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The distributive liberal social contract as definite norm of communicative action: A characterization through the Nash social welfare function

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Abstract: The distributive liberal social contract sets a norm of communicative action for the allocation and the redistribution of private wealth. It consists of activities that achieve a Pareto-efficient allocation of resources unanimously preferred to a hypothetical state of status quo. The status quo is the allocation of resources that would be obtained in the absence of the activities of the contract, under ideal conditions of perfect communication. We provide general sufficient conditions under which any allocation of the social contract maximizes some suitable Nash social welfare function in the set of attainable allocations. We show also that the correspondence between the allocations of the liberal social contract and the supporting Nash social welfare functions is one-to-one if, and in general only if, the distributive preferences of individuals are cardinal. In the latter case, the cooperative solution of Nash extended to any number of individuals provides a definite norm of communicative action for the allocation and the redistribution of private wealth.

Keywords: liberal social contract, redistribution, communicative action, Nash cooperative solution

JEL codes: D6, D7, H4

1-The liberal social contract as a norm of communicative action

The social contract that we consider in this article sets a norm for the allocation of scarce resources in a liberal democracy. It consists of a unanimous agreement about the activities of production, consumption and transfer of scarce resources, obtained under conditions designed to guarantee the validity of consent. These conditions are formulated below as applications of Habermas's principles of universalization (principle U) and discourse ethics (principle D) to this specific type of activities, that is: (i) "For a norm to be valid, the consequences and side effects that its general observance can be expected to have for the satisfaction of the particular interests of *each* person affected must be such that *all* affected can accept them freely" (principle U: Habermas (1990), p. 120); and (ii) "every valid norm would meet with the approval of all concerned if they could take part in a practical discourse" (principle D: Habermas (1990), p. 121). The liberal social contract is derived and interpreted here, in other words, as a norm of communicative action.

The activities regulated by the liberal social contract pertain, more specifically, in the present paper, to the allocation of private wealth, that is, to operations on the type of scarce resources that are owned and consumed by individuals, and whose consumption is an object of concern only for the individual consumer.[1] Such activities arouse common concerns typically through their consequences on the distribution (in both the practical and the statistical senses of the word) of individual wealth and welfare within the political society. Locke's justification of private property in chapter 5 of his *Second Treatise* (1690) gave a famous expression to common concerns of this type through his normative requirement that the appropriation of parcels of cultivable land by some leave "enough and as good" to all others. In the presence of widespread common concerns relative to the distribution of private wealth, the latter is formally analogous to a pure public good in the sense of contemporary economic theory (Kolm, 1966), the "production" or provision of which is one of the main objects of the variant of the liberal social contract that we consider here.

The liberal social contract selects activities that achieve a Pareto-efficient allocation of resources unanimously preferred to a hypothetical state of status quo. The status quo, or initial position or initial situation of the contract, is the allocation of resources that would be obtained in the absence of the activities of the contract, under ideal conditions of perfect communication. Mercier Ythier (2018) establishes that the maximization of a benevolent Nash social welfare function in the set of attainable allocations of the economy yields an allocation of the liberal social contract. The present article establishes a partial converse to this property, that is, we exhibit general sufficient conditions under which any allocation of the (distributive) liberal social contract maximizes some suitable Nash social welfare function in the set of attainable allocations. We show also that the correspondence between the allocations of the liberal social contract and the supporting Nash social welfare functions is one-to-one if, and in general only if, the distributive preferences of individuals are cardinal. In the latter case, the cooperative solution of Nash (1950) extended to any number of individuals provides a definite (i.e. unique) norm of communicative action for the allocation and the distribution of private wealth.

The paper is organized as follows. Section 2 derives the distributive liberal social contract from Habermas's principles U and D. The characterization of the corresponding norms of communicative action by means of the Nash social welfare function is performed in sections 3 and 4. Section 5 discusses the relevance of the normative solutions so obtained, in terms of the basic normative presuppositions of inclusiveness and procedural equality that they implement.

[1] Mercier Ythier (2018) extends the contract to the allocation of common scarce resources.

2-Liberal social contracts of competitive market economies with distributive externalities

The strategy that we retain here for developing an exact formulation of Habermas's principles U and D in the context of abstract processes of allocation of scarce resources relies on the distinction between two contrasted meanings of the notion of "approval" (or "acceptance") that the principles convey.

One meaning is simply considered agreement: the stakeholders acknowledge that, all things considered, the claims submitted to their appreciation are valid. Basic presuppositions for an agreement of this type are full inclusion, procedural fairness and non-coercion, that is, deliberation should be arranged in such a way that (i) all stakeholders can participate freely, (ii) with equal voice, and (iii) in so doing are exposed neither to coercion nor to deception (including self-deception).

The second meaning of approval is more narrowly connected to the conceptual apparatus of economic theory. A plan of action is approved of by stakeholders in this second sense if it is preferred by them in the following two complementary senses: (i) they all prefer the plan to status quo; and (ii) there is no other plan that all of them prefer to the approved plan.

The two notions of approval are jointly mobilized below in order to define first the norm of competitive market exchange, and next the norm of fair distribution of the liberal social contract.

2.1-Competitive market exchange as communicative action

We first translate principles U and D into the norm of competitive market exchange.

The theme of communicative action is, here, market exchange. It consists of a number of activities that individual owners are allowed to perform from the resources they own, namely, consumption, disposal and sale (the latter including gift, construed as the special case of a "sale" at null price). It may be further extended to the productive and commercial activities performed by privately owned firms.

Among these activities, those which are specifically "communicative", in the sense that they necessarily involve the consensual coordination of participants' plans, are individuals' sales and firms' activities.

Let us deal with individuals' sales first.

We consider an abstract process of market exchange of the type defined by Debreu as an economy of private property (1959, 5.5). It involves a fixed, finite number of private commodities, whose final destination in the process, if any, is their consumption by individuals. "Initially", that is, in the absence of the activities listed above (consumption, disposal, production and sale), they are available in fixed quantities privately owned by individuals. These commodities are private also in a second sense, namely, the consumption of any of them is an object of final concern only for the individual who consumes.

Formally, there are n individuals and l private commodities (at least two of them each), designated respectively by indices $i \in N = \{1,...,n\}$ and $h \in L = \{1,...,l\}$. ω_{ih} (resp. x_{ih}) denotes individual i's initial ownership (resp. final consumption) in private commodity h. Each individual i has well-defined, complete transitive preferences on his own final consumption of private commodities, represented by his private utility function u_i. Disposal is free, in the sense that any quantity disposed of is transferred to a pool of common resources that all individuals are free to consume, dispose of, or sell as they wish. We let $\omega_i = (\omega_{i1},...,\omega_{il}) \in \mathbb{R}^l_+$ and $x_i = (x_{i1},...,x_{il}) \in \mathbb{R}^l_+$.[2]

[2] We use the following standard notations for partial orderings of vectors of $\mathbb{R}^q$, $q \geq 2$:

The core of the exchange economy so outlined provides a first exact transcription, in the context of market exchange, of the second notion of approval distinguished above. Suppose for simplicity that quantities are perfectly divisible. The set of final outcomes accessible to any subset ("coalition") of individual owners $I \subset N$ may be described as $\left\{(x_i)_{i\in I} \in \mathbb{R}_+^{l.\#I} : \sum_{i\in I} x_i \leq \sum_{i\in I} \omega_i\right\}$, that is, the set of consumption allocations of its members that can be achieved from initial distribution $(\omega_i)_{i\in I}$ by means of voluntary transfers (barter or gifts) inside the coalition, or possibly also through disposal (i.e. voluntary transfers to "nature", that is, to the pool of common resources of the economy). The core of exchange economy $((u_i)_{i\in N}, (\omega_i)_{i\in N})$ then consists of the allocations x^* of $\left\{x \in \mathbb{R}_+^{l.n} : \sum_{i\in N} x_i \leq \sum_{i\in N} \omega_i\right\}$ that are unblocked in the sense that there exists no nonempty set of individual owners $I \subset N$ ("blocking coalition") and no consumption allocation in $\left\{(x_i)_{i\in I} \in \mathbb{R}_+^{l.\#I} : \sum_{i\in I} x_i \leq \sum_{i\in I} \omega_i\right\}$ such that $u_i(x_i) \geq u_i(x_i^*)$ for all $i \in I$, with a strict inequality for at least one individual. The core is therefore the set of allocations that achieve stakeholders' unanimous agreement, in the second sense that we gave to the notion of agreement: (i) stakeholders are individual owners, acting individually or in larger coalitions of traders, and ultimately concerned with their individual consumption of private commodities; (ii) any set of stakeholders I unanimously (weakly) prefers any allocation of the core to its status quo allocation $(\omega_i)_{i\in I}$; and (iii) no set of stakeholders can achieve a consumption allocation that all of its members prefer (and one of them strictly prefers) to an allocation of the core.

As is well-known, the core of a large exchange economy coincides with its set of competitive equilibria.[3] The economy is large, essentially, when individual demands and supplies have a null influence on equilibrium prices. The notion of large economy provides us, therefore, with a first natural formulation of the basic presupposition of equal voice, required for considered agreement in the context of market exchange: the partners in exchange have equal voice in the sense that none of them can individually influence the terms of trade through her/his quantitative demands and supplies of market commodities. Competitive equilibrium extends this basic feature of procedural fairness to exchange economies of any size if exchange partners freely endorse the norm of parametric prices, that is, if they freely, truly and truthfully renounce any manipulation of market prices through their individual or collective decisions of transfer and/or disposal of their initial resources.[4]

The characterization of competitive market equilibrium as a norm of communicative action is almost complete at this point. We still have to clarify the meaning of inclusiveness, non-coercion and non-deception in this context. This is done partially through the following assertoric propositions relative to the basic characteristics of the exchange economy $((u_i)_{i\in N}, (\omega_i)_{i\in N})$: (i) the demand function of each individual i reveals the consumption preferences represented by his utility function u_i; and (ii) all individuals agree that the initial distribution $(\omega_i)_{i\in N}$ is a distribution of legitimate property rights. Inclusiveness may be further

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$z = (z_1,...,z_q) \geq z' = (z'_1,...,z'_q)$ if $z_i \geq z'_i$ for all $i \in \{1,...,q\}$; $z > z'$ if $z \geq z'$ and $z \neq z'$; $z >> z'$ if $z_i > z'_i$ for all $i \in \{1,...,q\}$. We let: $\mathbb{R}_+^q = \left\{z \in \mathbb{R}^q : z \geq 0\right\}$; $\mathbb{R}_-^q = \left\{z \in \mathbb{R}^q : z \leq 0\right\}$; $\mathbb{R}_{++}^q = \left\{z \in \mathbb{R}^q : z >> 0\right\}$. If z and z' are vectors of $\mathbb{R}^q$, their inner product $z.z'$ is defined by $z.z' = \sum_{i=1}^{q} z_i.z'_i$.

[3] See Debreu and Scarf (1963), Aumann (1964) and the synthesis of Hildenbrand (1982).

[4] The manipulation of terms of trade through endowment transfers has been known as the transfer problem in economic theory since a famous controversy that opposed Keynes to Ohlin about the German war reparations after the first world war (1929).

guaranteed through the Lockean proviso, that is, the requirement that the property rights of $(\omega_i)_{i\in N}$ be distributed in such a way that any individual share ω_i leaves "enough and as good" for others.[5] Nevertheless, distributive issues are dealt with explicitly in the specification of the liberal social contract below, and we must therefore reserve to the latter a more complete formulation of the basic presupposition of inclusiveness that applies here.

The extension of this norm of communicative action to the for-profit production of private commodities is straightforward. In particular, individuals who endorse the norm of parametric prices and who own shares in firms' profits unanimously agree, in the second sense of the notion of agreement, that the firms they own should maximize their profits.

Formally, we consider a fixed finite number e of private firms (one at least), designated by index $j \in E = \{1,...,e\}$. The productive activities of each firm consume quantities of some private commodities (the "inputs" of the production process) in order to produce quantities of other private commodities (the "outputs") available for sale on the market or possibly also for disposal. The quantity of commodity h produced (resp. consumed or disposed of) by firm j is denoted by positive (resp. negative) real number y_{jh}. An activity of firm j is therefore described by a list $y_j = (y_{j1},..., y_{jl}) \in \mathbb{R}^l$, where the null components correspond to the commodities that do not appear in the process of production. The set of activities that are technically accessible to firm j is described by its production set Y_j. We suppose that non-activity is technically possible (i.e. $0 \in Y_j$) and that disposal is free (i.e. $\sum_{j\in E} Y_j \supset \mathbb{R}^l_-$).[6]

The profit (or loss) generated by activity y_j is the latter's money value, computed at market prices. Denoting by p_h the market price of commodity h, it is therefore $\sum_{h\in L} p_h.y_{jh}$. Any individual owner i of a private firm j is entitled to a definite share $\theta_{ij} \in [0,1]$ in j's profits, that is, individual i has a right to income $\theta_{ij}.\sum_{h\in L} p_h.y_{jh}$ if $\sum_{h\in L} p_h.y_{jh}$ is non-negative (and a symmetric obligation to pay $\theta_{ij}.\sum_{h\in L} p_h.y_{jh}$ in case of loss, i.e. in the case of a negative $\sum_{h\in L} p_h.y_{jh}$). His total wealth (resp. indirect utility), computed from the system of market prices $p = (p_1,..., p_l)$ and from production allocation $(y_1,..., y_e)$, reads then $r_i = p.\omega_i + \sum_{j\in E} \theta_{ij} p.y_j$ (resp. $v_i(p,r_i) = \sup\{u_i(x_i) : p.x_i \le r_i\}$). Individual i's indirect utility is non-decreasing in firm j's profit by definition. Subject to a mild assumption of local non-satiation of his consumption preferences, it is also strictly increasing in the latter whenever $v_i(p,r_i)$ is finite and $\theta_{ij} > 0$.[7] This implies stakeholders' unanimous agreement for profit

[5] Mercier Ythier (2018) develops an extensive, "all inclusive" interpretation of the Lockean proviso, including not only the norm of fair distribution above, but also non-coercion. This extensive interpretation was inspired notably by Gauthier's account of the proviso (1986), the destination of which is, in the variant of the liberal social contract that he considers, to set the best possible circumstances for getting unanimous agreement from the initial position. In Gauthier's game-theoretic construct, the proviso serves the purpose of creating the best prior conditions for cooperative interactions. We discuss extensively the differences between our interpretation of the proviso and Gauthier's in footnote 14 of the reference above. In the present paper, we are back to a more classical interpretation of Locke's text.

[6] $\sum_{j\in E} Y_j \supset \mathbb{R}^l_-$ means that it is technically possible, for the productive system as a whole, described through its aggregate production set $\sum_{j\in E} Y_j = \left\{\sum_{j\in E} y_j : y_j \in Y_j \text{ for all } j \in E\right\}$, to get rid of any quantity of any private commodity.

[7] In our setup, consumer i's preferences verify local non-satiation if, for all $x_i \in \mathbb{R}^l_+$ and all neighbourhood V of x_i in $\mathbb{R}^l_+$, there exists $x'_i \in \mathbb{R}^l_+$ such that $u_i(x'_i) > u_i(x_i)$. Suppose that $v_i(p,r_i)$ is finite, that $\theta_{ij} > 0$, and that i

maximization, that is, for the choice of a firm’s activity $y^*_j \in Y_j$ such that $p.y^*_j = \max\{p.y_j : y_j \in Y_j\}$ if any: (i) stakeholders are firm’s owners, ultimately concerned with their indirect utility; (ii) any set of stakeholders unanimously (weakly) prefers any profit-maximizing activity of the firm to non-activity (since $0 \in Y_j$); and (iii) there is no activity that all firm’s owners prefer, and at least one firm owner strictly prefers, to a profit-maximizing activity.[8]

Valid considered agreement relative to equity ownership is stated, finally, through the following assertoric proposition: all individuals agree that equity distribution $(\theta_{ij})_{(i,j)\in N\times E}$ is a distribution of legitimate property rights.

To summarize, we consider market economies of the type $((u_i)_{i\in N},(Y_j)_{j\in E},(\omega_i)_{i\in N},(\theta_{ij})_{(i,j)\in N\times E})$. A competitive equilibrium of $((u_i)_{i\in N},(Y_j)_{j\in E},(\omega_i)_{i\in N},(\theta_{ij})_{(i,j)\in N\times E})$ consists of a system of market prices $p \in \mathbb{R}^l$, a private consumption allocation $x = (x_1,...,x_n) \in \mathbb{R}^{nl}_+$ and a market production allocation $y = (y_1,...,y_e) \in \prod_{j\in E} Y_j$ such that: (i) $\sum_{i\in N}(x_i - \omega_i) = \sum_{j\in E} y_j$; (ii) y_j maximizes j’s profit over Y_j for all j; and (iii) x_i maximizes u_i over $\left\{x_i \in \mathbb{R}^l_+ : p.x_i \le p.\omega_i + \sum_{j\in E}\theta_{ij}p.y_j\right\}$ for all i. It makes a valid norm of communicative action in the field of market exchange if all individuals accept the property rights distributed through $(\omega_i)_{i\in N}$ and $(\theta_{ij})_{(i,j)\in N\times E}$ as legitimate.[9]

2.2-Distributive concerns: Private wealth as a public good

This norm of communicative action is a norm of commutative justice. It imposes only two types of limited conditions of distributive fairness, namely, that the initial distribution $((\omega_i)_{i\in N}, (\theta_{ij})_{(i,j)\in N\times E})$ should not be influenced by force or fraud, and that the final distribution of private wealth and welfare that one gets at market equilibrium should not be influenced by price manipulations. These conditions are not sufficient, in general, to guarantee that each and every individual enjoys a decent level of private wealth and welfare at competitive equilibrium.[10] It could be the case, for example, that an individual is unable to get a decent income on the labor market, because of his health condition, or because, for whatever reason, he didn’t achieve any suitable educational training.

’s wealth is increased to $r'_i > r_i$ due to a ceteris paribus increase in firm j’s profit. Local non-satiation then implies the existence of $x'_i \in \mathbb{R}^l_+$, which may be picked arbitrarily close to $x_i \in \mathbb{R}^l_+$, such that both $u_i(x'_i) > u_i(x_i)$ and $p.x'_i \le r'_i$. Therefore $v_i(p,r'_i) \ge u_i(x'_i) > v_i(p,r_i)$.

[8] Note, nevertheless, that the private ownership of productive capital can possibly undermine competitive pricing if it is very concentrated. Shareholders of a company that is large enough to exert market power, for example, on the price of its output (or of its labor inputs), confront a trade-off between the increased dividends they could get from an increase in the price of the firm’s product (or decrease in the firm’s wages) and the loss of purchasing power they could then suffer as consumers or as workers. The balance of effects will be positive, presumably, for large shareholders, so driving a wedge between shareholders’ interests according to their respective weights in the firm’s capital. The more capital ownership is concentrated, the less “spontaneous” unanimous agreement about competitive pricing appears plausible, in both senses of the notion of agreement that we distinguish in this paper.

[9] As is well-known, the demand functions used to define (and compute) a competitive equilibrium “reveal” individuals’ consumption preferences. Competitive market prices are “true”, in the sense that they are derived from the true and truthful consumption preferences of individuals.

[10] The (logical) fact that competitive equilibrium is compatible with large inequalities in the distribution of private wealth and welfare, including the extreme poverty of some individuals, is immediately apparent in an Edgeworth box.

Decency can be appreciated from the individual's perspective, by comparison with his own standards of expectation if any, or from a more collective perspective, by comparison with more or less extended common standards of personal achievement. It involves, in any case, a synthetic evaluation of the whole process of resource allocation, considered from the standpoint of individuals' achievements (i.e. whether each and every individual achieves a decent outcome or not). In terms of the basic presuppositions of discourse ethics, the violation of common standards of decent personal wealth and welfare can be analysed as a breach of inclusiveness. This is the case, conspicuously, when poverty is suffered and is so extreme as to result in premature death. This can be the case also in situations, much less easy to characterize, where poverty is attached to forms of social stigma, especially when it is inherited and thus becomes analogous to a form of long-lasting status inequality.[11]

Property rights that are legitimate in the sense that they were acquired through legitimate means may nevertheless result in an illegitimate distribution $((\omega_i)_{i\in N}, (\theta_{ij})_{(i,j)\in N\times E})$ if initial endowments do not leave "enough and as good" to some individuals, that is, if the corresponding competitive equilibrium fails to meet common standards of decent personal wealth and welfare for them. The present section applies principles U and D to the construction of a norm of communicative action for wealth redistribution.

The theme of communicative action is, now, the (final) distribution of private wealth and welfare. Formally, we suppose that individuals have complete, transitive preferences on the distribution of private welfare $u(x) = (u_1(x_1),...,u_n(x_n))$, represented by a utility function of the type $x \to w_i(u(x))$. Given market prices, these preferences induce preferences on the distribution of private wealth $r = (r_1,...,r_n)$, represented by utility function $r \to w_i(v_1(p,r_1),...,v_n(p,r_n))$. We call w_i the *distributive utility function* of individual i, and $w_i \circ u : x \to w_i(u(x))$ his *social utility function*.

Quite paradoxically, distributive concerns make private wealth a public good (or bad), as object of common concern (Kolm, 1966). This remark applies to each level of private welfare $u_i(x_i)$ or of private wealth r_i (provided that at least two individuals actually care about i's welfare, including presumably i himself), and to the whole distributions $(u_1(x_1),...,u_n(x_n))$ and $(r_1,...,r_n)$ as well. Moreover, any variation in the distribution of wealth is susceptible to induce variations in equilibrium market prices, which are themselves objects of common concern through indirect private utilities $v_i(p,r_n)$ as well as through distributive utilities $w_i(v_1(p,r_1),...,v_n(p,r_n))$. Put another way, the distribution of private wealth is a major source of externalities, through the distributive concerns of individuals, but also, still more pervasively perhaps, through the pecuniary externalities generated by redistribution. It must be characterized, therefore, as a pure public good, that is, as an object of universal common concern, inside the perimeter of the polity.[12]

[11] This difficult issue of the potentially stigmatizing effects of poverty as a relative phenomenon are discussed with subtlety in chapter 4 of Scanlon (2018). One reads notably on p. 37: "I would speculate that the economic inequality in our societies that causes status harms is mainly the inequality between people like me–successful educated professionals–and those who have less, especially, as I have noted above, those who are truly poor, and lack education. These effects may be reduced by the fact that we belong to non-comparing groups, but I doubt that they are totally erased".

[12] In spite of the general interconnection of market economies through the so-called "globalization", political societies maintain a substantial degree of autonomous existence, in terms of their ability to exert an influence on market prices and wealth distribution, notably through the existence of their own monetary institutions (i.e. their currency and their central bank, to begin with).

The activities regulated by the norm of communicative action that we construct are best construed, consequently, as redistributive taxation. Redistributive taxes must be lump-sum in order not to damage the (normative) process of competitive market exchange.[13] We assume also that the costs of administration of the tax system are null. This latter hypothesis requires special justification. It parallels the symmetric assumption of null transaction costs in the field of market exchange, implicit in the definition of the competitive equilibrium above. The two hypotheses participate in a general notion of perfect communication, which may be described metaphorically as "transparent" or "costless" communication. Put another way, communication such as construed here is *not* a productive or a consumptive *activity*, in the sense that, as a process, it does not consume any scarce resource. And the communication media (the "market" and the "State") are *not agents*, in the sense that they do not pursue any idiosyncratic objective. Communication through market prices and redistributive taxes is but an expression of the social contract as process and achievement, and the market and the State are but alternative names for the same thing in the subfields of private activities and public activities respectively.[14]

We denote by τ_i the lump-sum redistributive tax on individual i's private wealth. It corresponds to a tax proper if it is positive, to a subsidy if it is negative. Given τ_i, individual i's budget constraint reads $p.x_i \leq p.\omega_i + \sum_{j\in E} \theta_{ij} p.y_j - \tau_i$. From the definition of redistributive taxes and our assumption of null government costs, we deduce that $\sum_{i\in N} \tau_i = 0$. This equality means that the government's budget is balanced, as far as public redistribution is concerned. The consequences of lump-sum redistribution on competitive equilibrium are summarized through the following notion:

Definition 1: A *competitive equilibrium of* $((u_i)_{i\in N},(Y_j)_{j\in E},(\omega_i)_{i\in N},(\theta_{ij})_{(i,j)\in N\times E})$ *subject to system of lump-sum taxes* $\tau = (\tau_1,...,\tau_n)$ is a system of market prices $p \in \mathbb{R}^l$ and a market allocation $(x,y) \in \mathbb{R}_+^{nl} \times \prod_{j\in E} Y_j$ such that (i) $\sum_{i\in N}(x_i - \omega_i) = \sum_{j\in E} y_j$, (ii) y_j maximizes j's profit over Y_j for all j, and (iii) x_i maximizes u_i over $\left\{x_i \in \mathbb{R}_+^l : p.x_i \leq p.\omega_i + \sum_{j\in E} \theta_{ij} p.y_j - \tau_i\right\}$ for all i.

Finally, an allocation $(x,y) \in \mathbb{R}_+^{nl} \times \mathbb{R}^{el}$ is attainable (or accessible, or feasible) if it satisfies both firms' technical constraints and the aggregate resource constraint of the economy, that is, if $y_j \in Y_j$ for all $j \in E$ and $\sum_{i\in N}(x_i - \omega_i) = \sum_{j\in E} y_j$. The set of attainable allocations (or set of attainable states) of social system $(w,(u,(Y_j)_{j\in E},\omega,\theta))$ is defined, consequently, by $\left\{(x,y) \in \mathbb{R}_+^{nl} \times \prod_{j\in E} Y_j : \sum_{i\in N}(x_i - \omega_i) = \sum_{j\in E} y_j\right\}$. It is denoted by A.

We can now apply to the theme of wealth redistribution our basic pattern of construction of unanimous agreement, in the second sense of the notion of agreement. The stakeholders are the citizens, that is, the individual members of the polity. Status quo is competitive equilibrium

[13] Distortionary taxes are generally incompatible with the Pareto-efficiency of market equilibrium (they induce "dead-weight losses"). In other words, they perturb one of the major normative properties of perfect competition. Lump-sum taxes do not, as established in full generality by Samuelson (1947).

[14] Departure from this basic notion and assumption of perfect communication is the starting point of the influential streams of contemporary institutionalism initiated by the works and Coase (1937) and Buchanan and Tullock (1962). They derive their explanation of the existence of institutional *agents* (their "raison d'être et d'agir") from what may be called communication costs, namely, transactions costs in Coase's theory of the firm and the costs of collective decision-making in the theory of constitutional democracy of Buchanan and Tullock.

in the absence of redistributive taxation. A norm of communicative action relative to the distribution of private wealth and welfare consists of a system of redistributive taxes and an associate competitive equilibrium (i) that all citizens (weakly) prefer to status quo, and (ii) that is such that there is no attainable allocation that all citizens prefer and at least one citizen strictly prefers. It corresponds to the distributive liberal social contract relative to an initial situation, defined formally in Mercier Ythier (2010) and reformulated in Definitions 2 and 3 below.

We alleviate notations by setting $w=(w_i)_{i\in N}$, identified with $(w_1,...,w_n)$, $u=(u_i)_{i\in N}$, identified with $(u_1,...,u_n)$, $\omega=(\omega_i)_{i\in N}$, identified with $(\omega_1,...,\omega_n)$, and $\theta=(\theta_{ij})_{(i,j)\in N\times E}$. A social system is a list $(w,(u,(Y_j)_{j\in E},\omega,\theta))$ of fundamental characteristics relative to individual preferences, production techniques and endowments. List $(u_1,...,u_n)$ will be further identified below with function $u:\mathbb{R}_+^{nl}\to\mathbb{R}^n$ defined by $u(x)=(u_1(x_1),...,u_n(x_n))$ for any allocation $x=(x_1,...,x_n)$, that is, with the function that computes the distribution of private welfare from any given allocation. And list $(w_1\circ u,...,w_n\circ u)$ will be likewise identified with function $w\circ u:u(\mathbb{R}_+^{nl})\to\mathbb{R}^n$ defined by $w(u(x))=(w_1(u(x)),...,w_n(u(x)))$ for any allocation $x=(x_1,...,x_n)$, that is, with the function that computes the distribution of individual social welfare from any given allocation.

Definition 2: An initial situation of $(w,(u,(Y_j)_{j\in E},\omega,\theta))$ consists of a system of market prices p and an allocation (x,y) such that $(p,(x,y))$ is a competitive equilibrium of $(u,(Y_j)_{j\in E},\omega,\theta)$.

Definition 3: A liberal social contract of social system $(w,(u,(Y_j)_{j\in E},\omega,\theta))$ relative to initial situation $(p^0,(x^0,y^0))$ consists of a system of lump-sum taxes τ, and a competitive equilibrium $(p^*,(x^*,y^*))$ of $(u,(Y_j)_{j\in E},\omega,\theta)$ subject to τ such that: (i) $w_i(u(x^*))\geq w_i\left(u\left(x^0\right)\right)$ for all i ; and (ii) there exists no attainable allocation (x,y) of $(u,(Y_j)_{j\in E},\omega,\theta)$ such that $w_i(u(x))\geq w_i\left(u\left(x^*\right)\right)$ for all i and $w_i(u(x))>w_i\left(u\left(x^*\right)\right)$ for at least one i.

We complement this formal definition with an informal discussion of the basic presuppositions of inclusiveness, equal voice, non-coercion and non-deception.

Inclusiveness is implied by the formal definition, provided that the distributive preferences reported in w are true and truthful, and that they incorporate suitable common standards. The main issue in terms of inclusiveness, as far as distribution is concerned, is the satisfaction of the Lockean proviso. If individuals' distributive preferences truly and truthfully report their opinions relative to decent standards of living, then unanimous agreement in the second sense of the notion does imply the satisfaction of their common standards if any (that is, does imply that each and every individual gets "enough and as good" relative to such common standards).

Non-coercion is implied by the formal definition as well, if the competitive equilibrium of the initial position proceeds from legitimate property rights, that is, if all agree that initial endowments and equity shares have been acquired through legitimate ways or means.

The most difficult questions bear on equal voice. We established in Mercier Ythier (2010) that the allocations of the liberal social contract make a "large" set (a manifold) of dimension $n-1$ when the social contract requires some redistribution. These allocations being, by construction, Pareto-efficient relative to individuals' social preferences, the choice of anyone of them supposes a definite arbitrage between the conflicting interests of individuals as expressed through their social utility functions $x \to w_i(u(x))$. One may argue, in particular, that at most one of these allocations satisfies equal voice, for if such an allocation exists, then any move from it necessarily implies a breach of equality by favouring some (the better off) and penalizing others (the worse off) relative to the presupposed state of equal influence.[15]

It seems natural, at first sight, to try to solve this problem of abstract identification of "the" unique social contract that satisfies equal voice (if any) by replicating the procedure that proved successful in the case of market exchange, namely, coordination through competitive prices. The variant of Lindahl equilibrium developed in Mercier Ythier (2004) and (2010) would then provide a natural candidate.[16] Unfortunately, this is a (practical) dead-end, for the basic reason known as the preference revelation problem. Lindahl prices reflect individuals' marginal valuation of the public good. As such, they must be personalized, and they introduce incentives for individuals to report false preferences by understating their taste for the public good. As a consequence, we can neither assume that individuals' choices reveal their true preferences, nor suppose that prices are "competitive" (i.e. not manipulated). The remainder of this paper is devoted to the examination of this problem of characterization of the social contracts that satisfy equal voice relative to wealth redistribution.

3-The liberal social contract and Nash social welfare functions

In this section, we characterize the allocations of the distributive liberal social contract as maxima of suitable Nash social welfare functions in the set of attainable allocations of the economy.

The Nash social welfare functions that are relevant here are defined from the initial situation of the social system in the following way. Let $(p^0,(x^0,y^0))$ be an initial situation of $(w,(u,(Y_j)_{j\in E},\omega,\theta))$ (see Definition 2 above). The object of the social contract is to achieve an allocation that is both Pareto-efficient relative to individual social preferences and unanimously (weakly) preferred to the allocation of the initial situation. For any allocation (x,y) achieved cooperatively from (x^0,y^0), one may interpret $w_i(u(x))-w_i(u(x^0))$ as the cooperative surplus captured by individual i. The social welfare functions that we consider here evaluate such cooperative achievements by means of a definite geometric average of the individual surpluses they produce. Formally, let $\alpha=(\alpha_1,...,\alpha_n)\in\mathbb{R}^n_+$ be a fixed vector of non-negative "weights" adding up to 1. The geometric average of individual surpluses associated with these weights, computed at allocation (x,y), is the weighted product of individual surpluses $\Pi_{i\in N}(w_i(u(x))-w_i(u(x^0))^{\alpha_i}$. The (generalized) Nash social welfare function associated with α is function $\psi_{\alpha,w}:\mathbb{R}^{nl}_+\times\mathbb{R}^{el}\to\mathbb{R}_+$ defined by: (i) $\psi_{\alpha,w}(x,y)=\Pi_{i\in N}(w_i(u(x))-w_i(u(x^0))^{\alpha_i}$ if $w(u(x))\geq w(u(x^0))$; (ii) $\psi_{\alpha,w}(x,y)=0$ otherwise. We denote by $S_n=\left\{(\alpha_1,...,\alpha_n)\in\mathbb{R}^n_+:\sum_{i\in N}\alpha_i=1\right\}$ the unit simplex of $\mathbb{R}^n$, that is, the set of possible weight vectors.

[15] We ignore here, as inessential logical curiosities, the cases of unanimous indifference between two distinct allocations of the social contract.

[16] The dual core concept developed in Mercier Ythier (2004) bears some formal analogies with Nozick's utopian polity, developed in the last chapter of *Anarchy, State and Utopia* (1974) as an imaginative (and impracticable) extension of the basic norm of the minimal state that his volume promotes (see notably his formal framework, sketched on p. 301).

Theorem 2 of Mercier Ythier (2018) provides general sufficient conditions implying that a maximum of $\psi_{\alpha,w}$ in the set A of accessible allocations of the economy is an allocation of the liberal social contract. Leaving technicalities aside, these conditions, as far as wealth distribution is concerned, are essentially three: (i) an *inclusive* function $\psi_{\alpha,w}$, that is, positive weights α_i for all i; (ii) a *benevolent* function $\psi_{\alpha,w}$, that is, a social welfare $\Pi_{i\in N}(w_i(u(x))-w_i(u(x^0))^{\alpha_i}$ that is increasing in the private welfare $u_i(x_i)$ of each and every individual (whenever $\Pi_{i\in N}(w_i(u(x))-w_i(u(x^0))^{\alpha_i}>0$); and (iii) a *common standard of subsistence minimum*, embodied in the social preferences of all individuals.

Theorem 1 below establishes a partial converse. We provide a set of general sufficient conditions implying that, for any allocation (x,y) of the social contract, there exists a system of weights α in S_n such that (x,y) maximizes $\psi_{\alpha,w}$ in A. Leaving technicalities aside again, it is required here, essentially, that individual social preferences be convex. The hypotheses are detailed in Assumptions 1 and 2 and in Theorem 1 below.

Assumption 1: For all $i\in N$: (i) $u_i:\mathbb{R}^l\to\mathbb{R}$ is (a) strictly increasing[17], (b) strictly quasi-concave[18], and (c) continuous ; (ii) $w_i:u(\mathbb{R}^{nl})\to\mathbb{R}_+$ is (a) increasing in its i-th argument[19], and (b) continuous. (iii) For all $j\in E$: (a) $0\in Y_j$; and (b) Y_j is convex. (iv) $\sum_{j\in E}Y_j\supset\mathbb{R}^{nl}_-$.

Assumption 2: **Regular differentiable social system**: For all $i\in N$: (i) $u_i:\mathbb{R}^l\to\mathbb{R}$ is C^2; (ii) $w_i:u(\mathbb{R}^{nl})\to\mathbb{R}_+$ is C^2 in an open subset of $\mathbb{R}^n$ that contains $u(\mathbb{R}^n_+)$. (iii) $\partial\psi_{\alpha,w}(x^*,y^*)\neq 0$ for all $\alpha\in S_n$ and all $(x^*,y^*)\in\{(x,y)\in A: w(u(x))>>w(u(x^0))\}$. (iv) The aggregate production set reads $Y=\{z\in\mathbb{R}^l: F(z)\geq 0\}$, where $F=(F_1,...,F_m)$ is an m-dimensional transformation function $\mathbb{R}^l\to\mathbb{R}^m$ such that: For all $k\in\{1,...,m\}$, $F_k:\mathbb{R}^l\to\mathbb{R}$ is (a) C^2 and (b) quasi-concave; and (c) $\mu.\partial F(z)\neq 0$ for all $\mu\in\mathbb{R}^m_+$ such that $\mu>0$ and for all $z\in Y$.

Theorem 1: Let $(w,(u,(Y_j)_{j\in E},\omega,\theta))$ verify Assumption 1, (x^*,y^*) be an allocation of the distributive liberal social contract of $(w,(u,(Y_j)_{j\in E},\omega,\theta))$, and suppose moreover that: (i) either functions $x\to w_i(u(x))$ are concave in $\{x\in\mathbb{R}^{nl}:\exists y\in\mathbb{R}^{nl}$ such that $(x,y)\in A\}$ for all i; or (ii) functions $x\to w_i(u(x))$ are quasi-concave for all i, $(w,(u,(Y_j)_{j\in E},\omega,\theta))$ verifies Assumption 2, and $x^*>>0$. Then, there exists $\alpha\in S_n$ such that (x^*,y^*) maximizes $\psi_{\alpha,w}$ in A.

Note that the Theorem 2 of Mercier Ythier (2018) does not suppose the convexity of individual social preferences. Symmetrically, Theorem 1 above does not imply that the social

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[17] That is, $x_i>x'_i\Rightarrow u_i(x_i)>u_i(x'_i)$ for all $(x_i,x'_i)\in\mathbb{R}^l_+\times\mathbb{R}^l_+$.

[18] That is, $u_i(x_i)\geq u_i(x'_i)\Rightarrow u_i(\lambda x_i+(1-\lambda)x'_i)>u_i(x'_i)$ for all $(x_i,x'_i)\in\mathbb{R}^l_+\times\mathbb{R}^l_+$ such that $x_i\neq x'_i$ and all real number $\lambda\in]0,1[$.

[19] That is, $u_i(x_i)>u_i(x'_i)\Rightarrow w_i(u(x),\tilde{x})>w_i(u((x_{n/i},x'_i)),\tilde{x})$ for all $(x,x'_i)\in\mathbb{R}^{nl}_+\times\mathbb{R}^l_+$ such that $u(x)>>0$ and $u((x_{n/i},x'_i))>>0$ and for all $\tilde{x}\in\mathbb{R}^m_+$, where $(x_{n/i},x'_i)$ denotes the allocation of private commodities obtained from (x,x'_i) by substituting x'_i for x_i in x.

welfare function $\psi_{\alpha,w}$ that one gets as an outcome of the existence property is inclusive, or benevolent. And the proof does not make use of any common standard of subsistence minimum.

4-Cardinal utilities and the determinacy of the distributive liberal social contract

We briefly establish in Theorem 2 below that the correspondence between social contract allocations and weight vectors is exact if (and in general only if) individual social utilities are cardinal. By an exact correspondence, we mean that the set of weight vectors supporting any given social contract allocation is invariant to increasing affine transformations of individuals' social utility functions.

We use the following standard definitions: a property attached to $(w_1,...,w_n)$ is *invariant* to increasing affine transformations of distributive utility functions if it holds true for $(a_1.w_1+b_1,...,a_n.w_n+b_n)$ for all $(a_1,...,a_n)\in\mathbb{R}^n_{++}$ and all $(b_1,...,b_n)\in\mathbb{R}^n$; it is invariant to *any* (strictly) increasing transformations of distributive utility functions if it holds true for $(f_1\circ w_1,...,f_n\circ w_n)$ for all n-tuple $(f_1,...,f_n)$ of increasing functions $\mathbb{R}\to\mathbb{R}$; and weight vector α *supports* social contract allocation (x^*,y^*) if (x^*,y^*) maximizes $\psi_{\alpha,w}$ in A. We denote by L_w the set of allocations of the distributive liberal social contract of $(w,(u,(Y_j)_{j\in E},\omega,\theta))$ relative to (fixed) initial situation $(p^0,(x^0,y^0))$, and by $\mathrm{A}_w(x,y)$ the set of supporting weight vectors of $(x,y)\in L_w$.

Theorem 2: Let $w'=(f_1\circ w_1,...,f_n\circ w_n)$ for some n-tuple $(f_1,...,f_n)$ of increasing functions $\mathbb{R}\to\mathbb{R}$. Then: (i) $L_{w'}=L_w$; (ii) if, and in general only if, the functions of $(f_1,...,f_n)$ are affine, one gets $\mathrm{A}_{w'}(x,y)=\mathrm{A}_w(x,y)$ for all $(x,y)\in L_{w'}=L_w$.

Put another way, the *distributive liberal social contract* is an *ordinal* notion, in the sense that the associate set of social contract allocations is invariant to arbitrary increasing transformations of individuals' distributive (hence social)[20] utility functions. Whereas the *Nash social welfare function* $\psi_{\alpha,w}$ is a *cardinal* notion, in the sense that (i) the set of weight vectors supporting any given social contract allocation is invariant to increasing *affine* transformations of distributive utility functions, but (ii) is *not*, in general, invariant to non-affine increasing transformations.

If individual social utilities are cardinal, that is, if they are defined up to an increasing affine transformation, we get a natural candidate for an allocation of the liberal social contract that satisfies the basic presupposition of *equal voice*, namely, an attainable allocation that maximizes the Nash social welfare function that attributes *equal weights* to all individuals (i.e. that maximizes $\psi_{\alpha,w}$ such that $\alpha_i=1/n$ for all i).

Interestingly enough, we end up here with a solution that shares several important common features with Nash's well-known cooperative solution to the bargaining problem (1950). Nash derives his solution in a two-player setup from a set of axioms which include notably: (i) cardinal expected utilities of von Neumann-Morgenstern u_i, $i\in\{1,2\}$; (ii) a set S of attainable profiles of utility levels (u_1,u_2) that is compact, and convex; (iii) a set of (weakly)

[20] The cardinality of *distributive* utility functions w_i implies the cardinality of *social* utility functions $w_i\circ u$. It neither implies nor requires the cardinality of private utility functions u_i.

Pareto-efficient solutions $c(S)$ that verifies a property of stability by contraction of the opportunity set (i.e. $T \supset S$ and $c(T) \subset S$ together imply $c(S) = c(T)$); and finally (iv) a symmetry property of the solution, interpreted in terms of *equality of bargaining powers* of the two players (precisely: if S is symmetric, in the sense that $(u_1, u_2) \in S$ implies $(u_2, u_1) \in S$, then any solution $(u_1, u_2) \in c(S)$ must be such that $u_1 = u_2$). Status quo is the utility profile $(0,0) \in S$, and the solution is characterized as the (weak) Pareto-optimum that maximizes the product $u_1.u_2$ of individual utility levels. This construction, obtained by logical deduction, shares with our own, obtained through quite different means, at least four significant characteristic features: (i) cardinal utilities; (ii) a solution that is Pareto-efficient and unanimously preferred to status quo; (iii) a solution that maximizes the product of individuals' surpluses from cooperation; and (iv) a form of equality in stakeholders' influence ("bargaining power" and "voice" respectively).

5-Inclusiveness, equal voice and the definiteness of the distributive liberal social contract

In this last section, we return on the interpretation of the distributive liberal social contract in terms of the basic presuppositions of equal voice and inclusiveness. We focus on wealth distribution, and examine successively the procedural and the substantive aspects of the norm.

The procedural features are concentrated mainly in what we called the second notion of approval or agreement.

Inclusiveness translates there into stakeholders' unanimity, that is, in the requirement that all concerned prefer the chosen distribution to the distribution of the initial position, and in the subsidiary requirement that the chosen distribution be (strongly) Pareto-efficient relative to individuals' social preferences. Stakeholders' unanimity is full unanimity here, since the distribution of private wealth is a pure public good.

Procedural equality (i.e. equal voice) admits of two distinct interpretations at the same level, according to the methodological choice relative to the measurement of distributive welfare.

If distributive preferences are ordinal, so is the social contract. Equal voice is then reduced to an aspect of the rule of unanimity, namely, the possibility given to each and every individual to veto any proposal of redistribution by declaring that s/he prefers status quo (i.e. the distribution of the initial position). In other words, the unanimity rule implies equal individual veto rights, which is a clear case of equal voice, turned towards the conservative dimension of the deliberation process (the preference for status quo). The norm as specified in this paper is mute, in terms of procedural equality, on the progressive (i.e. active) side of the deliberation process. In the absence of procedural or substantive complements, the abstract norm typically maintains some degree of fundamental indeterminacy concerning the outcome of the deliberation process, except in the trivial, essentially uninteresting case where status quo is Pareto-efficient (i.e. where any redistribution from the initial position is vetoed by some).

With cardinal distributive utilities, equal voice translates into Nash's cooperative solution extended to any number of individuals. The abstract norm then yields a determinate, typically unique outcome. The difference with the ordinal variant should not be overstated, nevertheless. The claim that the abstract norm yields a valid definite outcome irrespective of the substantive preferences conveyed by distributive utilities seems prima facie exorbitant. We show below that this is actually the case. The argument relying principally on the consideration of the substantive (as distinguished from procedural) aspects of redistribution, I will only provide here a simple outlook of the conclusion relative to procedural equality. We argue in the paragraphs below that Nash's cooperative solution should be interpreted as a default solution (valid as such only if distributive utilities are cardinal, naturally). By this we mean that the individuals' surpluses should have equal weights in the social welfare function, *unless deliberation elicits some valid claims to the contrary*. Such claims would arise, typically, from the consideration that the initial position exerts an undue influence on the default solution, and that this influence could and should be corrected by selecting alternative (unequal) weights.

We now turn to the discussion of the substantive issues of social contract redistribution.

We recalled above that the distribution of private wealth (or welfare) is a pure public good, as an object of universal (distributive and/or pecuniary) common concerns. The distributive liberal social contract elicits and achieves individuals' common standards relative to this public good, if any. The abstract norm defined here, viewed as an ordinal notion, achieves a definite outcome in two types of substantive configurations only.

We briefly introduced one of them above, namely, the configuration where status quo is a distributive (Pareto) optimum. This happens either because individuals have no common distributive standard at all, or because their common standards are already satisfied at the initial position (everybody getting "enough and as good" from the beginning). The distributive social contract is pointless then, either because distributive preferences require no specific action, or because they leave no room for consensual action. In the first case, we get the trivial "conservative" solution to the substantive issue of redistribution, namely, consensual status quo: the norm of communicative action does not require any redistribution. In the second case, there is no agreement at all, that is, agreement in the formal (second) sense of the notion is incompatible with agreement in the first sense, notably because it contradicts any defensible substantive interpretation of the basic presupposition of inclusiveness: the norm of communicative action simply does not exist.

We now examine, at the opposite extreme, the configuration of exact unanimity. By this we mean a situation where all individuals have the same distributive preferences, conveying the same common distributive standards. It could be, for example, the conservative interpretation of the Lockean proviso in Nozick's minimal state (1974). Or, at the other end of the spectrum of the liberal democratic conceptions of distributive justice, Rawls's justice as fairness (1971). Or also, somewhere in-between the former two (but much closer to Rawls's than to Nozick's), Scanlon's egalitarian view that inequalities can be accepted only if either they "could not be eliminated without infringing important personal liberties" or "they are required in order for the economic system to function in a way that benefits all" (2018, p. 141). Or, alternatively, Kolm's "equal labor income equalization" (2004).[21] Let $\tilde{w}$ denote a distributive utility function, the same for all individuals, which captures anyone of the four substantive conceptions of distributive justice just outlined (or any other liberal democratic conception as well). The allocations of the distributive liberal social contract then reduce to the maxima of $\tilde{w} \circ u$ in the set of attainable allocations of the economy (i.e. in set A). They are also, equivalently, the maxima of $x \to \Pi_{i \in N}(\tilde{w}(u(x)) - \tilde{w}(u(x^0))^{1/n}$ in A. That is, the ordinal and cardinal versions of the norm merge into one and the same thing, *and the corresponding set of allocations is independent from the initial position*.

In intermediary configurations between the former two, involving both the Pareto-inefficiency of the initial position and some diversity in individual tastes for redistribution, the ordinal solution is typically indeterminate.[22] It can be characterized as the portion of the contract

[21] In Kolm's solution, the common standard is formulated in terms of labor time and income. In its simplest formulation ("equal duration income equalization"), a definite time of productive services is (notionally) owed by individuals over the lifecycle, and anyone who performs this labor time is paid on the basis of its average money value (labor duration times the average value of productive services per unit of time). Labor income is so equalized, for this definite, conventional common (and equal) labor time.

[22] See Mercier Ythier (2010) for a precise formulation, including formal assumptions of diversity of distributive preferences, their substantive interpretation and their discussion through characteristic examples.

locus (i.e. subset of distributive Pareto optima) contained in the Pareto lens anchored at the initial position.[23] It depends, therefore, on the initial position, in the sense that any variation of the latter typically induces variations in the solution set (by displacing the anchorage point of the Pareto lens). So does the cardinal solution, as the (typically unique) allocation that maximizes the average surplus $\Pi_{i\in N}(w_i(u(x))-w_i(u(x^0))^{1/n}$ generated by ordinal solutions. Their common dependence on the initial position exposes both the ordinal and the cardinal solutions to a criticism of *arbitrariness*, pointing out, in particular, some degree of inevitable arbitrariness in the initial distribution of rights (ω,θ), and also in the choice of a definite initial situation $(p^0,(x^0,y^0))$ when economy $(u,(Y_j)_{j\in E},\omega,\theta)$ admits of several stable competitive equilibria. In the ordinal framework, the stakes of this final aspect of overall deliberation about distribution consist of the choice of a definite solution in a large set of admissible solutions; and deliberation itself consists of the confrontation of claims and arguments about this definite choice. In the cardinal framework, the same closing step of deliberation can be formulated as the confrontation of arguments and claims relative to suitable definite deviations from the equal weights of the default Nash's solution. In both deliberative frameworks, this closure of the deliberation process supposes that an agreement be reached on a *definite* outcome, taking due consideration of the fact that deliberation proceeds from a *definite, thereby arbitrary, initial state of rights and position*. It must be an agreement in the first sense of the notion, that is, it must consist of the unanimous approval of choice, as object and as process, from considered individual appraisals built into a definite context. The last word about wealth redistribution, if any, normally belongs, in this normative framework, to individuals' reflective judgment confronting flat contingency. It must proceed, in other words, from their practical judgement in definite public choice situation.

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[23] See Figure 1 of Mercier Ythier (2010) for a geometric representation.

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Appendix: Proofs

Proof of Theorem 1: Let $(\tau,(p^*,(x^*,y^*)))$ be a liberal social contract of social system $(w,(u,(Y_j)_{j\in E},\omega,\theta))$ relative to initial situation $(p^0,(x^0,y^0))$, and suppose that $w(u(x^*)) >> w(u(x^0))$. We have to prove that there exists $\alpha \in S_n$ such that (x^*,y^*) maximizes $\psi_{\alpha,w}$ in A.

The canonical projection $(x,y)\to x$ is denoted by π. Notice that: set A is convex by convexity of production sets (Assumption 1-(iii)-(b)); and therefore $\pi(A)$ is convex by convexity of A and linearity of π.

(i) Suppose first that functions $x\to w_i(u(x))$ are concave in $\pi(A)$ for all i. For any i, define function $\varphi_i:\mathbb{R}^{nl}\to[-\infty,+\infty[$ such that: $\varphi_i(x)=\ln(w_i(u(x))-w_i(u(x^0)))$ whenever $x\in\mathbb{R}^{nl}$ is such that $w_i(u(x))>w_i(u(x^0))$; $\varphi_i(x)=-\infty$ otherwise, that is, if $w_i(u(x))\le w_i(u(x^0))$. Functions φ_i are concave in $\pi(A)$ as concave transformations of concave functions $x\to w_i(u(x))$. Let $\varphi=(\varphi_1,...,\varphi_n)$, and define set $S=\{\varphi(x):x\in\pi(A)\}$. S is the set of accessible vectors of logarithms of individual surpluses $\ln(w_i(u(x))-w_i(u(x^0)))$. Notice that if $\varphi(x)\in\mathbb{R}^n$, then $(x,y)\in A$ is Pareto-efficient relative to individual social utilities if and only if $\left[\{\varphi(x)\}+\mathbb{R}^n_+\right]\cap S=\{\varphi(x)\}$. Notice also that $w(u(x^*)) >> w(u(x^0))$ implies $\varphi_i(x^*)>-\infty$ for all i (i.e. $\varphi(x^*)\in\mathbb{R}^n$).

Denote by $\operatorname{co} S$ the convex hull of S (i.e. the smallest convex set containing S). We first show that $\left[\{\varphi(x^*)\}+\mathbb{R}^n_+\right]\cap\operatorname{co} S=\{\varphi(x^*)\}$. Clearly, $\varphi(x^*)\in\left[\{\varphi(x^*)\}+\mathbb{R}^n_+\right]\cap\operatorname{co} S$. Conversely, let $z\in\operatorname{co} S$, suppose that $z>\varphi(x^*)$, and let us derive a contradiction from this assumption. By definition of $\operatorname{co} S$, there exists some real number $\lambda\in[0,1]$ and some pair $(x,x')\in\pi(A)\times\pi(A)$ such that $z=\lambda\varphi(x)+(1-\lambda)\varphi(x')$. The concavity of functions φ_i then implies $\varphi(\lambda x+(1-\lambda)x')\ge\lambda\varphi(x)+(1-\lambda)\varphi(x')$, and therefore $w(u(\lambda x+(1-\lambda)x'))>w(u(x^*))$; and the convexity of $\pi(A)$ implies $\lambda x+(1-\lambda)x'\in\pi(A)$, which contradicts the Pareto efficiency of (x^*,y^*).

We have proved so far that $\left[\{\varphi(x^*)\}+\mathbb{R}^n_+\right]\cap\operatorname{co} S=\{\varphi(x^*)\}$, or equivalently that $\{\varphi(x^*)+z:z\in\mathbb{R}^n_+,z>0\}\cap\operatorname{co} S$ is empty. Sets $\{\varphi(x^*)+z:z\in\mathbb{R}^n_+,z>0\}$ and $\operatorname{co} S$ being both convex, the separating hyperplane theorem implies that there is $\alpha\ne 0$ in $\mathbb{R}^n$ and $c\in\mathbb{R}$ such that $\alpha.\{\varphi(x^*)+z:z\in\mathbb{R}^n_+,z>0\}\ge c\ge\alpha.\operatorname{co} S$. We must have $\alpha\ge 0$, for otherwise $\operatorname{Inf}\alpha.\{\varphi(x^*)+z:z\in\mathbb{R}^n_+,z>0\}=-\infty$. We can suppose that $\alpha\in S_n$ without loss of generality (if $\alpha\notin S_n$, replace it by $\alpha/\sum_{i\in N}\alpha_i$ and replace c by $c/\sum_{i\in N}\alpha_i$ in inequalities above).

$\left[\{\varphi(x^*)\}+\mathbb{R}^n_+\right]\cap\operatorname{co} S=\{\varphi(x^*)\}$ and $\alpha.\{\varphi(x^*)+z:z\in\mathbb{R}^n_+,z>0\}\ge c\ge\alpha.\operatorname{co} S$ together imply that $\alpha.\varphi(x^*)=c$. In particular, x^* maximizes $x\to\alpha.\varphi(x)$ in $\pi(A)$. Since

$\alpha.\varphi(x) = \ln \psi_{\alpha,w}(x,y)$ for all (x,y) (where $\ln$ is extended to 0 in the usual way) and function $\ln$ is strictly increasing, (x^*, y^*) maximizes $\psi_{\alpha,w}$ in A.

(ii) Suppose next that $(w,(u,(Y_j)_{j\in E},\omega,\theta))$ is a differentiable social system of the type described by Assumption 2, that functions $x \to w_i(u(x))$ are quasi-concave, and that $x^* >> 0$.

Let $X = \{(x,y) \in \mathbb{R}^{nl} \times \mathbb{R}^{el} : w(u(x)) \geq w(u(x^0))\}$, and denote by $\tilde{\varphi}_i$ function $(x,y) \to \varphi_i(x)$. $\tilde{\varphi}_i$ is a function $\mathbb{R}^{nl} \times \mathbb{R}^{el} \to [-\infty,+\infty[$. It is quasi-concave, as a simple concave transformation of quasi-concave functions. We have $\operatorname{Int} X = \{(x,y) \in \mathbb{R}^{nl} \times \mathbb{R}^{el} : w(u(x)) >> w(u(x^0))\}$ by continuity of utility functions (Assumption 1). $\tilde{\varphi}_i$ is C^2 in $\operatorname{Int} X$ by Assumption 2-(i) and -(ii). We let $\tilde{\varphi} = (\tilde{\varphi}_1,...,\tilde{\varphi}_n)$.

Subject to free disposal (i.e. Assumption 1-(iv)) and Assumption 2-(iii), set A reads $\{(x,y) \in \mathbb{R}^{nl} \times \mathbb{R}^{el} : x \geq 0, \sum_{i\in N}(\omega_i - x_i) + \sum_{j\in E} y_j \geq 0 \text{ and } F(\sum_{j\in E} y_j) \geq 0\}$.

We first show that (x^*, y^*) is a weak maximum of $\tilde{\varphi}$ in A, as a simple consequence of the definition of a liberal social contract. Suppose the contrary, that is, suppose that there exists $(x,y) \in A$ such that $\tilde{\varphi}(x,y) >> \tilde{\varphi}(x^*, y^*)$. This readily implies that $w(u(x)) >> w(u(x^*))$. (x,y) being accessible by construction, the latter inequalities imply in turn that (x^*, y^*) is not Pareto-efficient relative to individuals' social preferences, and therefore that (x^*, y^*) is not an allocation of the liberal social contract.

It results from assumption $x^* >> 0$ and from the former two paragraphs that (x^*, y^*) is a maximum of $\tilde{\varphi}$ in $\operatorname{Int} X$, subject to the following $l+m$-dimensional system of constraints $\mathbb{R}^{nl} \times \mathbb{R}^{el} \to \mathbb{R}$: (i) $\sum_{i\in N}(\omega_i - x_i) + \sum_{j\in E} y_j \geq 0$; and (ii) $F(\sum_{j\in E} y_j) \geq 0$.

The linear constraint functions $(x,y) \to \sum_{i\in N}(\omega_i - x_{ih}) + \sum_{j\in E} y_{jh}$, $h \in \{1,...,l\}$, are clearly C^∞ and quasi-concave in $\mathbb{R}^{nl} \times \mathbb{R}^{el}$. Constraint functions $(x,y) \to F_k(\sum_{j\in E} y_j)$, $k \in \{1,...,m\}$, are C^2 and quasi-concave in $\mathbb{R}^{nl} \times \mathbb{R}^{el}$ by Assumption 2-(iv)-(a) and -(b).

The differentiability properties of the objective and constraint functions imply that (x^*, y^*) verifies the necessary first-order conditions (f.o.c.) for a local weak maximum of the program above (e.g. Mas-Colell (1985), D.3). That is, there exists $(\lambda, p, \mu) \in \mathbb{R}^n_+ \times \mathbb{R}^l_+ \times \mathbb{R}^m_+$ such that: (i) $(\lambda, p, \mu) \neq 0$; (ii) $\lambda.\partial_{x_i}\tilde{\varphi}(x^*, y^*) = p = -\mu.\partial F(\sum_{j\in E} y_j^*)$ for all i; and (iii) $p.\left(\sum_{i\in N}(\omega_i - x_i^*) + \sum_{j\in E} y_j^*\right) = 0$ and $\mu.F(\sum_{j\in E} y_j^*) = 0$.

Note that $\lambda = 0$ implies $0 = p = -\mu.\partial F(\sum_{j\in E} y_j^*)$ by f.o.c. (ii). Assumption 2-(iv)-(c) then implies that $\mu = 0$. But $\lambda = p = \mu = 0$ contradicts f.o.c. (i). Therefore $\lambda > 0$ in the necessary first-order conditions of this program. We can suppose without loss of generality that $\lambda \in S_n$ (if $\lambda \notin S_n$, replace (λ, p, μ) by $\frac{1}{\sum_{i\in N}\lambda_i}.(\lambda, p, \mu)$ in the f.o.c. above). The f.o.c. above imply a system of Kuhn and Tucker first-order conditions for a local maximum of $(x,y) \to \sum_{i\in N}\lambda_i.\tilde{\varphi}_i(x,y)$ in $\{(x,y) \in \operatorname{Int} X : \sum_{i\in N}(\omega_i - x_i) + \sum_{j\in E} y_j \geq 0 \text{ and } F(\sum_{j\in E} y_j) \geq 0\}$

That is, there exists $(p,\mu) \in \mathbb{R}^l_+ \times \mathbb{R}^m_+$ such that: (i) $\lambda.\partial_{x_i}\tilde{\varphi}(x^*, y^*) = p = -\mu.\partial F(\sum_{j\in E} y_j^*)$ for all $i \in N$; and (ii) $p.\left(\sum_{i\in N}(\omega_i - x_i^*) + \sum_{j\in E} y_j^*\right) = 0$ and $\mu.F(\sum_{j\in E} y_j^*) = 0$.

$x^* >> 0$, Assumption 2-(iii) and the differentiability and quasi-concavity properties of objective and constraint functions then imply that (x^*, y^*) is a global maximizer of $(x,y) \to \sum_{i\in N} \lambda_i.\tilde{\varphi}_i(x,y)$ in $\left\{(x,y) \in \text{Int } X : \sum_{i\in N}(\omega_i - x_i) + \sum_{j\in E} y_j \geq 0 \text{ and } F(\sum_{j\in E} y_j) \geq 0\right\}$ by the proof of Theorem 1-(a) and -(b) of Arrow and Enthoven (1962).[24] We can conclude as in the last sentence of part (i) of the proof above, by setting $\alpha = \lambda$.■

Proof of Theorem 2: The Theorem is a simple consequence of definitions. Part (i) follows immediately from the definition of the liberal social contract and the simple fact that the sets of Pareto-optima relative to individuals' social preferences of $(w,(u,(Y_j)_{j\in E},\omega,\theta))$ and $(w',(u,(Y_j)_{j\in E},\omega,\theta))$ are identical. To establish part (ii), let w'_i be an affine increasing transformation $a_i.w_i + b_i$, $a_i > 0$, for all *i*. Then $\psi_{\alpha,w'}(x,y) = \left(\prod_{i\in N} a_i^{\alpha_i}\right).\psi_{\alpha,w}(x,y)$ for all (x,y) and all $\alpha \in S_n$. Since the multiplicative constant $\prod_{i\in N} a_i^{\alpha_i}$ is positive, functions $\psi_{\alpha,w'}$ and $\psi_{\alpha,w}$ have the same maxima in set A. This fact and part (i) imply that $\mathrm{A}_{\alpha,w'}(x,y) = \mathrm{A}_{\alpha,w}(x,y)$ for all $(x,y) \in L_{w'} = L_w$. Of course, this is not true anymore if we substitute arbitrary non-affine increasing transformations for the affine transformations above (we leave the interested reader check this by himself by constructing simple calculated counterexamples). ■

[24] We cannot apply the theorem of Arrow and Enthoven as such here, because the objective function of the program need not be differentiable everywhere in the positive orthant. Nevertheless, the argument developed in the proof of parts (a) and (b) of this theorem applies to our setup (substituting our x^* for the x^0 of Arrow and Enthoven, pp. 784-85). Note, in particular, that assumption $x^* >> 0$ implies that all x_{ih} are relevant variables in the sense of these authors.