{C}} = \int_0^{\mathcal{E}} \left(p_{l1}(t)\left(\tilde{\delta}_l\sqrt{R(t)} - \tilde{\beta}_l p_{l1}(t)\right) - \frac{\omega}{2}a_1^2(t)\right) dt + W_l(\mathcal{E}, R(\mathcal{E})).$$

subject to the reputation dynamics

$$\dot{R}(t) = k a_1(t) - \sigma R(t),\ R(0) = R_0.$$

Observe that this optimization problem is very similar to the one solved in the scenario without counterfeiting. The main difference is the duration of the planning horizon and of the transversality condition. Adapting the proof of Proposition 1, we get the following optimal solution on $[0, \mathcal{E}]$:

Proposition 5. *The optimal pricing and advertising policies are given by*

$$\begin{aligned} p_{l1}^{\mathcal{C}}\left(t, R_1\left(t\right)\right) &= p_{l1}^{\mathcal{C}}\left(R_1\left(t\right)\right) = \frac{\tilde{\delta}_l}{2\tilde{\beta}_l}\sqrt{R_1\left(t\right)}, \\ a_1^{\mathcal{C}}\left(t, R_1\left(t\right)\right) &= a_1^{\mathcal{C}}\left(t\right) = \frac{k}{4\sigma\tilde{\beta}_l\omega}\left(\tilde{\delta}_l^2\left(1-e^{\sigma(t-\mathcal{E})}\right) + 4\sigma\tilde{\beta}_l(\Gamma + (s-\Gamma)\,e^{-\sigma(T-\mathcal{E})})e^{\sigma(t-\mathcal{E})}\right), \end{aligned}$$

and the reputation stock by

$$R_1^{\mathcal{C}}\left(t\right) = R_0 e^{-\sigma t} + \frac{k^2}{\omega}\frac{4\sigma\tilde{\beta}_l x(\mathcal{E}) - \tilde{\delta}_l^2}{8\sigma^2\tilde{\beta}_l}\left(e^{\sigma(t-\mathcal{E})} - e^{-\sigma(\mathcal{E}+t)}\right) + \frac{k^2}{\omega}\frac{\tilde{\delta}_l^2}{4\sigma^2\tilde{\beta}_l}\left(1-e^{-\sigma t}\right).$$

Proof. See Appendix B. ■

The same comments made after Proposition 1 remain valid, qualitatively speaking, and therefore there is no need to repeat them. Substituting for $x(\mathcal{E})$ in $R_1^{\mathcal{C}}\left(t\right)$ we obtain

$$R_1^{\mathcal{C}}\left(t\right) = R_0 e^{-\sigma t} + \frac{k^2}{\omega}\frac{4\sigma\tilde{\beta}_l\left(\Gamma + (s-\Gamma)\,e^{-\sigma(T-\mathcal{E})}\right) - \tilde{\delta}_l^2}{8\sigma^2\tilde{\beta}_l}\left(e^{\sigma(t-\mathcal{E})} - e^{-\sigma(\mathcal{E}+t)}\right) + \frac{k^2}{\omega}\frac{\tilde{\delta}_l^2}{4\sigma^2\tilde{\beta}_l}\left(1-e^{-\sigma t}\right), \tag{18}$$

and in particular, the following value for reputation at the counterfeiter's entry date:

$$R_1^{\mathcal{C}}\left(\mathcal{E}\right) = R_0 e^{-\sigma\mathcal{E}} + \frac{k^2}{8\sigma^2\tilde{\beta}_l\omega}\left(\left(4\sigma\tilde{\beta}_l\left(\Gamma + (s-\Gamma)\,e^{-\sigma(T-\mathcal{E})}\right) - \tilde{\delta}_l^2\right)\left(1-e^{-2\sigma\mathcal{E}}\right) + 2\tilde{\delta}_l^2\left(1-e^{-\sigma\mathcal{E}}\right)\right).$$

The reputation by the end of the planning horizon is

$$R_2^{\mathcal{C}}\left(T\right) = R_1^{\mathcal{C}}\left(\mathcal{E}\right)e^{-\sigma(T-\mathcal{E})} + \frac{k^2\Gamma}{\sigma\omega}\left(1-e^{-\sigma(T-\mathcal{E})}\right) + \frac{k^2\left(s-\Gamma\right)}{2\sigma\omega}\left(1-e^{-2\sigma(T-\mathcal{E})}\right).$$

It is shown in Appendix B that the first-stage value function $Z_l\left(t, R\left(t\right)\right)$ is linear, that is,

$$Z_l\left(t, R\left(t\right)\right) = m\left(t\right)R\left(t\right) + n\left(t\right),$$

where the coefficients $m\left(t\right)$ and $n\left(t\right)$ are given by

$$\begin{aligned} m\left(t\right) &= \frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l} + \frac{4\sigma\tilde{\beta}_l x(\mathcal{E}) - \tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{\sigma(t-\mathcal{E})}, \\ n\left(t\right) &= -\frac{k^2}{4\sigma\omega}\left(\frac{\tilde{\delta}_l^4}{8\sigma\tilde{\beta}_l^2}t + \tilde{\delta}_l^2\left(\frac{4\sigma\tilde{\beta}_l x(\mathcal{E}) - \tilde{\delta}_l^2}{4\sigma^2\tilde{\beta}_l^2}e^{-\sigma\mathcal{E}}\right)e^{\sigma t} + \left(\frac{4\sigma\tilde{\beta}_l x(\mathcal{E}) - \tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{-\sigma\mathcal{E}}\right)^2 e^{2\sigma t}\right) \\ &+ \frac{k^2}{16\sigma^3\omega\tilde{\beta}_l^2}\left(\frac{\sigma\tilde{\delta}_l^4}{2}\mathcal{E} + \tilde{\delta}_l^2(4\sigma\tilde{\beta}_l x(\mathcal{E}) - \tilde{\delta}_l^2) + \frac{(4\sigma\tilde{\beta}_l x(\mathcal{E}) - \tilde{\delta}_l^2)^2}{4}\right) + v\left(\mathcal{E}\right) \end{aligned}$$

Note that the above coefficients involve $x\left(\mathcal{E}\right)$ and $v\left(\mathcal{E}\right)$, that is, the coefficients of the second-stage value function evaluated at entry time $\mathcal{E}$. As alluded to it earlier, $W_l\left(\mathcal{E}, R\left(\mathcal{E}\right)\right)$ plays the role of a salvage value in the first-stage optimization problem of the legal firm. Substituting for $x(\mathcal{E})$

and $v(\mathcal{E})$, and next for $m(t)$ and $n(t)$ in $Z_l(t, R(t))$, we obtain the value function for the legal firm on $[0, \mathcal{E}]$, that is,

$$\begin{aligned} Z_l(t, R(t)) &= \frac{1}{4\sigma\tilde{\beta}_l}\left(\tilde{\delta}_l^2 + \Lambda e^{\sigma(t-\mathcal{E})}\right) R(t) + \frac{k^2\tilde{\delta}_l^4(\mathcal{E}-t)}{32\sigma^2\omega\tilde{\beta}_l^2} \\ &+ \frac{k^2\Lambda\left(1-e^{\sigma(t-\mathcal{E})}\right)}{64\sigma^3\omega\tilde{\beta}_l^2}\left(4\tilde{\delta}_l^2 + \Lambda\left(1+e^{\sigma(t-\mathcal{E})}\right)\right) \\ &+ \frac{k^2}{2\omega}\left(\Gamma^2(T-\mathcal{E}) + \frac{(s-\Gamma)^2}{2\sigma}(1-e^{2\sigma(\mathcal{E}-T)}) + \frac{2\Gamma(s-\Gamma)}{\sigma}(1-e^{\sigma(\mathcal{E}-T)})\right), \end{aligned}$$

where

$$\Lambda = 4\sigma\tilde{\beta}_l\left(\Gamma + (s-\Gamma)e^{-\sigma(T-\mathcal{E})}\right) - \tilde{\delta}_l^2.$$

To obtain the total profit that the legal firm gets in the game with counterfeiting, it suffices to evaluate the above value function at $(0, R(0))$, which yields

$$\begin{aligned} Z_l(0, R(0)) &= \frac{1}{4\sigma\tilde{\beta}_l}\left(\tilde{\delta}_l^2 + \Lambda e^{-\sigma\mathcal{E})}\right) R_0 + \frac{k^2\tilde{\delta}_l^4\mathcal{E}}{32\sigma^2\omega\tilde{\beta}_l^2} \\ &+ \frac{k^2\Lambda\left(1-e^{-\sigma\mathcal{E}}\right)}{64\sigma^3\omega\tilde{\beta}_l^2}\left(4\tilde{\delta}_l^2 + \Lambda\left(1+e^{-\sigma\mathcal{E}}\right)\right) \\ &+ \frac{k^2}{2\omega}\left(\Gamma^2(T-\mathcal{E}) + \frac{(s-\Gamma)^2}{2\sigma}(1-e^{2\sigma(\mathcal{E}-T)}) + \frac{2\Gamma(s-\Gamma)}{\sigma}(1-e^{\sigma(\mathcal{E}-T)})\right). \quad (19) \end{aligned}$$

Before comparing the results of the two scenarios, it is of particular interest to look at is the impact of the counterfeiter's entry date on the legal firm's pricing and advertising policies and on the reputation of the brand. As we shall see, this impact hinges on the sign of the difference between the instantaneous (static) revenue of the legal firm without counterfeiting (which we denote by $r_l^{\mathcal{N}}(t)$) and its revenue with counterfeiting (denoted $r_l^{\mathcal{C}}(t)$) for any given reputation level $R(t)$. Substituting for $p_l^{\mathcal{N}}(t)$ from (8) and for $p_{l2}^{\mathcal{C}}(t)$ and $p_c^{\mathcal{C}}(t)$ from (12) and (13) in the relevant revenue functions, we get

$$\begin{aligned} r_l^{\mathcal{N}}(t) &= p_l^{\mathcal{N}}(t)\left(\tilde{\delta}_l\sqrt{R(t)} - \tilde{\beta}_l p_l^{\mathcal{N}}(t)\right) = \frac{\tilde{\delta}_l^2}{4\tilde{\beta}_l}R(t), \\ r_l^{\mathcal{C}}(t) &= \frac{\beta_l(2\beta_c\delta_l + \delta_c\gamma)^2}{(4\beta_c\beta_l - \gamma^2)^2}R(t). \end{aligned}$$

We have the following result.

Lemma 1. *For any given reputation level $R(t)$, the revenue of the legal firm without counterfeiting $r_l^{\mathcal{N}}(t)$) is higher than its revenue with counterfeiting (denoted $r_l^{\mathcal{C}}(t)$). More formally, the following inequality holds true:*

$$\Delta = \frac{\tilde{\delta}_l^2}{4\tilde{\beta}_l} - \beta_l\left(\frac{2\beta_c\delta_l + \delta_c\gamma}{4\beta_c\beta_l - \gamma^2}\right)^2 > 0. \quad (20)$$

Proof. See Appendix B. ■

The proof of the above lemma relies on the general result that in imperfect competition, firms realize higher profits when they compete in quantities à la Cournot than in prices à la Bertrand. This result also strongly depends on the micro-foundations for the demand functions.

Noting that Δ can also be written as

$$\Delta = \frac{1}{4\tilde{\beta}_l}\left(\tilde{\delta}_l^2 - 4\sigma\tilde{\beta}_l\Gamma\right),$$

the effect of the counterfeiter's entry date on the legal firm's pricing and advertising policies and on the reputation of the brand is given in the following result.

Proposition 6. *On $[0,\mathcal{E}]$, the legal firm's advertising, pricing, and reputation are increasing in the counterfeiter's entry date $\mathcal{E}$.*

Proof. It suffices to compute the derivatives

$$\begin{aligned}
\frac{\partial a_1^{\mathcal{C}}(t)}{\partial \mathcal{E}} &= \frac{k}{\omega}\Delta e^{\sigma(t-\mathcal{E})}, \\
\frac{\partial R_1^{\mathcal{C}}(t)}{\partial \mathcal{E}} &= \frac{k^2}{2\sigma\omega}\left(e^{\sigma(t-\mathcal{E})} - e^{-\sigma(\mathcal{E}+t)}\right)\Delta, \\
\frac{\partial p_{l1}^{\mathcal{C}}(t)}{\partial \mathcal{E}} &= \frac{\tilde{\delta}_l}{4\tilde{\beta}_l\sqrt{R_1(t)}}\frac{\partial R_1^{\mathcal{C}}(t)}{\partial \mathcal{E}},
\end{aligned}$$

and to use Lemma 1 to get the result. ■

Intuitively, one would expect the price to be increasing in $\mathcal{E}$, as the need to face price competition is less urgent for the legal firm when the entry date is later. Further, during the monopoly period $[0,\mathcal{E}]$, the legal firm is the only beneficiary from advertising investment in reputation, and therefore, the later is the counterfeiter's entry date, the higher is the incentive to invest in advertising to raise the value of the (private good) reputation.

Remark 3. *During the duopoly period $[\mathcal{E},T]$, the advertising, reputation and pricing trajectories vary as follows in terms of entry date $\mathcal{E}$:*

$$\begin{aligned}
\frac{\partial a_2^{\mathcal{C}}(t)}{\partial \mathcal{E}} &= 0, \\
\frac{\partial R_2^{\mathcal{C}}(t)}{\partial \mathcal{E}} &= \frac{k^2}{2\omega}e^{-\sigma(t-\mathcal{E})}\left(2e^{-\sigma(T-\mathcal{E})}s + \Gamma e^{-2\sigma\mathcal{E}}\left(1-2e^{-\sigma(T+\mathcal{E})}\right) + \frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l} - \Gamma\right) > 0, \\
\frac{\partial p_{l2}^{\mathcal{C}}(t)}{\partial \mathcal{E}} &= \frac{2\beta_c\delta_l + \delta_c\gamma}{4\beta_c\beta_l - \gamma^2}\frac{1}{2\sqrt{R(t)}}\frac{\partial R_2^{\mathcal{C}}(t)}{\partial \mathcal{E}} > 0.
\end{aligned}$$

The reputation and the counterfeiter's price are increasing with respect to the date of entry $\mathcal{E}$. As shown above, the later the date of entry, the higher the values of advertising and reputation *before* entry. Since reputation after $\mathcal{E}$ depends on the level achieved at this date, the later the date of entry, the higher the level of reputation *after* entry. And since the legal firm's price increases with its reputation, the later the entry date, the higher is this price. Observe also that advertising does not depend on the date of entry. This is because advertising does not depend on the legal firm's reputation but only on the date at which it is carried out and the final date (to put it differently, advertising does not depend on a state variable, which would take into account what happened at date $\mathcal{E}$). Notice that this property also holds for the case where there is no counterfeiting.

Of particular interest is the impact of the counterfeiter's entry date on the legal firm's total profit.

Proposition 7. *The impact of the counterfeiter's entry date on the legal firm's total profit is positive and given by*

$$\frac{\partial Z_l(0,R_0;\mathcal{E})}{\partial \mathcal{E}} = \pi_1\left(R_1^{\mathcal{C}}(\mathcal{E};\mathcal{E}), a_1^{\mathcal{C}}(\mathcal{E};\mathcal{E}), p_{l1}^{\mathcal{C}}(\mathcal{E};\mathcal{E})\right) - \pi_2\left(R_2^{\mathcal{C}}(\mathcal{E};\mathcal{E}), a_2^{\mathcal{C}}(\mathcal{E};\mathcal{E}), p_{l2}^{\mathcal{C}}(\mathcal{E};\mathcal{E})\right) > 0$$

Proof. See Appendix B. ■

The proposition first establishes that the impact of the counterfeiter's entry date on the legal firm's total profit is equal to the difference between the instantaneous profit of the legal firm just before the counterfeiter's entry, denoted by $\pi_1\left(R_1^{\mathcal{C}}(\mathcal{E};\mathcal{E}), a_1^{\mathcal{C}}(\mathcal{E};\mathcal{E}), p_{l1}^{\mathcal{C}}(\mathcal{E};\mathcal{E})\right)$, and its instantaneous profit just after the counterfeiter's entry, denoted by $\pi_2\left(R_2^{\mathcal{C}}(\mathcal{E};\mathcal{E}), a_2^{\mathcal{C}}(\mathcal{E};\mathcal{E}), p_{l2}^{\mathcal{C}}(\mathcal{E};\mathcal{E})\right)$.[17] Since $R_1^{\mathcal{C}}(\mathcal{E};\mathcal{E}) = R_2^{\mathcal{C}}(\mathcal{E};\mathcal{E})$, and since, from Lemma 1, we know that the instantaneous profit before entry is higher than the instantaneous profit after entry, we see that the earlier the counterfeiter enters the market, the greater is the legal firm's loss, which is intuitive, as entry changes the market from a monopoly to a duopoly.

Finally, as we assumed that the entry date is exogenous, it is of interest to check how the counterfeiter's equilibrium payoff varies with this parameter. The total counterfeiter's payoff is given by

$$\Pi_c^{\mathcal{E}} = \int_{\mathcal{E}}^{T} p_c(t)\left(\delta_c\sqrt{R(t)} - \beta_c p_c(t) + \gamma p_{l2}(t)\right)dt.$$

Substituting for the equilibrium values for $p_c(t)$and $p_{l2}(t)$ we get

$$\Pi_c^{\mathcal{C}} = \beta_c\left(\frac{2\beta_l\delta_c + \delta_l\gamma}{4\beta_c\beta_l - \gamma^2}\right)^2 \int_{\mathcal{E}}^{T} R(t)\,dt.$$

Taking the derivative with respect to $\mathcal{E}$, we have

$$\frac{\partial \Pi_c^{\mathcal{C}}}{\partial \mathcal{E}} = \beta_c\left(\frac{2\beta_l\delta_c + \delta_l\gamma}{4\beta_c\beta_l - \gamma^2}\right)^2 \frac{\partial\left(\int_{\mathcal{E}}^{T} R(t)\,dt\right)}{\partial \mathcal{E}},$$

$$\frac{\partial\left(\int_{\mathcal{E}}^{T} R(t)\,dt\right)}{\partial \mathcal{E}} = -R(\mathcal{E}) + \int_{\mathcal{E}}^{T} \frac{\partial R(t)}{\partial \mathcal{E}}dt$$

The above equality has the following interpretation: On the one hand, an increase in $\mathcal{E}$ leads to the loss of the profit at date $\mathcal{E}$. On the other hand, from Remark 3, the value of the goodwill is higher at any date after $\mathcal{E}$, and so is the price of the legal firm. This directly increases the counterfeiter's demand.

Now, from equation (18) in the text, we have

$$R(\mathcal{E}) = R_1^{\mathcal{C}}(\mathcal{E}) = R_0 e^{-\sigma\mathcal{E}} + \frac{k^2}{8\sigma^2\tilde{\beta}_l\omega}\left(\left(4\sigma\tilde{\beta}_l\left(\Gamma + (s-\Gamma)e^{-\sigma(T-\mathcal{E})}\right) - \tilde{\delta}_l^2\right)\left(1 - e^{-2\sigma\mathcal{E}}\right) + 2\tilde{\delta}_l^2\left(1 - e^{-\sigma\mathcal{E}}\right)\right),$$

and from Remark 3, we know that

$$\frac{\partial R_2^{\mathcal{C}}(t)}{\partial \mathcal{E}} = \frac{k^2}{2\omega}e^{-\sigma(t-\mathcal{E})}\left(2e^{-\sigma(T-\mathcal{E})}s + \Gamma e^{-2\sigma\mathcal{E}}\left(1 - 2e^{-\sigma(T+\mathcal{E})}\right) + \frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l} - \Gamma\right).$$

[17] The argument $(\mathcal{E};\mathcal{E})$ of the reputation, advertising and pricing variables is to specify that these variables depend on the entry date $\mathcal{E}$ and that this date is also a parameter.

Therefore,

$$\int_{\mathcal{E}}^{T} \frac{\partial R_2^c}{\partial \mathcal{E}} dt = \frac{1-e^{-\sigma(T-\mathcal{E})}}{\sigma} \frac{k^2}{2\omega} \left(2e^{-\sigma(T-\mathcal{E})} s + \Gamma e^{-2\sigma\mathcal{E}} \left(1 - 2e^{-\sigma(T+\mathcal{E})} \right) + \frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l} - \Gamma \right).$$

After some algebra we find that

$$\frac{\partial \left(\int_{\mathcal{E}}^{T} R(t)\, dt \right)}{\partial \mathcal{E}} = -R_0 e^{-\sigma\mathcal{E}} - \frac{sk^2}{2\omega\sigma} e^{-\sigma(T-\mathcal{E})} \left(2e^{-\sigma(T-\mathcal{E})} - e^{-2\sigma\mathcal{E}} - 1 \right) \tag{21}$$

$$- \frac{k^2 \tilde{\delta}_l^2}{8\omega\sigma^2\tilde{\beta}_l} \left(e^{-2\sigma\mathcal{E}} - 2e^{-\sigma\mathcal{E}} + e^{-\sigma(T-\mathcal{E})} \right) \tag{22}$$

$$- \frac{\Gamma k^2}{\omega\sigma} (1 - e^{-\sigma(T-\mathcal{E})}) \left(1 - e^{-2\sigma\mathcal{E}} + e^{-3\sigma\mathcal{E}-\sigma T} \right). \tag{23}$$

The right-hand side of the above equation is highly nonlinear in all model's parameters and cannot be utterly signed. However, we see that for an R_0 high enough, we have $\frac{\partial \left(\int_{\mathcal{E}}^{T} R(t) dt \right)}{\partial \mathcal{E}} < 0$, that is, the counterfeiter's equilibrium payoff is decreasing in the entry date.

5 Comparison

In this section, we compare the strategies and outcomes in the two scenarios. Further, we determine the cost of counterfeiting to the legal firm and to the consumer.

5.1 Profit comparison

We shall first compare the advertising policies with and without counterfeiting.

Proposition 8. *The legal firm advertises more when there is no counterfeiting. That is, $a^{\mathcal{N}}(t) > a^{\mathcal{C}}(t)$, for all t in $[0, T]$*

Proof. See Appendix B. ■

Before interpreting the above result, we shall next compare the trajectories of reputation and the prices in the two scenarios.

Proposition 9. *At each instant of time, the legal brand enjoys a higher reputation when there is no counterfeiting, and the legal firm sells throughout the whole planning horizon at a higher price. That is, $R^{\mathcal{N}}(t) > R^{\mathcal{C}}(t)$, and $p_l^{\mathcal{N}}(t) > p_l^{\mathcal{C}}(t)$ for all t in $[0, T]$.*

Proof. See Appendix B. ■

Proposition 9 shows that the impact of entry on reputation is felt at any instant of time throughout the planning horizon, and not only after entry actually occurs. The fact that a counterfeiter will enter the market influences the advertising behavior of the legal firm during the monopoly period and this results in a loss of reputation even before entry takes place.

The interpretation of these results is as follows: Counterfeiting induces a competitive pressure on the legal firm pushing it to lower its price. Further, the legal firm invests less in advertising because the consequent reward, namely, a higher reputation and larger market size, is not fully appropriable in the counterfeiting scenario since the illegal firm benefits for free from the advertising investments and the brand's reputation. This is a typical case where the counterfeiter enjoys a positive externality without contributing at all to the building of reputation.

The above result differs from some of the findings in the literature, according to which counterfeiting may stimulate innovation or the quality of the genuine good (see Zhang et al. (2012)). This occurs notably when there are network externalities and R&D competition (Banerjee (2013)) or imperfect information (Qian (2014), Qian et al. (2014)). A common feature of these results is that the legal firm is able to sustainably differentiate the quality of its product from that of the counterfeiters. This, however, probably overlooks the case where the counterfeiters interact *repeatedly* with the legal firm. In such a case, it makes sense for counterfeiters to react to the differentiation efforts of the legal firm by adapting their own products. Here, we capture this reaction by assuming that the reputation of the genuine good always positively affects the reputation of the counterfeited product.

The following proposition shows that, for any given value of reputation $R(t)$, the legal firm obtains a higher total payoff in the no-counterfeiting case than in the counterfeiting scenario.

Proposition 10. *For any $R(t)$ and all $t \in [\mathcal{E}, T]$, we have $W_l(t, R(t)) < V_l(t, R(t))$.*

Proof. See Appendix B. ■

The two preceding propositions imply the following corollary:

Corollary 1. *We have $W_l(\mathcal{E}, R^{\mathcal{C}}(\mathcal{E})) < V_l(\mathcal{E}, R^{\mathcal{N}}(\mathcal{E}))$.*

Proof. From Proposition 10, we have

$$W_l(t, R^{\mathcal{C}}(t)) < V_l(t, R^{\mathcal{C}}(t))$$

and from Proposition 8, we have $R^{\mathcal{N}}(\mathcal{E}) > R^{\mathcal{C}}(\mathcal{E})$, so $W_l(\mathcal{E}, R^{\mathcal{C}}(\mathcal{E})) < V_l(\mathcal{E}, R^{\mathcal{N}}(\mathcal{E}))$. ■

The impact of counterfeiting on total profit is given in the following result.

Proposition 11. *The total profit of the legal firm calculated by starting at any date t in $[0, \mathcal{E}]$ is higher in the absence of counterfeiting. That is, $V_l(t, R^{\mathcal{N}}(t)) > Z_l(t, R^{\mathcal{C}}(t))$.*

Proof. Denote by $\left(R^{\mathcal{C}}(s), a^{\mathcal{C}}(s), p_l^{\mathcal{C}}(s)\right)$ the equilibrium trajectory in the presence of the counterfeiter and by $\pi_1\left(R^{\mathcal{C}}(s), a^{\mathcal{C}}(s), p_s^{\mathcal{C}}(s)\right)$ the corresponding instantaneous profit of the legal firm before the counterfeiter's entry. The total payoff that the legal firm realizes in the game starting at any t in $[0, \mathcal{E}]$ can be written as

$$\begin{aligned} Z_l\left(t, R^{\mathcal{C}}(t)\right) &= \int_t^{\mathcal{E}} \pi_1\left(p_l^{\mathcal{C}}(s), a^{\mathcal{C}}(s), R^{\mathcal{C}}(t)\right) ds + W_l(\mathcal{E}, R^{\mathcal{C}}(\mathcal{E})), \\ &\leq \int_t^{\mathcal{E}} \pi_1\left(p_l^{\mathcal{C}}(s), a^{\mathcal{C}}(s), R^{\mathcal{C}}(s)\right) ds + V_l(\mathcal{E}, R^{\mathcal{C}}(\mathcal{E})), \\ &\leq V_l(t, R^{\mathcal{N}}(t)). \end{aligned}$$

The first inequality is due to Proposition 10, and the second inequality follows from the optimality principle of dynamic programming. In particular, the total payoff in the whole game is higher in the absence of counterfeiting, that is, $Z_l(0, R_0) \leq V_l(0, R_0)$. ■

Independently of the fact that counterfeiting is illegal, its very presence means competition for the legal firm, and consequently, the above result is not surprising. A relevant question is how much counterfeiting costs the legal firm and how this loss varies with the parameter values. The total loss is given by $\Delta\Pi = V_l(0, R(0)) - Z_l(0, R(0))$. We note that $\Delta\Pi$ is increasing in R_0, which means that a company having a high initial brand equity (or reputation) suffers more from counterfeiting than a firm with a lower value.[18]

[18] This assertion can be established using equations (11) and (16), and Lemma 1.

The main message from the above comparisons is that counterfeiting is under no circumstances beneficial to the legal firm. Although these results sometimes involved complicated proofs, they are somewhat expected. If this were not the case, then legal firms would not invest much effort in deterring counterfeiting.[19] In the next subsection, we shift the focus from the firm to the consumer.

5.2 Consumer Welfare comparison

Standard consumer measures of surplus are difficult to use here since, in our setting, there are two goods whose prices change over time. It is then better to study the welfare effect of counterfeiting by comparing the equilibrium value of the consumer's utility function with and without counterfeiting.

First, at any $t \in [0, \mathcal{E}]$, the consumer's optimization problem is

$$\max_{q_l} U(q_l, 0, y) = \sigma_l \sqrt{R} q_l - \frac{\kappa_l q_l^2}{2} + I - p_l q_l.$$

From the first-order optimality condition, we obtain $q_l = \frac{\sigma_l \sqrt{R}}{2\kappa_l}$ and, for any $t \in [0, \mathcal{E}]$, the equilibrium (indirect) utility value

$$U(q_l, 0, y) = \frac{\kappa_l}{2}(q_l)^2 + I.$$

The above expression is the same with and without counterfeiting (only the value of brand reputation and the quantity q_l are different). Knowing that the brand's reputation is lower under counterfeiting, we conclude unambiguously that the counterfeiter causes a loss in welfare even during the monopoly period, that is, even before it enters into play.

Now at any $t \in [\mathcal{E}, T]$, the consumer's optimization problem is

$$\max_{q_l, q_c} U(q_l, q_c, y) = \left(\sigma_l \sqrt{R} q_l + \sigma_c \sqrt{R} q_c - \frac{\kappa_l q_l^2}{2} - \frac{\kappa_c q_c^2}{2} - \psi q_l q_c + I - p_l q_l - p_c q_c \right).$$

Assuming an interior solution, we can show that the equilibrium value of the demand for the legal product and the counterfeit are respectively given as follows:

$$q_l^{\mathcal{C}} = \frac{\kappa_c \left(2\kappa_c \kappa_l \sigma_l - \psi \sigma_c \kappa_l - \psi^2 \sigma_l \right)}{\left(4\kappa_c \kappa_l - \psi^2 \right) \left(\kappa_c \kappa_l - \psi^2 \right)} \sqrt{R},$$
$$q_c^{\mathcal{C}} = \frac{\kappa_l \left(2\kappa_c \kappa_l \sigma_c - \psi \sigma_l \kappa_c - \psi^2 \sigma_c \right)}{\left(4\kappa_c \kappa_l - \psi^2 \right) \left(\kappa_c \kappa_l - \psi^2 \right)} \sqrt{R}.$$

Inserting these demands in $U(q_l, q_c, y)$, it is easy to show that the equilibrium value of the consumer (indirect) utility function can be written as $U(q_l^{\mathcal{C}}, q_c^{\mathcal{C}}, y^{\mathcal{C}}) = \chi^{\mathcal{C}} R^{\mathcal{C}}$, where

$$\chi^{\mathcal{C}} = \frac{\kappa_c^2 X_1^2 + \kappa_l^2 X_2^2 + 2\psi \kappa_c \kappa_l X_1 X_2}{2 \left(4\kappa_c \kappa_l - \psi^2 \right)^2 \left(\kappa_c \kappa_l - \psi^2 \right)^2},$$

and

$$X_1 = \left(2\kappa_c \kappa_l \sigma_l - \psi \sigma_c \kappa_l - \psi^2 \sigma_l \right),$$
$$X_2 = \left(2\kappa_c \kappa_l \sigma_c - \psi \sigma_l \kappa_c - \psi^2 \sigma_c \right).$$

[19] See El Harbi and Grolleau (2008), however, for a review of some cases where counterfeiting can be profit enhancing for the legal firm.

We first want to compare $\chi^{\mathcal{C}}$ with $\chi^{\mathcal{N}}$ where we recall that

$$\chi^{\mathcal{N}} = \frac{\sigma_l^2}{8\kappa_l}.$$

Assume that $R^{\mathcal{C}} = R^{\mathcal{N}} = R$. We know, of course, that this is false in equilibrium, but it does not matter as we are dealing with variables that are solutions to static optimization problems. We know that the equilibrium price of the legal good is higher without counterfeiting than with counterfeiting. Therefore, the equilibrium value of the consumer's utility function with counterfeiting is no lower (and indeed is higher) than this value when there is counterfeiting. This is because the consumer can always buy the same quantity of the legal good that he bought when there was no counterfeiting, at a *lower* price. Since his income is constant, he can also buy the fake good, and this increases his utility. This leads to the following:

Proposition 12. *We have* $\chi^{\mathcal{N}} < \chi^{\mathcal{C}}$.

The next result gives a sufficient condition for counterfeiting to be welfare improving for any $t \in [\mathcal{E}, T]$, that is, $\chi^{\mathcal{N}} R^{\mathcal{N}}(t) < \chi^{\mathcal{C}} R^{\mathcal{C}}(t)$.

Proposition 13. *There exists* $\underline{\omega}$*, such that, for all* ω*, such that* $\underline{\omega} \leq \omega$*, counterfeiting is welfare improving for all* t *in* $[\mathcal{E}, T]$.

Proof. See Appendix B. ■

One explanation of this result is the following: When the advertising cost is high, the legal firm invests less in this activity, which results in a lower value for the brand's reputation, and consequently, the market size is smaller. This in turn increases competition between the two firms, and prices are lower, which is good news for the consumer. In this case, the positive effect of price competition on welfare more than compensates for the negative effect of the decrease in the legal firm's reputation (since accumulating reputation is costly, even in the absence of counterfeiting, the negative effect of counterfeiting on reputation is small).

Though counterfeiting may enhance consumer welfare on the interval $[\mathcal{E}, T]$, we have seen that counterfeiting is unambiguously welfare decreasing on the interval $[0, \mathcal{E}]$. The question of the global impact of counterfeiting on welfare is thus pending. The next result extends Proposition 13 to ensure that counterfeiting may improve consumer welfare on the whole horizon.

Proposition 14. *There exists* $\underline{\omega}'$*, such that, for all* ω*, such that* $\underline{\omega}' \leq \omega$*, counterfeiting is welfare improving* $[0, T]$ *in the sense that*

$$\int_0^T \chi^{\mathcal{N}} R^{\mathcal{N}}(t)dt < \int_0^{\mathcal{E}} \chi^{\mathcal{N}} R^{\mathcal{C}}(t)dt + \int_{\mathcal{E}}^T \chi^{\mathcal{C}} R^{\mathcal{C}} dt.$$

6 Concluding remarks

To the best of our knowledge, this is the first attempt to analyze the impact of counterfeiting in a fully dynamic context with micro-founded demand functions. The decision variables, that is, price and advertising, are clearly the most relevant ones for well-known brands that eventually end up being copied by illegal producers. In one sentence, the main takeaway of our paper is that counterfeiting is under no circumstances beneficial to the legal producer, but it can suit consumers under some conditions. Further, we showed that brand equity is always lower in the presence of counterfeiting. This implies that this illegal activity has a really damaging effect on the legal firm over the long-term.

This last effect clearly supports prosecuting counterfeiters, as it is currently done in many countries. By contrast, only a few countries like France and Italy penalize consumers who purchase counterfeits. Our finding that counterfeiting can benefit consumers suggests acting with caution with regard to the introduction of consumer liability.[20] That is because, it may be difficult to actually identify the goods for which counterfeiting is detrimental to consumers from the others.

As in any modeling effort, some simplifying assumptions have been made here, and it would clearly be advantageous to relax them in future work. First, we assumed that the counterfeiter's entry date is known, which in practice may be hard to predict precisely. It would not really be conceptually difficult to keep the same framework and consider a case where this date is random. However, one can expect this to potentially lead to equilibria that cannot be either fully characterized analytically or not be compared analytically.

Second, we have implicitly assumed that the legal producer cannot deter entry. In the absence of efficient institutions to combat counterfeiting, one intuitive option for private firms to prevent illegal producers from entering the market is to sell at a lower price to reduce the temptation of consumers to buy the illegal product. (The assumption here is that the attractiveness of going illegal depends on the gap in prices.) For this to work, we minimally need to assume that the illegal producer faces a fixed cost. The relevance and the level of such cost is an empirical matter. Indeed, the fixed cost that needs to be paid to be able to start selling an illegal version of software is not the same as producing a fake *Lancel* bag.

Third, we assumed that the product is normal. An interesting question that we did not address is what would happen if the product had a network externality value. For instance, the value that a person derives from a video game may depend on the number of individuals in the person's circle who own the product. Here, the illegal demand may have a positive effect on the brand's reputation, that is, illegal demand works as an additional advertising activity that feeds the brand equity. In such a case, one expects very different results from those obtained here, and it is surely of interest to investigate such a context.

7 Appendices

7.1 Appendix A: Derivation of the demand functions

Assume that the utility function of the representative consumer is given by the following quadratic function:

$$U(q_l, q_c, y) = \sigma_l\sqrt{R}q_l + \sigma_c\sqrt{R}q_c - \frac{\kappa_l q_l^2}{2} - \frac{\kappa_c q_c^2}{2} - \psi q_l q_c + y,$$

where y is a composite good, and $\sigma_l, \sigma_c, \psi, \kappa_l$ and κ_c are positive parameters, with

$$\sigma_l\kappa_c - \sigma_c\psi > 0, \tag{24}$$

$$\sigma_c\kappa_l - \sigma_l\psi > 0, \tag{25}$$

$$\psi > 0, \tag{26}$$

The budget constraint is given by

$$p_l q_l + p_c q_c + y = I,$$

where p_j is the price of product $j = l, c$ and I is the income.

[20] For a defense of consumer liability in the U.S., see, e.g., Orscheln (2015) or Riso (2015). According to Orschel (2015), in 1993, Ms. Chin, a representative for District 1 of New York City, proposed to adapt New York legislation to prosecute consumers for purchasing counterfeit goods. The proposal appears to be laid over.

Suppose now that there is no counterfeit good, i.e., $q_c = 0$. Then, the representative consumer solves the following problem:

$$\max_{q_l} \left(\sigma_l \sqrt{R} q_l - \frac{\kappa_l q_l^2}{2} + I - p_l q_l \right).$$

We easily find that the demand function is

$$q_l = \frac{\sigma_l \sqrt{R} - p_l}{\kappa_l}. \tag{27}$$

By contrast, when there *is* a counterfeiter, the representative consumer solves the following program:

$$\max_{q_l, q_c} \left(\sigma_l \sqrt{R} q_l + \sigma_c \sqrt{R} q_c - \frac{\kappa_l q_l^2}{2} - \frac{\kappa_c q_c^2}{2} - \psi q_l q_c + I - p_l q_l - p_c q_c \right).$$

Assuming an interior solution, then the first-order optimality conditions are given by

$$\sigma_l \sqrt{R} - \kappa_l q_l - \psi q_c - p_l = 0, \tag{28}$$

$$\sigma_c \sqrt{R} - \kappa_c q_c - \psi q_l - p_c = 0. \tag{29}$$

Solving for q_l and q_c, we obtain

$$q_l = \frac{\kappa_c \sigma_l \sqrt{R} - \psi \sigma_c \sqrt{R} - \kappa_c p_l + \psi p_c}{\kappa_l \kappa_c - \psi^2}, \tag{30}$$

$$q_c = \frac{\kappa_l \sigma_c \sqrt{R} - \psi \sigma_l \sqrt{R} - \kappa_l p_c + \psi p_l}{\kappa_l \kappa_c - \psi^2}. \tag{31}$$

We see at once that the demand functions for the legal good are structurally different in the two cases. Setting $p_c = 0$ in (30) does not yield (27). We shall then assume that the demand functions for the legal good and the counterfeit good are given by the next expressions:

$$q_l(t) = \begin{cases} \tilde{\delta}_l \sqrt{R(t)} - \tilde{\beta}_l p_l(t), & t \in [0, \mathcal{E}), \\ \delta_l \sqrt{R(t)} - \beta_l p_l(t) + \gamma p_c(t), & t \in [\mathcal{E}, T], \end{cases}$$

$$q_c(t) = \delta_c \sqrt{R(t)} - \beta_c p_c(t) + \gamma p_l(t), \quad t \in [\mathcal{E}, T],$$

where $\beta_j > 0$ and $\gamma \geq 0$, with $\beta_j > \gamma$, $j \in \{l, c\}$, and

$$\begin{aligned} \tilde{\delta}_l &= \frac{\sigma_l}{\kappa_l}, \quad \delta_l = \frac{\kappa_c \sigma_l - \psi \sigma_c}{\kappa_c \kappa_l - \psi^2}, \quad \delta_c = \frac{\kappa_l \sigma_c - \psi_l \sigma_l}{\kappa_c \kappa_l - \psi^2}, \\ \tilde{\beta}_l &= \frac{1}{\kappa_l}, \quad \beta_l = \frac{\kappa_c}{\kappa_c \kappa_l - \psi^2}, \quad \beta_c = \frac{\kappa_l}{\kappa_c \kappa_l - \psi^2}, \quad \gamma = \frac{\psi}{\kappa_c \kappa_l - \psi^2}. \end{aligned}$$

We notice that

$$\delta_l = \frac{\kappa_c \sigma_l - \psi \sigma_c}{\kappa_c \kappa_l - \psi^2} < \tilde{\delta}_l = \frac{\sigma_l}{\kappa_l},$$

if and only if $\sigma_c \kappa_l - \sigma_l \psi > 0$ which holds true by assumption.

Moreover, we have

$$\tilde{\beta}_l = \frac{1}{\kappa_l} < \beta_l = \frac{\kappa_c}{\kappa_c \kappa_l - \psi^2}.$$

To ensure that in equilibrium the price of the good produced by the legal firm is higher than the price of the counterfeit good, that is,

$$\frac{p_l^{\mathcal{C}}\left(R\left(t\right)\right)}{p_c^{\mathcal{C}}\left(R\left(t\right)\right)}=\frac{2\beta_c\delta_l+\delta_c\gamma}{2\beta_l\delta_c+\delta_l\gamma}>1,$$

we assume that $\sigma_l-\sigma_c>0$ and

$$\left(2\kappa_l\kappa_c-\psi^2\right)\left(\sigma_l-\sigma_c\right)+\psi(\kappa_c\sigma_l-\kappa_l\sigma_c)>0.$$

7.2 Appendix B: Proofs

7.2.1 Proof of Proposition 1

Denote by $V_l\left(t,R\left(t\right)\right):[0,T]\times\mathbb{R}_+\to\mathbb{R}_+$ the value function of the legal firm. The Hamilton-Jacobi-Bellman (HJB) equation reads as follows:

$$\begin{aligned}-\frac{\partial V_l}{\partial t}\left(t,R\left(t\right)\right) &= \max_{p_l,a}\left(\left(p_l\left(t\right)\left(\tilde{\delta}_l\sqrt{R\left(t\right)}-\tilde{\beta}_l p_l\left(t\right)\right)-\frac{\omega}{2}a^2\left(t\right)\right)\right.\\ &\quad\left.+\frac{\partial V_l}{\partial R}\left(t,R\left(t\right)\right)\left(ka\left(t\right)-\sigma R\left(t\right)\right)\right).\end{aligned}$$

Assuming an interior solution, the first-order optimality conditions are

$$\begin{aligned}\frac{\partial RHS}{\partial p_l} &= \tilde{\delta}_l\sqrt{R}-2\tilde{\beta}_l p_l=0\Leftrightarrow p_l=\frac{\tilde{\delta}_l}{2\tilde{\beta}_l}\sqrt{R},\\ \frac{\partial RHS}{\partial a} &= -\omega a+k\frac{\partial V_l}{\partial R}=0\Leftrightarrow a=\frac{k}{\omega}\frac{\partial V_l}{\partial R}\end{aligned}$$

Substitute in the HJB equation to get

$$-\frac{\partial V_l}{\partial t}=\left(\frac{\tilde{\delta}_l}{2\tilde{\beta}_l}\sqrt{R\left(t\right)}\left(\tilde{\delta}_l\sqrt{R\left(t\right)}-\tilde{\beta}_l\frac{\tilde{\delta}_l}{2\tilde{\beta}_l}\sqrt{R\left(t\right)}\right)-\frac{\omega}{2}\left(\frac{k}{\omega}\frac{\partial V_l}{\partial R}\right)^2\right)+\frac{\partial V_l}{\partial R}\left(k\frac{k}{\omega}\frac{\partial V_l}{\partial R}-\sigma R\left(t\right)\right),$$

which simplifies to

$$-\frac{\partial V_l}{\partial t}=\frac{\tilde{\delta}_l^2}{4\tilde{\beta}_l}R+\frac{k^2}{2\omega}\left(\frac{\partial V_l}{\partial R}\right)^2-\sigma R\frac{\partial V_l}{\partial R}. \tag{32}$$

Conjecture that the value function is linear, i.e.,

$$\begin{aligned}V_l\left(t,R\left(t\right)\right) &= z\left(t\right)R\left(t\right)+y\left(t\right),\\ V_l\left(T,R\left(T\right)\right) &= sR\left(T\right),\end{aligned}$$

where $z\left(t\right)$ and $y\left(t\right)$ are the coefficient to be identified. Substituting in (36) yields

$$-\left(\dot{z}R+\dot{y}\right)=\left(\frac{\tilde{\delta}_l^2}{4\tilde{\beta}_l}-\sigma z\right)R+\frac{k^2}{2\omega}z^2.$$

By identification, we have

$$\begin{aligned}-\dot{z} &= \frac{\tilde{\delta}_l^2}{4\tilde{\beta}_l}-\sigma z,\\ -\dot{y}\left(t\right) &= \frac{k^2}{2\omega}\left(z\left(t\right)\right)^2.\end{aligned}$$

Solving the two above differential equations, we obtain

$$z(t) = \frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l} + C_1 e^{\sigma t}, \tag{33}$$

$$y(t) = -\frac{k^2}{4\sigma\omega}\left(\frac{\tilde{\delta}_l^4}{8\sigma\tilde{\beta}_l^2}t + \frac{\tilde{\delta}_l^2 C_1}{\sigma\tilde{\beta}_l}e^{\sigma t} + C_1^2 e^{2\sigma t}\right) + C_2, \tag{34}$$

where C_1 and C_2 are integration constants.

Using the terminal condition

$$V_l(T, R(T)) = sR(T),$$

we conclude that

$$\begin{aligned} z(T) &= s, \\ y(T) &= 0. \end{aligned}$$

Consequently,

$$z(T) = \frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l} + C_1 e^{\sigma T} = s \Leftrightarrow C_1 = \frac{4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{-\sigma T}.$$

Further, we have

$$\begin{aligned} y(T) &= -\frac{k^2}{4\sigma\omega}\left(\frac{\tilde{\delta}_l^4}{8\sigma\tilde{\beta}_l^2}T + \frac{\tilde{\delta}_l^2 C_1}{\sigma\tilde{\beta}_l}e^{\sigma T} + C_1^2 e^{2\sigma T}\right) + C_2 = 0, \\ \frac{4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{-\sigma T} &= -\frac{k^2}{4\sigma\omega}\left(\frac{\tilde{\delta}_l^4}{8\sigma\tilde{\beta}_l^2}T + \frac{\tilde{\delta}_l^2\frac{4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{-\sigma T}}{\sigma\tilde{\beta}_l}e^{\sigma T} + (\frac{4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{-\sigma T})^2 e^{2\sigma T}\right) + C_2 = 0, \\ &\Leftrightarrow \quad C_2 = \frac{k^2}{4\sigma\omega}\left(\frac{\tilde{\delta}_l^4}{8\sigma\tilde{\beta}_l^2}T + \tilde{\delta}_l^2\frac{4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2}{4\sigma^2\tilde{\beta}_l^2} + \left(\frac{4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}\right)^2\right) \\ &= \frac{k^2}{16\sigma^3\omega\tilde{\beta}_l^2}\left(\frac{\tilde{\sigma\delta}_l^4}{2}T + \tilde{\delta}_l^2(4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2) + \frac{(4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2)^2}{4}\right) \end{aligned}$$

Substituting for C_1 and C_2 in (33) and (34) yields

$$\begin{aligned} z(t) &= \frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l} + \frac{4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{\sigma(t-T)}, \\ y(t) &= -\frac{k^2}{4\sigma\omega}\left(\frac{\tilde{\delta}_l^4}{8\sigma\tilde{\beta}_l^2}t + \frac{\tilde{\delta}_l^2(\frac{4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{-\sigma T})}{\sigma\tilde{\beta}_l}e^{\sigma t} + \left(\frac{4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{-\sigma T}\right)^2 e^{2\sigma t}\right) \\ &+ \frac{k^2}{16\sigma^3\omega\tilde{\beta}_l^2}\left(\frac{\tilde{\sigma\delta}_l^4}{2}T + \tilde{\delta}_l^2(4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2) + \frac{(4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2)^2}{4}\right) \\ &= \frac{k^2}{16\sigma^3\omega\tilde{\beta}_l^2}\left(\frac{\tilde{\sigma\delta}_l^4}{2}(T-t) + \tilde{\delta}_l^2(4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2)(1 - e^{\sigma(t-T)}) + \frac{(4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2)^2}{4}(1 - e^{2\sigma(t-T)})\right) \end{aligned}$$

Now,

$$a = \frac{k}{\omega}\frac{\partial V_l}{\partial R} = \frac{k}{\omega}z(t) = \frac{k}{4\sigma\tilde{\beta}_l\omega}(\tilde{\delta}_l^2 + (4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2)e^{\sigma(t-T)}).$$

Inserting in the dynamics and solving the differential equation, we obtain the following trajectory:

$$R(t) = R_0 e^{-\sigma t} + \frac{k^2}{\omega}\frac{4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2}{8\sigma^2\tilde{\beta}_l}\left(e^{\sigma(t-T)} - e^{-\sigma(T+t)}\right) + \frac{k^2}{\omega}\frac{\tilde{\delta}_l^2}{4\sigma^2\tilde{\beta}_l}\left(1 - e^{-\sigma t}\right).$$

Substituting for $z(t)$ and $y(t)$ in $V_l(t, R(t))$ yields the following value:

$$\begin{aligned} V_l(t, R(t)) &= \left(\frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l} + \frac{4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{\sigma(t-T)}\right) R(t) \\ &\quad + \frac{k^2}{16\sigma^3\omega\tilde{\beta}_l^2}\left(\frac{\tilde{\sigma\delta}_l^4}{2}(T-t) + \tilde{\delta}_l^2(4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2)(1 - e^{\sigma(t-T)})\right. \\ &\quad \left. + \frac{(4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2)^2}{4}(1 - e^{2\sigma(t-T)})\right). \end{aligned}$$

The total payoff is obtained by evaluating the above value function at $(0, R(0))$, that is,

$$\begin{aligned} V_l(0, R(0)) &= z(0) R(0) + y(0), \\ &= \left(\frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l} + \frac{4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{-\sigma T}\right) R_0 \\ &\quad + \frac{k^2}{16\sigma^3\omega\tilde{\beta}_l^2}\left(\frac{\tilde{\sigma\delta}_l^4}{2}T + \tilde{\delta}_l^2(4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2)(1 - e^{-\sigma T}) + \frac{(4\sigma\tilde{\beta}_l s - \tilde{\delta}_l^2)^2}{4}(1 - e^{-2\sigma T})\right). \end{aligned}$$

Payoff starting from $(\mathcal{E}, R(\mathcal{E}))$ is given by

$$V_l(\mathcal{E}, R(\mathcal{E})) = z(\mathcal{E}) R(\mathcal{E}) + y(\mathcal{E}).$$

7.2.2 Proof of Proposition 4

Denote by $W_l(t, R(t)) : [\mathcal{E}, T] \times \mathbb{R}_+ \to \mathbb{R}$ the legal firm's value function. The HJB equation of the legal firm is given by

$$\begin{aligned} -\frac{\partial W_l}{\partial t}(t, R(t)) &= \max_{p_l, a}\left(\left(p_l(t)\left(\delta_l\sqrt{R(t)} - \beta_l p_l(t) + \gamma p_c(t)\right) - \frac{\omega}{2}a^2(t)\right)\right. \\ &\quad \left. + \frac{\partial W_l}{\partial R}(t, R(t))\left(ka(t) - \sigma R(t)\right)\right). \end{aligned}$$

The counterfeiter's optimization problem is

$$\max_{p_c(t)} \pi_c(t) = \max_{p_c(t)} p_c(t)\left(\delta_c\sqrt{R(t)} - \beta_c p_c(t) + \gamma p_l(t)\right), \quad \forall t \in [\mathcal{E}, T].$$

Assuming an interior solution, the first-order equilibrium conditions are

$$\begin{aligned} \frac{\partial RHS}{\partial p_l} &= \delta_l\sqrt{R} - 2\beta_l p_l + \gamma p_c = 0, \\ \frac{\partial RHS}{\partial a} &= -\omega a + k\frac{\partial W_l}{\partial R} = 0, \\ \frac{\partial \pi_c}{\partial p_c} &= \delta_c\sqrt{R} - 2\beta_c p_c + \gamma p_l = 0 \Leftrightarrow p_c = \frac{\delta_c\sqrt{R} + \gamma p_l}{2\beta_c}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} p_l &= \frac{2\beta_c\delta_l + \delta_c\gamma}{4\beta_c\beta_l - \gamma^2}\sqrt{R}, \\ p_c &= \frac{2\delta_c\beta_l + \gamma\delta_l}{4\beta_c\beta_l - \gamma^2}\sqrt{R}, \\ a &= \frac{k}{\omega}\frac{\partial W_l}{\partial R}. \end{aligned}$$

Substituting in the HJB yields

$$-\frac{\partial W_l}{\partial t} = \beta_l\left(\frac{2\beta_c\delta_l + \delta_c\gamma}{4\beta_c\beta_l - \gamma^2}\right)^2 R + \frac{\omega}{2}\left(\frac{k}{\omega}\frac{\partial W_l}{\partial R}\right)^2 - \sigma R\frac{\partial W_l}{\partial R}. \tag{35}$$

Conjecture the following linear form for the value function:

$$W_l\left(t, R\left(t\right)\right) = x\left(t\right)R\left(t\right) + v\left(t\right),$$

then

$$\begin{aligned} a &= \frac{k}{\omega}x, \\ \frac{\partial W_l}{\partial t} &= \dot{x}R + \dot{v} \end{aligned}$$

Substituting in (35), we obtain

$$-\left(\dot{x}R + \dot{v}\right) = \left(\beta_l\left(\frac{2\beta_c\delta_l + \delta_c\gamma}{4\beta_c\beta_l - \gamma^2}\right)^2 - \sigma x\right)R + \frac{k^2x^2}{2\omega}.$$

By identification of terms in order of R, we have

$$\begin{aligned} -\dot{x} + \sigma x &= \beta_l\left(\frac{2\beta_c\delta_l + \delta_c\gamma}{4\beta_c\beta_l - \gamma^2}\right)^2, \\ \dot{v} &= -\frac{k^2}{2\omega}x^2. \end{aligned}$$

Solving the two above differential equations, we get

$$\begin{aligned} x\left(t\right) &= \Gamma + C_1e^{\sigma t}, \\ v\left(t\right) &= -\frac{k^2}{2\omega}\left(\Gamma^2t + \frac{C_1^2}{2\sigma}e^{2\sigma t} + \frac{2\Gamma C_1}{\sigma}e^{\sigma t}\right) + C_{2,} \end{aligned}$$

where C_1 and C_2 are integration constants and

$$\Gamma = \frac{\beta_l}{\sigma}\left(\frac{2\beta_c\delta_l + \delta_c\gamma}{4\beta_c\beta_l - \gamma^2}\right)^2.$$

Using the boundary condition

$$W_l\left(T, R\left(T\right)\right) = sR\left(T\right),$$

yields

$$\begin{aligned} C_1 &= \left(s - \Gamma\right)e^{-\sigma T}, \\ C_2 &= \frac{k^2}{2\omega}\left(\Gamma^2T + \frac{(s-\Gamma)^2}{2\sigma} + \frac{2\Gamma(s-\Gamma)}{\sigma}\right), \end{aligned}$$

and consequently

$$\begin{aligned} x(t) &= \Gamma + (s-\Gamma)\, e^{-\sigma(T-t)} \\ v(t) &= -\frac{k^2}{2\omega}\left(\Gamma^2 t + \frac{((s-\Gamma)\, e^{-\sigma T})^2}{2\sigma} e^{2\sigma t} + \frac{2\Gamma((s-\Gamma)\, e^{-\sigma T})}{\sigma} e^{\sigma t}\right) \\ &\quad + \frac{k^2}{2\omega}\left(\Gamma^2 T + \frac{(s-\Gamma)^2}{2\sigma} + \frac{2\Gamma(s-\Gamma)}{\sigma}\right) \\ &= \frac{k^2}{2\omega}\left(\Gamma^2(T-t) + \frac{(s-\Gamma)^2}{2\sigma}(1-e^{2\sigma(t-T)}) + \frac{2\Gamma(s-\Gamma)}{\sigma}(1-e^{\sigma(t-T)})\right) \end{aligned}$$

Recalling that $a = \frac{k}{\omega}x$, we then have

$$a = \frac{k}{\omega}\left(\Gamma + (s-\Gamma)\, e^{-\sigma(T-t)}\right).$$

Substituting for a in the dynamics and solving the differential equation with $R(\mathcal{E})$ as initial condition, we get

$$R^{\mathcal{C}}(t) = R(\mathcal{E})\, e^{-\sigma(t-\mathcal{E})} + \frac{k^2\Gamma}{\sigma\omega}\left(1-e^{-\sigma(t-\mathcal{E})}\right) + \frac{k^2(s-\Gamma)}{2\sigma\omega}\left(1-e^{-2\sigma(t-\mathcal{E})}\right) e^{-\sigma(T-t)}.$$

7.2.3 Proof of Proposition 5

Denote by $Z_l(t, R(t)) : [0,T]\times\mathbb{R}_+ \to \mathbb{R}_+$ the value function of the legal firm. The Hamilton-Jacobi-Bellman (HJB) equation reads as follows:

$$\begin{aligned} -\frac{\partial Z_l}{\partial t}(t, R(t)) &= \max_{p_l, a}\left(\left(p_l(t)\left(\tilde{\delta}_l\sqrt{R(t)} - \tilde{\beta}_l p_l(t)\right) - \frac{\omega}{2}a^2(t)\right)\right. \\ &\quad \left. + \frac{\partial Z_l}{\partial R}(t, R(t))\,(ka(t) - \sigma R(t))\right). \end{aligned}$$

Assuming an interior solution, the first-order optimality conditions are

$$\begin{aligned} \frac{\partial RHS}{\partial p_l} &= \tilde{\delta}_l\sqrt{R} - 2\tilde{\beta}_l p_l = 0 \Leftrightarrow p_l = \frac{\tilde{\delta}_l}{2\tilde{\beta}_l}\sqrt{R}, \\ \frac{\partial RHS}{\partial a} &= -\omega a + k\frac{\partial Z_l}{\partial R} = 0 \Leftrightarrow a = \frac{k}{\omega}\frac{\partial Z_l}{\partial R}. \end{aligned}$$

Substitute in the HJB equation to get

$$-\frac{\partial Z_l}{\partial t} = \frac{\tilde{\delta}_l}{2\tilde{\beta}_l}\sqrt{R(t)}\left(\tilde{\delta}_l\sqrt{R(t)} - \tilde{\beta}_l\frac{\tilde{\delta}_l}{2\tilde{\beta}_l}\sqrt{R(t)}\right) - \frac{\omega}{2}\left(\frac{k}{\omega}\frac{\partial Z_l}{\partial R}\right)^2 + \frac{\partial Z_l}{\partial R}\left(k\frac{k}{\omega}\frac{\partial Z_l}{\partial R} - \sigma R(t)\right),$$

which simplifies to

$$-\frac{\partial Z_l}{\partial t} = \frac{\tilde{\delta}_l^2}{4\tilde{\beta}_l}R + \frac{k^2}{2\omega}\left(\frac{\partial Z_l}{\partial R}\right)^2 - \sigma R\frac{\partial Z_l}{\partial R}. \tag{36}$$

Conjecture that the value function is linear, i.e.,

$$\begin{aligned} Z_l(t, R(t)) &= m(t)\, R(t) + n(t), \\ Z_l(\mathcal{E}, R(\mathcal{E})) &= W_l(\mathcal{E}, R(\mathcal{E})) \end{aligned} \quad ,$$

where $m(t)$ and $n(t)$ are the coefficients to be identified. Substituting in (36) yields

$$-(\dot{m}R+\dot{n})=\left(\frac{\tilde{\delta}_l^2}{4\tilde{\beta}_l}-\sigma m\right)R+\frac{k^2}{2\omega}m^2.$$

By identification, we have

$$\begin{aligned}-\dot{m} &= \frac{\tilde{\delta}_l^2}{4\tilde{\beta}_l}-\sigma m\\ -\dot{n} &= \frac{k^2}{2\omega}m^2\end{aligned}$$

Solving the two above differential equations, we obtain

$$m(t) = \frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}+C_1e^{\sigma t}, \tag{37}$$

$$n(t) = -\frac{k^2}{4\sigma\omega}\left(\frac{\tilde{\delta}_l^4}{8\sigma\tilde{\beta}_l^2}t+\frac{\tilde{\delta}_l^2C_1}{\sigma\tilde{\beta}_l}e^{\sigma t}+C_1^2e^{2\sigma t}\right)+C_2, \tag{38}$$

where C_1 and C_2 are integration constants.

Using the terminal condition

$$Z_l(\mathcal{E},R(\mathcal{E}))=W_l(\mathcal{E},R(\mathcal{E}))=x(\mathcal{E})R(\mathcal{E})+v(\mathcal{E}),$$

we conclude that

$$\begin{aligned}m(\mathcal{E})=x(\mathcal{E}) &= \Gamma+(s-\Gamma)e^{-\sigma(T-\mathcal{E})},\\ n(\mathcal{E})=v(\mathcal{E}) &= \frac{k^2}{2\omega}\left(\Gamma^2(T-\mathcal{E})+\frac{(s-\Gamma)^2}{2\sigma}(1-e^{2\sigma(\mathcal{E}-T)})+\frac{2\Gamma(s-\Gamma)}{\sigma}(1-e^{\sigma(\mathcal{E}-T)})\right)\end{aligned}$$

Consequently,

$$m(\mathcal{E})=\frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}+C_1e^{\sigma\mathcal{E}}=x(\mathcal{E})\Leftrightarrow C_1=\frac{4\sigma\tilde{\beta}_lx(\mathcal{E})-\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{-\sigma\mathcal{E}}.$$

Further, we have

$$n(\mathcal{E})=-\frac{k^2}{4\sigma\omega}\left(\frac{\tilde{\delta}_l^4}{8\sigma\tilde{\beta}_l^2}\mathcal{E}+\frac{\tilde{\delta}_l^2C_1}{\sigma\tilde{\beta}_l}e^{\sigma\mathcal{E}}+C_1^2e^{2\sigma\mathcal{E}}\right)+C_2=v(\mathcal{E}),$$

$$=-\frac{k^2}{16\sigma^3\omega\tilde{\beta}_l^2}\left(\frac{\sigma\tilde{\delta}_l^4}{2}\mathcal{E}+\tilde{\delta}_l^2(4\sigma\tilde{\beta}_lx(\mathcal{E})-\tilde{\delta}_l^2)+\frac{(4\sigma\tilde{\beta}_lx(\mathcal{E})-\tilde{\delta}_l^2)^2}{4}\right)+C_2=v(\mathcal{E}).$$

Substituting for C_1 and C_2 in (37) and (38) yields

$$\begin{aligned}m(t) &= \frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}+\frac{4\sigma\tilde{\beta}_lx(\mathcal{E})-\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{\sigma(t-\mathcal{E})},\\ n(t) &= -\frac{k^2}{4\sigma\omega}\left(\frac{\tilde{\delta}_l^4}{8\sigma\tilde{\beta}_l^2}t+\frac{\tilde{\delta}_l^2C_1}{\sigma\tilde{\beta}_l}e^{\sigma t}+C_1^2e^{2\sigma t}\right)+C_2\end{aligned}$$

Now,

$$a=\frac{k}{\omega}\frac{\partial Z_l}{\partial R}=\frac{k}{\omega}m(t)=\frac{k}{4\sigma\tilde{\beta}_l\omega}\left(\tilde{\delta}_l^2\left(1-e^{\sigma(t-\mathcal{E})}\right)+4\sigma\tilde{\beta}_lx(\mathcal{E})e^{\sigma(t-\mathcal{E})}\right)$$

$$= \frac{k}{4\sigma\tilde{\beta}_l\omega}\left(\tilde{\delta}_l^2\left(1-e^{\sigma(t-\mathcal{E})}\right)+4\sigma\tilde{\beta}_l(\Gamma+(s-\Gamma)\,e^{-\sigma(T-\mathcal{E})})e^{\sigma(t-\mathcal{E})}\right).$$

Inserting in the dynamics and solving the differential equation, we obtain the following trajectory:

$$R\left(t\right)=R_0e^{-\sigma t}+\frac{k^2}{\omega}\frac{4\sigma\tilde{\beta}_l x(\mathcal{E})-\tilde{\delta}_l^2}{8\sigma^2\tilde{\beta}_l}\left(e^{\sigma(t-\mathcal{E})}-e^{-\sigma(\mathcal{E}+t)}\right)+\frac{k^2}{\omega}\frac{\tilde{\delta}_l^2}{4\sigma^2\tilde{\beta}_l}\left(1-e^{-\sigma t}\right).$$

The total payoff is given by

$$\begin{aligned} Z_l\left(0,R\left(0\right)\right) &= \left(\frac{\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}+\frac{4\sigma\tilde{\beta}_l x(\mathcal{E})-\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{-\sigma\mathcal{E}}\right)R_0 \\ &\quad -\frac{k^2}{4\sigma\omega}\left(\frac{\tilde{\delta}_l^2(\frac{4\sigma\tilde{\beta}_l x(\mathcal{E})-\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{-\sigma\mathcal{E}})}{\sigma\tilde{\beta}_l}+(\frac{4\sigma\tilde{\beta}_l x(\mathcal{E})-\tilde{\delta}_l^2}{4\sigma\tilde{\beta}_l}e^{-\sigma\mathcal{E}})^2\right)+C_2 \end{aligned}$$

7.2.4 Proof of Lemma 1

To prove the Lemma, we have to establish that the legal firm's profit in the monopoly case is higher than the profit under Bertrand competition. To do this, we shall rely on the micro-foundations of the demand functions. From Appendix A, we know that when the consumptions of the three goods, q_l, q_c and y, are positive (where we recall that y is the composite good), the next conditions hold:

$$\sigma_l-\kappa_l q_l-\psi q_c=p_l, \tag{39}$$
$$\sigma_c-\kappa_l q_c-\psi q_l=p_c. \tag{40}$$

To derive the demand functions used in the paper, we have solved the consumer's maximization problem for quantities q_l and q_c (as a function of the prices) and we have studied the Bertrand competition case. We could also have considered Cournot competition where the legal firm (resp. the counterfeiter) maximizes $p_l q_l$ (resp. $p_c q_c$) with respect to q_l (resp. q_c), p_l and p_c being given by (39)-(40).

The quantities associated to a Cournot equilibrium satisfy the next conditions:

$$\sigma_l-2\kappa_l q_l-\psi q_c=0, \tag{41}$$
$$\sigma_c-2\kappa_l q_c-\psi q_l=0, \tag{42}$$

and are given by

$$\bar{q}_l=\frac{2\kappa_c\sigma_l-\psi\sigma_c}{4\kappa_c\kappa_l-\psi^2}, \tag{43}$$
$$\bar{q}_c=\frac{2\kappa_l\sigma_c-\psi\sigma_l}{4\kappa_c\kappa_l-\psi^2}. \tag{44}$$

Using equations (39), (40), (41), and (42) we notice that, in a Cournot equilibrium,

$$\bar{p}_l=\kappa\bar{q}_l, \tag{45}$$
$$\bar{p}_c=\kappa\bar{q}_c. \tag{46}$$

• Now recall that, in the monopoly case, the demand for the legal product is obtained from the condition $\sigma_l-\kappa_l q_l-p_l=0$. In this case, the legal firm chooses its price so as to maximize its profit $p_l q_l$, and we obtain that $q_c^*=\frac{\sigma_l}{2\kappa_l}$ and $p_l^*=\frac{\sigma_l}{2}$.

• Next, we shall rely on Proposition 1 of Singh and Vives (1984), p. 549, which asserts that the profit of each firm under Cournot competition is higher than the profit obtained under Bertrand Competition (which is the case considered in the paper).

• We shall now prove that the monopoly profit is higher than the Cournot profit. To do this, we only have to show that $q_l^* > \bar{q}_l$ (see equations (45) and (46)). But we can check that the condition $q_l^* > \bar{q}_l$, that is,

$$\frac{\sigma_l}{2\kappa_l} > \frac{2\kappa_c\sigma_l - \psi\sigma_c}{4\kappa_c\kappa_l - \psi^2} \tag{47}$$

is equivalent to

$$2\kappa_l\sigma_c > \sigma_l\psi.$$

This last condition is always met since we have assumed that $\kappa_l\sigma_c > \sigma_l\psi$. In the model's notation, the inequality in (47) corresponds to the inequality in the Lemma.

7.2.5 Proof of Proposition 7

By the dynamic programming optimality principle, we have, along an optimal path (here it is unique) for the legal firm, that

$$Z_l\left(0, R_0\right) = \int_0^{\mathcal{E}} \pi_1\left(R^{\mathcal{C}}(t;\mathcal{E}), a^{\mathcal{C}}(t;\mathcal{E}), p_l^{\mathcal{C}}(t;\mathcal{E})\right) dt + W_l(\mathcal{E}, R^{\mathcal{C}}(\mathcal{E};\mathcal{E})).$$

Notice that the optimal path $\left(R^{\mathcal{C}}(t;\mathcal{E}), a^{\mathcal{C}}(t;\mathcal{E}), p_l^{\mathcal{C}}(t;\mathcal{E})\right)$ a priori depends on $\mathcal{E}$.

Differentiating with respect to $\mathcal{E}$, we get

$$\frac{\partial Z_l(0,R_0;\mathcal{E})}{\partial\mathcal{E}} = \int_0^{\mathcal{E}} \left\{ \frac{\partial\pi_1}{\partial R}\frac{\partial R}{\partial\mathcal{E}} + \frac{\partial\pi_1}{\partial a}\frac{\partial a}{\partial\mathcal{E}} + \frac{\partial\pi_1}{\partial p_l}\left(\frac{\partial\pi_l}{\partial R}\frac{\partial R}{\partial\mathcal{E}} + \frac{\partial p_l}{\partial\mathcal{E}}\right)\right\} dt \tag{48}$$

$$+\pi_1\left(R^{\mathcal{C}}(\mathcal{E};\mathcal{E}), a^{\mathcal{C}}(\mathcal{E};\mathcal{E}), p_l^{\mathcal{C}}(\mathcal{E};\mathcal{E})\right) + \frac{\partial W_l}{\partial t}(\mathcal{E}; R^{\mathcal{C}}(\mathcal{E};\mathcal{E})) + \frac{\partial W_l}{\partial R}\left(\frac{\partial R}{\partial t}(\mathcal{E};\mathcal{E}) + \frac{\partial R}{\partial\mathcal{E}}(\mathcal{E};\mathcal{E})\right). \tag{49}$$

Now, by the Pontryagin maximum principle, there exists an adjoint variable $\lambda(t;\mathcal{E})$, such that, for all t in $[0,\mathcal{E}]$, the (unique) optimal path $\left(R^{\mathcal{C}}(t;\mathcal{E}), a^{\mathcal{C}}(t;\mathcal{E}), p_l^{\mathcal{C}}(t;\mathcal{E})\right)$ maximizes the Hamiltonian

$$\pi_1(R(t), a(t), p_l(t)) + \lambda(t)[ka(t) - \sigma R(t)].$$

Moreover the adjoint variable $\lambda(t)$ also satisfies

$$\dot{\lambda}(t;\mathcal{E}) = -\left(\frac{\partial\pi_1}{\partial R} - \sigma\lambda(t,\mathcal{E})\right),$$
$$\lambda(\mathcal{E};\mathcal{E}) = \frac{\partial W_l}{\partial R}(\mathcal{E}; R^{\mathcal{C}}(\mathcal{E};\mathcal{E})).$$

Therefore, the next conditions must hold at each date t:

$$\frac{\partial\pi_1}{\partial a} + \lambda(t;\mathcal{E})k = 0, \tag{50}$$

$$\frac{\partial\pi_1}{\partial p_l} = 0. \tag{51}$$

Following an argument in the proof of the Dynamic Envelope Theorem (Th. 9.1, pp 233) in Caputo (2005), we first differentiate the following dynamics equation:

$$\dot{R}(t,\mathcal{E}) = ka(t;\mathcal{E}) - \sigma R(t;\mathcal{E}),$$

with respect to $\mathcal{E}$ to obtain

$$\frac{\partial \dot{R}(t;\mathcal{E})}{\partial \mathcal{E}} = k\frac{\partial a(t;\mathcal{E})}{\partial \mathcal{E}} - \sigma\frac{\partial R(t,\mathcal{E})}{\partial \mathcal{E}}.$$

Let us now add the following quantity

$$\lambda(t,\mathcal{E})\left(k\frac{\partial a(t;\mathcal{E})}{\partial \mathcal{E}} - \sigma\frac{\partial R(t,\mathcal{E})}{\partial \mathcal{E}} - \frac{\partial \dot{R}(t;\mathcal{E})}{\partial \mathcal{E}}\right) = 0,$$

to the integrand of the integral in (48). Using (51) we get

$$\begin{aligned}\frac{\partial Z_l(0,R_0;\mathcal{E})}{\partial \mathcal{E}} = \int_0^{\mathcal{E}} \Bigg\{ &\frac{\partial \pi_1}{\partial R}\frac{\partial R}{\partial \mathcal{E}} + \frac{\partial \pi_1}{\partial a}\frac{\partial a}{\partial \mathcal{E}} \\ &+\lambda(t;\mathcal{E})\left(k\frac{\partial a(t;\mathcal{E})}{\partial \mathcal{E}} - \sigma\frac{\partial R(t,\mathcal{E})}{\partial \mathcal{E}} - \frac{\partial \dot{R}(t;\mathcal{E})}{\partial \mathcal{E}}\right)\Bigg\} dt \\ &+\pi_1\left(R^{\mathcal{C}}(\mathcal{E};\mathcal{E}), a^{\mathcal{C}}(\mathcal{E};\mathcal{E}), p_l^{\mathcal{C}}(\mathcal{E};\mathcal{E})\right) \\ &+\frac{\partial W_l}{\partial t}(\mathcal{E}; R(\mathcal{E};\mathcal{E})) + \frac{\partial W_l}{\partial R}\left(\frac{\partial R}{\partial t}(\mathcal{E};\mathcal{E}) + \frac{\partial R}{\partial \mathcal{E}}(\mathcal{E};\mathcal{E})\right)\end{aligned} \tag{52}$$

To simplify the above expression, we integrate

$$\int_0^{\mathcal{E}} \lambda(t;\mathcal{E})\frac{\partial \dot{R}(t;\mathcal{E})}{\partial \mathcal{E}} dt,$$

by parts to obtain

$$\int_0^{\mathcal{E}} \lambda(t,\mathcal{E})\frac{\partial \dot{R}(t;\mathcal{E})}{\partial \mathcal{E}} dt = \lambda(\mathcal{E};\mathcal{E})\frac{\partial R}{\partial \mathcal{E}}(\mathcal{E};\mathcal{E}) - \lambda(0;\mathcal{E})\frac{\partial R}{\partial \mathcal{E}}(0;\mathcal{E}) - \int_0^{\mathcal{E}} \dot{\lambda}(t,\mathcal{E})\frac{\partial R(t;\mathcal{E})}{\partial \mathcal{E}} dt.$$

We observe that: $\frac{\partial R}{\partial \mathcal{E}}(0;\mathcal{E}) = 0$. Substituting the above expression in (52) we get after a little algebra

$$\begin{aligned}\frac{\partial Z_l(0,R_0;\mathcal{E})}{\partial \mathcal{E}} = \int_0^{\mathcal{E}} &\left\{\left(\frac{\partial \pi_1}{\partial R} - \sigma\lambda(t;\mathcal{E}) + \dot{\lambda}(t;\mathcal{E})\right)\frac{\partial R}{\partial \mathcal{E}} + \left(\frac{\partial \pi_1}{\partial a} + k\lambda(t;\mathcal{E})\right)\frac{\partial a}{\partial \mathcal{E}}\right\} dt \\ &-\lambda(\mathcal{E};\mathcal{E})\frac{\partial R}{\partial \mathcal{E}}(\mathcal{E};\mathcal{E}) + \pi_1\left(R^{\mathcal{C}}(\mathcal{E};\mathcal{E}), a^{\mathcal{C}}(\mathcal{E};\mathcal{E}), p_l^{\mathcal{C}}(\mathcal{E};\mathcal{E})\right) \\ &+\frac{\partial W_l}{\partial t}(\mathcal{E}; R^{\mathcal{C}}(\mathcal{E};\mathcal{E})) + \frac{\partial W_l}{\partial R}\left(\frac{\partial R}{\partial t}(\mathcal{E};\mathcal{E}) + \frac{\partial R}{\partial \mathcal{E}}(\mathcal{E};\mathcal{E})\right).\end{aligned} \tag{53}$$

Using the Pontryagin maximum principle (and notably the fact that $\lambda(\mathcal{E};\mathcal{E}) = \frac{\partial W_l}{\partial R}(\mathcal{E}; R^{\mathcal{C}}(\mathcal{E};\mathcal{E}))$) the above expression reduces to

$$\frac{\partial Z_l(0,R_0;\mathcal{E})}{\partial \mathcal{E}} = \pi_1\left(R^{\mathcal{C}}(\mathcal{E};\mathcal{E}), a^{\mathcal{C}}(\mathcal{E};\mathcal{E}), p_l^{\mathcal{C}}(\mathcal{E};\mathcal{E})\right) + \frac{\partial W_l}{\partial t}(\mathcal{E}; R^{\mathcal{C}}(\mathcal{E};\mathcal{E})) + \frac{\partial W_l}{\partial R}\frac{\partial R}{\partial t}(\mathcal{E};\mathcal{E}).$$

Now, we use the Hamilton-Jacobi-Bellman equation, which holds at date $\mathcal{E}$, that is,

$$-\frac{\partial W_l(\mathcal{E}, R^{\mathcal{C}}(\mathcal{E};\mathcal{E}))}{\partial t} = \pi_2\left(R^{\mathcal{C}}(\mathcal{E};\mathcal{E}), a^{\mathcal{C}}(\mathcal{E};\mathcal{E}), p_l^{\mathcal{C}}(\mathcal{E};\mathcal{E})\right) + \frac{\partial W_l(\mathcal{E}; R^{\mathcal{C}}(\mathcal{E};\mathcal{E}))}{\partial R}\dot{R}(\mathcal{E};\mathcal{E}).$$

Substituting the above equation in equation (53) yields

$$\frac{\partial Z_l(0,R_0;\mathcal{E})}{\partial \mathcal{E}} = \pi_1\left(R^{\mathcal{C}}(\mathcal{E};\mathcal{E}), a^{\mathcal{C}}(\mathcal{E};\mathcal{E}), p_l^{\mathcal{C}}(\mathcal{E};\mathcal{E})\right) - \pi_2\left(R^{\mathcal{C}}(\mathcal{E};\mathcal{E}), a^{\mathcal{C}}(\mathcal{E};\mathcal{E}), p_l^{\mathcal{C}}(\mathcal{E};\mathcal{E})\right).$$

A more direct route consists in directly computing $\frac{\partial Z_l(0,R_0;\mathcal{E})}{\partial \mathcal{E}}$. Indeed, we have:

$$\begin{aligned}
\frac{\partial Z_l(0,R_0;\mathcal{E})}{\partial \mathcal{E}} &= \frac{1}{4\sigma\tilde{\beta}_l}\left(\Lambda'^{-\sigma\mathcal{E}} - \sigma\Lambda e^{-\sigma\mathcal{E}}\right)R_0 + \frac{k^2\tilde{\delta}_l^4}{32\sigma^2\omega\tilde{\beta}_l^2} + \frac{k^2\tilde{\delta}_l^2}{16\sigma^3\omega\tilde{\beta}_l^2}(\Lambda'\left(1-e^{-\sigma\mathcal{E}}\right) + \Lambda\sigma e^{-\sigma\mathcal{E}}) \\
&\quad + \frac{k^2}{64\sigma^3\omega\tilde{\beta}_l^2}\left(2\Lambda\Lambda'\left(1-e^{-2\sigma\mathcal{E}}\right) + \Lambda^2 2\sigma e^{-2\sigma\mathcal{E}}\right) \\
&\quad - \frac{k^2}{2\omega}\left(\Gamma + (s-\Gamma)e^{\sigma(\mathcal{E}-T)}\right)^2, \\
&= \frac{R_0}{4\sigma\tilde{\beta}_l}(\tilde{\delta}_l^2 - 4\sigma\tilde{\beta}_l\Gamma)e^{-\sigma\mathcal{E}} + \frac{k^2\tilde{\delta}_l^4}{32\sigma^2\omega\tilde{\beta}_l^2} + \frac{k^2\tilde{\delta}_l^2}{16\sigma^3\omega\tilde{\beta}_l^2}\left(\sigma(\Lambda+\tilde{\delta}_l^2 - 4\sigma\tilde{\beta}_l\Gamma) + \sigma(4\sigma\tilde{\beta}_l\Gamma - \tilde{\delta}_l^2)e^{-\sigma\mathcal{E}}\right), \\
&\quad + \frac{k^2\Lambda}{32\sigma^3\omega\tilde{\beta}_l^2}\left(\sigma(\Lambda+\tilde{\delta}_l^2 - 4\sigma\tilde{\beta}_l\Gamma) + \sigma(4\sigma\tilde{\beta}_l\Gamma - \tilde{\delta}_l^2)e^{-2\sigma\mathcal{E}}\right) - \frac{k^2}{32\sigma^2\omega\tilde{\beta}_l^2}(\Lambda+\tilde{\delta}_l^2)^2, \\
&= \frac{R_0}{4\sigma\tilde{\beta}_l}(\tilde{\delta}_l^2 - 4\sigma\tilde{\beta}_l\Gamma)e^{-\sigma\mathcal{E}} + \frac{k^2\tilde{\delta}_l^2}{16\sigma^3\omega\tilde{\beta}_l^2}\left(\sigma(\tilde{\delta}_l^2 - 4\sigma\tilde{\beta}_l\Gamma)(1-e^{-\sigma\mathcal{E}})\right) \\
&\quad + \frac{k^2\Lambda}{32\sigma^3\omega\tilde{\beta}_l^2}\left(\sigma(\tilde{\delta}_l^2 - 4\sigma\tilde{\beta}_l\Gamma)(1-e^{-2\sigma\mathcal{E}})\right), \\
&= \frac{R_0}{4\sigma\tilde{\beta}_l}(\tilde{\delta}_l^2 - 4\sigma\tilde{\beta}_l\Gamma)e^{-\sigma\mathcal{E}} \\
&\quad + \frac{k^2(\sigma(\tilde{\delta}_l^2 - 4\sigma\tilde{\beta}_l\Gamma)}{32\sigma^3\omega\tilde{\beta}_l^2}(4\sigma\tilde{\beta}_l\left(\Gamma + (s-\Gamma)\,e^{-\sigma(T-\mathcal{E})}\right)(1-e^{-2\sigma\mathcal{E}}) + \tilde{\delta}_l^2(1+e^{-2\sigma\mathcal{E}} - 2e^{-\sigma\mathcal{E}})) > 0.
\end{aligned}$$

7.2.6 Proof of Proposition 8

We must prove the statement for the two periods, that is, before and after entry of the counterfeiter.

During the interval $[0,\mathcal{E})$, the difference in advertising is given by

$$a^{\mathcal{N}}(t) - a_1^{\mathcal{C}}(t) = \frac{k\Delta}{\sigma\omega}\left(e^{\sigma(t-\mathcal{E})} - e^{\sigma(t-T)}\right) \geq 0.$$

During the interval $[\mathcal{E},T]$, the difference in advertising is given by

$$a^{\mathcal{N}}(t) - a_2^{\mathcal{C}}(t) = \frac{k\Delta}{\sigma\omega}\left(1 - e^{-\sigma(T-t)}\right) \geq 0.$$

7.2.7 Proof of Proposition 9

On $[0,\mathcal{E}]$ the difference in reputation is given by

$$R^{\mathcal{N}}(t) - R^{\mathcal{C}}(t) = \frac{k^2}{2\sigma^2\omega}\Delta\left(e^{-\sigma\mathcal{E}} - e^{-\sigma T}\right)\left(e^{\sigma t} - e^{-\sigma t}\right),$$

which is clearly always positive for all $t \in [0,\mathcal{E}]$.

To check that the difference in reputation is positive on $[\mathcal{E},T]$, we consider the following differential equations:

$$\begin{aligned}
\dot{R}^{\mathcal{N}}(t) &= ka^{\mathcal{N}}(t) - \sigma R^{\mathcal{N}}(t), \\
\dot{R}^{\mathcal{C}}(t) &= ka^{\mathcal{C}}(t) - \sigma R^{\mathcal{C}}(t),
\end{aligned}$$

with $R^{\mathcal{N}}(\mathcal{E}) > R^{\mathcal{C}}(\mathcal{E})$ from the above result. Moreover

$$a^{\mathcal{N}}(t) - a_2^{\mathcal{C}}(t) \geq 0,$$

from the previous proposition. Set $D(t) = R^{\mathcal{N}}(t) - R^{\mathcal{C}}(t)$ and $b(t) = a^{\mathcal{N}}(t) - a^{\mathcal{C}}(t)$, thus D satisfies

$$\begin{aligned} \dot{D}(t) &= kb(t) - \sigma D(t) \\ D(\mathcal{E}) &> 0 \end{aligned}$$

and $b(t) \geq 0$, so we have

$$D(t) = e^{-\sigma(t-\mathcal{E})} D(\mathcal{E}) + ke^{-\sigma t} \int_{\mathcal{E}}^{t} b(s) e^{\sigma s} ds.$$

Clearly $D(t) > 0$. Hence the result.

During the interval $[0, \mathcal{E})$, the difference in price is given by

$$p_l^{\mathcal{N}}\left(R^{\mathcal{N}}(t)\right) - p_{l1}^{\mathcal{C}}\left(t, R^{\mathcal{C}}(t)\right) = \frac{\tilde{\delta}_l}{2\tilde{\beta}_l}\left(\sqrt{R^{\mathcal{N}}(t)} - \sqrt{R^{\mathcal{C}}(t)}\right).$$

By the above result, $\sqrt{R^{\mathcal{N}}(t)} > \sqrt{R^{\mathcal{C}}(t)}$ and consequently, $p_l^{\mathcal{N}}\left(R^{\mathcal{N}}(t)\right) > p_{l1}^{\mathcal{C}}\left(t, R^{\mathcal{C}}(t)\right)$ for all $t \in [0, \mathcal{E})$.

During the interval $[\mathcal{E}, T]$, the difference in price is given by

$$p_l^{\mathcal{N}}(R(t)) - p_{l2}^{\mathcal{C}}(R(t)) = \frac{\tilde{\delta}_l}{2\tilde{\beta}_l}\sqrt{R^{\mathcal{N}}(t)} - \frac{2\beta_c\delta_l + \delta_c\gamma}{4\beta_c\beta_l - \gamma^2}\sqrt{R^{\mathcal{C}}(t)}.$$

Given that $\sqrt{R^{\mathcal{N}}(t)} > \sqrt{R^{\mathcal{C}}(t)}$ by the above result, to prove that $p_l^{\mathcal{N}}(R(t)) > p_{l2}^{\mathcal{C}}(R(t))$, it suffices to show that

$$\frac{\tilde{\delta}_l}{2\tilde{\beta}_l} > \frac{2\beta_c\delta_l + \delta_c\gamma}{4\beta_c\beta_l - \gamma^2}.$$

By Lemma 1, we have

$$\frac{\tilde{\delta}_l^2}{4\tilde{\beta}_l} > \beta_l\left(\frac{2\beta_c\delta_l + \delta_c\gamma}{4\beta_c\beta_l - \gamma^2}\right)^2 \Leftrightarrow \frac{\tilde{\delta}_l^2}{4\tilde{\beta}_l^2} > \frac{\beta_l}{\tilde{\beta}_l}\left(\frac{2\beta_c\delta_l + \delta_c\gamma}{4\beta_c\beta_l - \gamma^2}\right)^2.$$

Since $\tilde{\beta}_l < \beta_l$, the above inequality implies

$$\frac{\tilde{\delta}_l^2}{4\tilde{\beta}_l^2} > \left(\frac{2\beta_c\delta_l + \delta_c\gamma}{4\beta_c\beta_l - \gamma^2}\right)^2.$$

Taking the square root of both side yields

$$\frac{\tilde{\delta}_l}{4\tilde{\beta}_l} > \left(\frac{2\beta_c\delta_l + \delta_c\gamma}{4\beta_c\beta_l - \gamma^2}\right),$$

which concludes the proof.

7.2.8 Proof of Proposition 10

We have

$$W_l(t,R(t)) = \max_{p_{l2}(.),a_2(.)} \int_t^T \left(p_{l2}(h) \left(\delta_l \sqrt{R(h)} - \beta_l p_{l2}(h) + \gamma p_c(h) \right) - \frac{\omega}{2} a_2^2(h) \right) dh + sR(T), \quad (54)$$

$$\text{subject to (4) and } R^{\mathcal{C}}(t). \quad (55)$$

Let $p_{l2}^{\mathcal{C}}(t,R(t))$, $p_c^{\mathcal{C}}(t,R(t))$, $a_2^{\mathcal{C}}(t,R(t))$ be the feedback Nash equilibrium. Let also $R^{\mathcal{C}}(.)$ be the induced path of the legal firm's reputation. We can then compute the values of the sales given the value of $R^{\mathcal{C}}(.)$. Using our notations, we get

$$r_l^{\mathcal{C}}(h) = p_{l2}^{\mathcal{C}}(h) \left(\delta_l \sqrt{R^{\mathcal{C}}(h)} - \beta_l p_{l2}^{\mathcal{C}}(h) + \gamma p_c^{\mathcal{C}}(h) \right) \quad (56)$$

$$= \frac{\beta_l (2\beta_c \delta_l + \delta_c \gamma)^2}{(4\beta_c \beta_l - \gamma^2)^2} R^{\mathcal{C}}(h) \quad (57)$$

$$< \hat{r}_l^{\mathcal{N}}(h) \quad (58)$$

$$= p_l^{\mathcal{N}}(h) \left(\tilde{\delta}_l \sqrt{R^{\mathcal{C}}(h)} - \tilde{\beta}_l p_l^{\mathcal{N}}(h) \right) \quad (59)$$

$$= \frac{\tilde{\delta}_l^2}{4\tilde{\beta}_l} R^{\mathcal{C}}(h), \quad (60)$$

where $\hat{r}_l^{\mathcal{N}}(h)$ is the maximum value of the sales of the legal firm at date h *along* the reputation path chosen when there is counterfeiting. The above inequality implies that:

$$W_l(t,R(t)) = \int_t^T \left(p_{l2}^{\mathcal{C}}(h) \left(\delta_l \sqrt{R^{\mathcal{C}}(h)} - \beta_l p_{l2}^{\mathcal{C}}(h) + \gamma p_c^{\mathcal{C}}(h) \right) - \frac{\omega}{2} (a_2^{\mathcal{C}})^2(h) \right) dh + sR^{\mathcal{C}}(T), \quad (61)$$

$$< \int_t^T \left(\frac{\tilde{\delta}_l^2}{4\tilde{\beta}_l} R^{\mathcal{C}}(h) - \frac{\omega}{2} (a_2^{\mathcal{C}})^2(h) \right) dh + sR^{\mathcal{C}}(T) \quad (62)$$

But by definition of $V_l(t,R(t))$, we have

$$V_l(t,R(t)) = \max_{p_l(.),a(.)} \int_t^T \left(p_l(h) \left(\tilde{\delta}_l \sqrt{R(h)} - \tilde{\beta}_l p_l(h) \right) - \frac{\omega}{2} a^2(h) \right) dt + sR(T), \quad (63)$$

$$= \max_{a(.)} \int_t^T \left(\frac{\tilde{\delta}_l^2}{4\tilde{\beta}_l} R(h) - \frac{\omega}{2} a^2(h) \right) dt + sR(T), \quad (64)$$

Therefore $W_l(t,R(t)) < V_l(t,R(t))$.

7.2.9 Proof of Proposition 13

Notice that we can write

$$R^{\mathcal{C}}(t) = R_0 e^{-\sigma t} + \frac{G(t)}{\omega},$$
$$D(t) = R^{\mathcal{N}}(t) - R^{\mathcal{C}}(t) = \frac{F(t)}{\omega},$$

where G and F do not depend on ω.

Now let $z \in [\mathcal{E}, T]$ be the value at which F (and therefore D) reaches its maximum value on $[\mathcal{E}, T]$, and $y \in [\mathcal{E}, T]$ the value at which G reaches its minimum value on that interval. These values exist, since F and G are continuous on $[\mathcal{E}, T]$.

We have

$$\lim_{\omega \to +\infty} (R^{\mathcal{N}}(z) - R^{\mathcal{C}}(z)) = 0,$$

$$\lim_{\omega \to +\infty} R^{\mathcal{C}}(t) \geq \lim_{\omega \to +\infty} R_0 e^{-T} + \frac{G(y)}{\omega} \geq R_0 e^{-\sigma T} > 0.$$

Further, for all $t \in [\mathcal{E}, T]$, we have

$$\begin{aligned} \chi^{\mathcal{C}} R^{\mathcal{C}}(t) - \chi^{\mathcal{N}} R^{\mathcal{N}}(t) &= \left(\chi^{\mathcal{C}} - \chi^{\mathcal{N}}\right) R^{\mathcal{C}}(t) + \chi^{\mathcal{N}} \left(R^{\mathcal{C}}(t) - R^{\mathcal{N}}(t)\right) \\ &> \left(\chi^{\mathcal{C}} - \chi^{\mathcal{N}}\right) \left(R_0 e^{-T} + \frac{G(y)}{\omega}\right) + \chi^{\mathcal{N}} \left(R^{\mathcal{C}}(z) - R^{\mathcal{N}}(z)\right) \end{aligned}$$

which implies

$$\lim_{\omega \to +\infty} (\chi^{\mathcal{C}} R^{\mathcal{C}}(t) - \chi^{\mathcal{N}} R^{\mathcal{N}}(t)) > 0.$$

And so the proposition follows.

7.2.10 Proof of Proposition 14

Following the proof of Proposition 13 the condition

$$\int_0^T \chi^{\mathcal{N}} R^{\mathcal{N}}(t) dt < \int_0^{\mathcal{E}} \chi^{\mathcal{N}} R^{\mathcal{C}}(t) dt + \int_{\mathcal{E}}^T \chi^{\mathcal{C}} R^{\mathcal{C}} dt,$$

is satisfied whenever

$$\mathcal{E} \chi^{\mathcal{N}} \frac{\sigma_l^2}{8\kappa_l} \max_{t \in [0, \mathcal{E}]} \frac{F(t)}{\omega} < (T - \mathcal{E}) \left(\chi^{\mathcal{C}} - \chi^{\mathcal{N}}\right) R_0^{-\sigma T}.$$

This condition is indeed satisfied for ω higher than a certain threshold $\underline{\omega}'$.

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