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Optimal Dynamic Management of a Charity

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Abstract

Since nonprofit organizations play an important role in providing goods and services in all countries, this paper aims at determining optimal policies for a charity. The starting point of our analysis is that the amount of donations received by a charity are function of its reputation, which is an asset that can be built up over time, not overnight. To account for this important aspect, we propose a dynamic model where the charity can allocate its revenues to three main activities, namely, program expenses (charitable projects), information (promotion of its causes, website, etc.) and administration (worker/manager salaries and other administrative costs). We assume that the donors are sensitive to the way in which the charity is managed. If the administrative expenses are above a socially accepted norm, then the charity's reputation suffers. The opposite occurs when the charity is efficient and keeps its administrative costs below the norm. We prove that depending on the parameter values, there exist different optimal policies involving either positive or nil advertising and administrative expenses. We discuss some policy implications for each case and assess the impact of the norm on the results.

Keywords: Charities; Dynamic Optimization; Management Style; Advertising; Reputation.

1 Introduction

The objective of this paper is to determine the optimal expenditures policies for a nonprofit organization, or charity, over time. Our starting point is that donations (revenues) depend on the reputation or goodwill of the charity. This goodwill can be increased by investing in advertising to inform the public about the charity, e.g., its causes and success stories, and by operating the organization in an efficient way, i.e., keeping managers' salaries and other administrative costs below an acceptable norm, which is established by the public or a consultancy.

Private provision of public goods in modern economies is organized to a large extent by nonprofit organizations. This provision contributes to a greater supply of goods and services, supplementing government efforts. In the US, charitable donations increased from \$133 billion in 1990 to more than \$390 billion in 2016; however, in terms of percentage of the GDP, donations decreased from about 2.5% to just above 2% in recent years (Monnet and Panizza (2016)). Also in the US, the nonprofit sector has been growing steadily in size: between 2000 and 2010, the number of registered nonprofits increased 24%. More than 70% of donations come from individuals, 15% from foundations, 8% from bequests, and the remaining are from corporate donations (ibid). In terms of recipient sectors, about one-third of donations are directed to religious organizations. The second- and third-largest recipients are educational institutions (15%) and humanitarian services (12%), the latter including charities, such as food banks and homeless shelters.

Nonprofit organizations (NPOs) have been studied in economics (see, e.g., Andreoni and Payne (2013)), operations research (e.g., Feng and Shanthikumar (2016)), and philanthropic studies (e.g., Bekkers and Wiepking (2011)). We briefly report the main issues discussed in the literature. A first issue concerns the donors' motivations. The following questions are typically raised: (i) Is giving an altruistic action? (ii) Is it driven by peer pressure, influenced by the power of the ask?; (iii) How do donations depend on tax reductions, charity auctions, and so on (Andreani and Payne (*ibid*))? A second issue is the charitable entrepreneurs' aims. It is indeed important to know to what extent they are altruistic (namely, output maximizers), budget maximizers, or trying to divert funds for consumption perquisites (or a combination of the three). For instance, Okten and Weisbrod (2000) show that charities do not behave as budget maximizers: the marginal return of fundraising far exceeds its cost. On the other hand, Glaeser (2003) demonstrates that nonprofit organizations, typically wealthy ones, will likely conform to the objectives of elite workers, rather than of donors or other constituents. Competition between NPOs is a third issue. Nonprofit organizations compete for donations through fundraising activities. When the aggregate amount of donations is relatively inelastic to fundraising, competition is socially wasteful (Rose-Ackerman (1982) and Aldashev et al. (2010)).[1] State intervention is then called for but it is unlikely to work since NPOs do not depend on the public sector. Bottom-up cooperation is a better way to address this problem. Indeed, unlike competition in the for-profit sector, cooperation is not illegal. The scope for (Pareto-optimal) sustainable cooperation is notably studied in Aldashev et al. (2014). A fourth, final issue addresses the measure of NPO performance, which requires the design of efficiency ratios to assess the way NPOs are run and to detect perquisite consumption.

The present paper departs from the literature by adopting a dynamic viewpoint of NPOs. Feng and Shanthikumar (2016) note that "... nonprofit organizations should take a dynamic view of their operations. There is a delicate balance in how much resources to allocate in the current period to generate potential current and future funds while not hurting the efficiency measured in the current period to reduce the funders, giving incentive." We do this by paying attention to the fact that donors are able to observe a variety of publicized efficiency ratios. These ratios are provided by watchdog groups, through charity ratings (Yörük (2016)), and by the nonprofits themselves, through the mandatory public disclosure of their annual returns.[2] The flow of information through the press and social networks (Shang and Croson (2009)) also enables donors to learn about different nonprofits, evaluate them, and decide where and how much to contribute. As a result, an NPO should take into account the impact of its funding allocation (and in particular of perquisite consumption, notably through abnormally high wages), as well as of its fundraising expenses on the dynamics of their goodwill. In this regard, it is interesting to note that the assumption that donors dislike high levels of both fundraising and administrative expense ratios is not always empirically supported. Frumpkin and Kim (2001), for instance, find no statistically significant relationship between administrative expense ratios and public donations. Moreover, as observed by Calabresi (2016, p.110) people understand that the amount of altruism in society can be increased "by paying, and paying very well, those who invent, create, or even simply manage altruistic structures. One example comes immediately to mind: the extraordinarily high salaries that are paid to the CEOs of certain not-for-profit firms." Yet there is no denying that abnormal expenses (and behavior) are conducive to a decrease in donations.[3] Advertising, however, can lessen the loss of reputation experienced by a charity whose expenses are above the socially acceptable level.[4] But exemplary behavior can also

[1]That donations are relatively inelastic to fundraising is not clear. For instance, Meer (2017) finds no evidence that giving to a particular charity is reduced by the presence of inducements to give to others.

[2]A typical recommendation for the allocation of total expenditures consists of program expenses at between 70–85%, administrative expenses at between 10–15% and fundraising expenses at between 5–10%. Often the ratio of variable to fixed costs is taken as an indicator of provision efficiency (see, for example, the charity rating system adopted by MoneySense, a not-for-profit Canadian consumer advocacy organization).

[3]Refering to the UK, Fabsikova and Stranak (2018) write: "Since 2013, both trust and donations significantly dropped at three distinct times and for each of these periods, we could find charity scandals which attracted substantial media attention: surpluses and dubious investments of Comic Relief (December 2013), aggressive fundraising practices blamed for the death of poppy collector Olive Cooke and dramatic closure of Kids Company (summer 2015), and the Oxfam crisis last February."

[4]There is a similarity here with the way a polluting firm reacts to a Public Disclosure Progam revealing its environmental record to the public (see, e.g., André et al. (2011)).

be a substitute for advertising.

In this paper, we address the following research questions:

1. How should a charity allocate its revenues to its different activities, that is, to information (advertising), administration, and program expenses?

2. What is the impact on the optimal policies of varying the norm for administrative expenses?

More specifically, we identify the conditions under which the optimal solution involves either positive or zero advertising and administrative expenses. An optimal administrative cost of zero is interpreted as a case where the charity is operated by volunteer workers.

The rest of the paper is organized as follows: In Section 2, we introduce the model. In Section 3, we state the general constrained optimization problem. Sections 4 and 5 present the optimal solutions with paid and volunteer workers, respectively. In Section 6, we present the impact of varying the administrative cost norm on the results. Section 7 briefly concludes.

2 Model

We consider a charity managing its operations over an infinite planning horizon. Denote time by $t \in [0, \infty)$ and the revenues or donations received by the charity by $R(t)$. We assume that these revenues are given by a concave function of the charity's goodwill (or reputation) with the public, that is,

$$R(t) = \max\left\{0, \theta_1 G(t) - \frac{\theta_2}{2} G^2(t)\right\},$$

where $G(t)$ is the goodwill at time t, and θ_1 and θ_2 are nonnegative parameters.

Denote by $c(t)$ the administrative expenses of the charity at time t, and let $\bar{c}$ be a positive parameter measuring what the public considers to be a reasonable cost for running a charity. We shall refer to $\bar{c}$ as the acceptable norm in the charity sector.

Remark 2.1. For simplicity, we let $\bar{c}$ be constant over time. There is no conceptual difficulty in extending our model to a case where the norm varies exogenously over time, e.g., $\bar{c}(t) = (1+\tau)^t \bar{c}(0)$, where $\bar{c}(0)$ is the norm at the initial instant of time and τ is a given growth rate.

Let $a(t)$ be the promotional activities undertaken by the charity to increase its reputation with the public. These activities include, e.g., website, brochures, information kiosks, and documentaries showing achievements in the community. To keep it compact, we shall generically refer to these activities as the advertising effort. We suppose that the advertising cost is convex increasing and given by the following quadratic function:

$$g(a(t)) = \omega_1 a(t) + \frac{\omega_2}{2} a^2(t),$$

where ω_1 and ω_2 are nonnegative parameters. Note that $g(0) = 0$.

The evolution of the goodwill over time is governed by the following linear-differential equation:

$$\dot{G}(t) = \alpha a(t) - \gamma(c(t) - \bar{c}) - \delta G(t), \qquad G(0) = G_0, \tag{1}$$

where α and γ are positive scaling parameters, $\delta > 0$ is the decay rate in goodwill due to forgetting, and $G_0 > 0$ is the initial value of the charity's reputation. The interpretation of the above dynamics is straightforward. Advertising effort has a positive effect on the charity's goodwill:

this assumption is made in all dynamic models of advertising (see the surveys in Huang et al. (2012) and Jørgensen and Zaccour (2014)). Further, if the administrative cost (or consumption by the charity) is lower than $\bar{c}$, then this reflects virtuous behavior (or efficient management) by the charity, and consequently, its reputation is enhanced. By contrast, a consumption above $\bar{c}$ signals poor management, and this harms the charity's reputation.

Denote by $e(t)$ the program expenses in, e.g., aid relief at time t. The budget constraint can then be written as follows:[5]

$$R(t) = \omega_1 a(t) + \frac{\omega_2}{2} a^2(t) + c(t) + e(t), \tag{2}$$

that is, the donations (revenues) are allocated to three activities, namely, advertising, administrative and programs.

Suppose that the utility of the charity's managers depends on their administration and program expenses, which we denote by $U(c(t), e(t))$. The rationale for including $c(t)$ is that this variable is a proxy for the managers' salaries and other, prestige-based activities, e.g., staying at international hotels when they visit project sites, attending conferences, etc. We assume that $U(c(t), e(t))$ is concave in $c(t)$ and linear in $e(t)$.[6] More precisely, we adopt the following specification:

$$U(c(t), e(t)) = \varphi c(t) - \frac{\mu}{2} c^2(t) + e(t),$$

where φ and μ are positive scaling parameters. We make the following assumptions.

Assumption 1. The marginal utility of consumption, evaluated at $\bar{c}$, is larger than the marginal utility derived from spending on charitable projects, that is,

$$\left.\frac{\partial U}{\partial c}\right|_{c=\bar{c}} > \frac{\partial U}{\partial e} \Leftrightarrow \varphi - \mu\bar{c} - 1 > 0. \tag{3}$$

This condition means that for any consumption level no higher than $\bar{c}$, the manager would be better off increasing his or her own consumption rather than the program expenses. In this sense, the consumption norm is binding. If the condition were not verified, then satisfying the norm would not be difficult since the manager would find it worthwhile to spend more on project expenses than on consumption for low levels of this consumption. To put it differently, the consumption norm would be relatively lax.

Denoting by ρ the discount rate, and assuming that the managers aim at maximizing their stream of utilities over an infinite horizon, then their objective functional is given by

$$J = \int_0^\infty e^{-\rho t} \left(\varphi c(t) - \mu \frac{c^2(t)}{2} + e(t) \right) dt. \tag{4}$$

To recapitulate, by (1), (2), and (4), we have defined a constrained infinite-horizon optimal-control problem with three control variables $(a(t) \geq 0, c(t) \geq 0, e(t) \geq 0)$ and one state variable $G(t)$.

To simplify the presentation of the results, without any loss of qualitative insights, we normalize the values of some parameters.

Assumption 2. Let

$$\alpha = \omega_2 = \mu = \theta_2 = 1. \tag{5}$$

[5] For simplicity, we assume that the charity cannot borrow or accumulate assets.

[6] If we think of $c(t)$ as the consumption by managers of the resource, then it is standard to assume concavity.

Further, we make the following

Assumption 3. Let

$$G_0 < \theta_1. \tag{6}$$

This assumption means that the charity's marginal revenue at the initial time is positive, that is,

$$\left.\frac{dR\,(t)}{dG\,(t)}\right|_{t=0} = \theta_1 - G_0 > 0.$$

3 A General Constrained Problem

To solve the dynamic constrained-optimization problem, we introduce the Hamiltonian

$$\begin{aligned} H\left(a\,(t)\,, c\,(t)\,, e\,(t)\,, \lambda^0, \lambda(t), \psi(t), s(t), x(t), y(t)\right) &= \lambda^0 \left(\varphi c(t) - \frac{c^2(t)}{2} + e(t)\right) + \\ &\lambda(t)\left(a(t) - \gamma\left(c(t) - \bar{c}\right) - \delta G(t)\right) \\ &+\psi(t)\left(\theta_1 G(t) - \frac{G^2(t)}{2} - c(t) - e(t) - \omega_1 a(t) - \frac{a^2(t)}{2}\right) \\ &+s(t)a(t) + x(t)c(t) + y(t)e(t), \end{aligned} \tag{7}$$

where: $\lambda\,(t)$ is the adjoint variable appended to the dynamics in (1); $\psi(t)$ is the Lagrange multiplier appended to the budget constraint in (2); $s(t), x(t)$, and $y(t)$ are Lagrange multipliers corresponding to the nonnegativity constraints $a(t) \geq 0, c(t) \geq 0$, and $e(t) \geq 0$, respectively; and finally, $\lambda^0 \geq 0$ is a Lagrange multiplier appended to the instantaneous objective.

The first-order conditions provide the existence of $\lambda^0, \lambda(t), \psi(t), s(t), x(t), y(t)$, not all zero, which are as follows:

Optimality conditions:

$$\frac{\partial H}{\partial c\,(t)} = \lambda^0\left(\varphi - c(t)\right) - \gamma\lambda(t) - \psi(t) + x(t) = 0, \tag{8}$$

$$\frac{\partial H}{\partial a\,(t)} = \lambda(t) - \psi(t)\left(\omega_1 + a(t)\right) + s(t) = 0, \tag{9}$$

$$\frac{\partial H}{\partial e\,(t)} = \lambda^0 - \psi(t) + y(t) = 0. \tag{10}$$

Complementarity conditions:

$$c\,(t) \geq 0, \quad x\,(t) \geq 0, \quad x(t)c(t) = 0, \tag{11}$$

$$a\,(t) \geq 0, \quad s(t) \geq 0, \quad s(t)a(t) = 0, \tag{12}$$

$$e\,(t) \geq 0, \quad y(t) \geq 0, \quad y(t)e(t) = 0, \tag{13}$$

$$\lambda^0 \geq 0. \tag{14}$$

Budget constraint:

$$\theta_1 G(t) - \frac{G^2(t)}{2} = c(t) + e(t) + \omega_1 a(t) + \frac{a^2(t)}{2}, \tag{15}$$

and

State and co-state equations:

$$\dot{G}(t) = a(t) - \gamma\big(c(t) - \bar{c}\big) - \delta G(t), \qquad G(0) = G_0, \tag{16}$$

$$\dot{\lambda}(t) = \psi\big(G(t) - \theta_1\big) + (\delta + \rho)\lambda(t). \tag{17}$$

Remark 3.1. When $e(t) > 0$ for all t, then $\lambda^0 \neq 0$ and we set it equal to one. Indeed, if $\lambda^0 = 0$ then, noticing that $y(t) = 0$ (since $e(t) > 0$), we obtain $\psi(t) = 0$. Therefore, (8) and (9) imply $\lambda(t) \geq 0$ and $\lambda(t) \leq 0$ so $\lambda(t) = 0$, which implies $x(t) = 0$ and $s(t) = 0$. But to have all the multipliers equal to zero is a contradiction.

In the next sections, we shall analyze four cases where the optimal solution displays different combinations of positive and zero values for the administrative and advertising expenses and interpret the conditions under which each of these cases can happen. These cases are described in Table 1.

Table 1: Description of the different cases

Superscript of optimal values	$a(t)$	$c(t)$	$e(t)$
N : Non-volunteer managers	> 0	> 0	> 0
N_0 : Non-volunteer managers, no advertising	0	> 0	> 0
V : Volunteer managers	> 0	0	> 0
V_0 : Volunteer, no advertising	0	0	> 0

Remark 3.2. The propositions stated in the sequel are (each) valid under some specific conditions, which are defined explicitly in the proofs in the Appendix.

4 Non-Volunteer Managers

In many realistic instances, professional managers are needed to run the charity, and other administrative costs are unavoidable. We have two subcases: (i) the charity invests in advertising to inform the public about its projects; and (ii) the charity does not wish to allocate any budget to advertising. In both solutions, we have $e(t) > 0$; otherwise, the charity looses its *raison d'être*. The first case is the interior solution, where all control variables assume positive values.

4.1 Interior Solution

Denote by

$$\begin{aligned} \Phi &= \frac{\theta_1\left(1+\gamma^2\right) - (\delta+\rho)\left(\omega_1 + \gamma\left(\varphi - 1 - \bar{c}\right)\right)}{\delta(\delta+\rho) + 1 + \gamma^2}, \\ \Omega &= (\rho + 2\delta)^2 + 4\left(1+\gamma^2\right) > 0. \end{aligned}$$

The following proposition gives the optimal interior solution.

Proposition 4.1. *There exists an interior solution, where the optimal administrative, advertising, and program expenses and the goodwill trajectories are given by*

$$
\begin{aligned}
c^N(t) &= \varphi - 1 - \gamma\lambda^N(t), && (18)\\
a^N(t) &= \lambda^N(t) - \omega_1, && (19)\\
e^N(t) &= \theta_1 G^N(t) - \frac{\left(G^N(t)\right)^2}{2} - \omega_1 a^N(t) - \frac{\left(a^N(t)\right)^2}{2} - c^N(t), && (20)\\
G^N(t) &= (G_0 - \Phi)\, e^{\frac{\rho-\sqrt{\Omega}}{2}t} + \Phi, \quad G^N(\infty) = \Phi, && (21)
\end{aligned}
$$

and the adjoint variable appended to the goodwill dynamics by

$$
\lambda^N(t) = \frac{2(\Phi - G_0)}{2\delta + \rho + \sqrt{\Omega}} e^{\frac{\rho-\sqrt{\Omega}}{2}t} + \frac{\theta_1 - \Phi}{\delta + \rho}, \qquad \lambda^N(\infty) = \frac{\theta_1 - \Phi}{\delta + \rho}. \tag{22}
$$

Proof. In an interior solution, we have $x(t) = y(t) = s(t) = 0$. From the optimality conditions (9)-(8), we get

$$
\begin{aligned}
c(t) &= \varphi - 1 - \gamma\lambda(t),\\
a(t) &= \lambda(t) - \omega_1,\\
\psi(t) &= 1.
\end{aligned}
$$

Inserting in the budget constraint and in the dynamic equations, we obtain

$$
\begin{aligned}
e(t) &= \theta_1 G(t) - \frac{G^2(t)}{2} - \varphi + \gamma\lambda(t) + 1 - \omega_1(\lambda(t) - \omega_1) - \frac{(\lambda(t) - \omega_1)^2}{2},\\
\dot{G}(t) &= \lambda(t) - \omega_1 - \gamma(\varphi - \gamma\lambda(t) - 1 - \bar{c}) - \delta G(t), \qquad G(0) = G_0,\\
\dot{\lambda}(t) &= (G(t) - \theta_1) + (\delta + \rho)\lambda(t).
\end{aligned}
$$

Solving the pair of differential equations (see Appendix) and substituting for the optimal $G(t)$ and $\lambda(t)$ in the optimality conditions, yields the optimal control trajectories. □

The above proposition is stated under the assumption of an interior solution. In the Appendix, we provide the conditions under which this solution exists. A few comments can be made on the results. First, the optimal advertising level is determined by the familiar rule equating marginal cost, given by $\omega_1 + a^N(t)$, to marginal revenue, given by $\lambda^N(t)$, which is the shadow price of the goodwill. So, the advertising level is positive whenever this shadow price is larger than the marginal advertising cost at zero, i.e., $a(t) > 0 \Leftrightarrow \lambda^N(t) > \omega_1$.

Second, the optimal level of $c^N(t)$ is similarly obtained by equating its marginal utility, given by $\varphi - c^N(t)$, to its cost, which is measured by the sum of two items: (i) the marginal utility of charity expenses, that is, $\frac{\partial U}{\partial e} = 1$; and (ii) the marginal loss in goodwill, which is measured by $\gamma\lambda^N(t)$. Consequently, we see that $c^N(t)$ is strictly positive when the marginal utility of consumption at zero (i.e., φ) is larger than its cost, given by $1 + \gamma\lambda^N(t)$. Putting together the two conditions, we conclude that advertising and consumption are strictly positive when the shadow price of goodwill satisfies the following restrictions:

$$
\frac{\varphi - 1}{\gamma} > \lambda^N(t) > \omega_1. \tag{23}
$$

Finally, program expenses $e^N(t)$ are obtained by substracting administrative and advertising costs from the revenues.

Proposition 4.2. *The evolution over time of the goodwill, its shadow price, advertising, and administrative expenses is as follows:*

$$\textit{sign}\ (G_0 - \Phi) = -\ \textit{sign}\ \dot{G}^N(t) = \textit{sign}\ \dot{\lambda}^N(t) = \textit{sign}\ \dot{a}^N(t) = -\textit{sign}\ \dot{c}^N(t).$$

Proof. Differentiating $G^N(t), \lambda^N(t), a^N(t)$, and $c^N(t)$ with respect to time, we get

$$\begin{aligned}
\dot{G}^N(t) &= (G_0 - \Phi)\left(\frac{\rho - \sqrt{\Omega}}{2}\right) e^{\frac{\rho-\sqrt{\Omega}}{2}t} \quad \text{is} \begin{cases} \leq 0, & \text{if } G_0 - \Phi \geq 0, \\ \geq 0, & \text{if } G_0 - \Phi \leq 0, \end{cases} \\
\dot{\lambda}^N(t) &= -\frac{(G_0 - \Phi)\left(\rho - \sqrt{\Omega}\right)}{2\delta + \rho + \sqrt{\Omega}} e^{\frac{\rho-\sqrt{\Omega}}{2}t} \quad \text{is} \begin{cases} \geq 0, & \text{if } G_0 - \Phi \geq 0, \\ \leq 0, & \text{if } G_0 - \Phi \leq 0, \end{cases} \\
\dot{a}^N(t) &= \dot{\lambda}^N(t), \\
\dot{c}^N(t) &= -\gamma \dot{\lambda}^N(t),
\end{aligned}$$

and hence the result. □

When convergence to the goodwill steady-state value is from above, that is, $G_0 - \Phi \geq 0$, the shadow price $\lambda^N(t)$ is increasing over time. Advertising, which contributes positively to the goodwill, and administrative cost, which has a negative impact on goodwill, have the same and opposite sign of $\dot{\lambda}^N(t)$, respectively. The results are in the opposite direction when the initial value of the goodwill is low and convergence to the steady state is from below.

The evolution of the program expenditures over time is less straightforward to characterize. Indeed, differentiating $e^N(t)$ with respect to time, we get

$$\begin{aligned}
\dot{e}^N(t) &= -\left(\frac{\left(2\delta + \rho + \sqrt{\Omega}\right)}{2}\left(\theta_1 - G^N(t)\right) + \lambda^N(t) - \gamma\right)\dot{\lambda}^N(t), \\
&\triangleq -\Psi \dot{\lambda}^N(t).
\end{aligned}$$

The coefficient Ψ of $\dot{\lambda}^N(t)$ is the sum of three terms. The first term is the positive coefficient $\frac{\left(2\delta+\rho+\sqrt{\Omega}\right)}{2}$ multiplying the marginal revenues $\left(\theta_1 - G^N(t)\right)$; the second term is the shadow price of the goodwill, which is also equal to the marginal advertising cost $\left(\omega_1 + a^N(t)\right)$; finally, the third term $-\gamma$ is equal to $\frac{\partial \dot{G}(t)}{\partial c(t)}$, that is, the marginal loss in future goodwill due to the administrative cost. Clearly, the first two terms are positive and the third is negative. Consequently, the sign of Ψ cannot be definitively determined, and we have the following characterization:

$$\text{If } \Psi \geq 0, \text{ then sign } \dot{e}^N(t) = \text{sign } \left(-\dot{\lambda}^N(t)\right) \text{ and is } \begin{cases} \leq 0, & \text{if } G_0 - \Phi \geq 0, \\ \geq 0, & \text{if } G_0 - \Phi \leq 0. \end{cases}$$

$$\text{If } \Psi \leq 0, \text{ then sign } \dot{e}^N(t) = \text{sign } \dot{\lambda}^N(t) \text{ and is } \begin{cases} \leq 0, & \text{if } G_0 - \Phi \geq 0, \\ \geq 0, & \text{if } G_0 - \Phi \leq 0. \end{cases}$$

To recapitulate, the state, costate, and control trajectories are monotone over time, with their direction (increasing or decreasing) depending essentially on the location of the steady-state value of the goodwill with respect to its initial value.

Finally, we look at the variations of the steady-state goodwill with respect to the parameter values, which are given by

$$\begin{aligned}
\frac{dG^N(\infty)}{d\bar{c}} &= \frac{\gamma(\delta+\rho)}{\delta(\delta+\rho)+1+\gamma^2} > 0, \\
\frac{dG^N(\infty)}{d\theta_1} &= \frac{\gamma^2+1}{\delta(\delta+\rho)+1+\gamma^2} > 0, \\
\frac{dG^N(\infty)}{d\omega_1} &= -\frac{\delta+\rho}{\delta(\delta+\rho)+1+\gamma^2} < 0, \\
\frac{dG^N(\infty)}{d\rho} &= -\frac{\left(1+\gamma^2\right)\left(\omega_1+\gamma\left(\varphi-\bar{c}-1\right)+\delta\theta_1\right)}{\left(\delta(\delta+\rho)+1+\gamma^2\right)^2} < 0, \\
\frac{dG^N(\infty)}{d\varphi} &= -\frac{\gamma(\delta+\rho)}{\delta(\delta+\rho)+1+\gamma^2} < 0, \\
\frac{dG^N(\infty)}{d\delta} &= -\frac{\left(1+\gamma^2-(\delta+\rho)^2\right)\left(\gamma\left(\varphi-1-\bar{c}\right)+\omega_1\right)+(2\delta+\rho)\left(1+\gamma^2\right)\theta_1}{\left(\delta(\delta+\rho)+1+\gamma^2\right)^2}, \\
\frac{dG^N(\infty)}{d\gamma} &= \frac{(\delta+\rho)\left(\left(\gamma^2-1-\delta(\delta+\rho)\right)\left(\varphi-1-\bar{c}\right)+2\gamma\left(\omega_1+\delta\theta_1\right)\right)}{\left(\delta(\delta+\rho)+1+\gamma^2\right)^2}.
\end{aligned}$$

The results that $G^N(\infty)$ is increasing in $\bar{c}$ and θ_1 is hardly surprising. Indeed, the higher is $\bar{c}$, the easier it is to have $c(t) \leq \bar{c}$ and to increase the charity's reputation. Similarly, the higher are the marginal revenues at zero goodwill (given by θ_1), the higher the incentive will be to invest in the goodwill by advertising or keeping the management cost below $\bar{c}$. Further $G^N(\infty)$ is decreasing in the marginal cost of advertising at zero, given by ω_1, the discount rate ρ and in φ, which gives the marginal utility of consumption at zero. The higher is φ, the higher the incentive to consume, which negatively affects the goodwill. Under the sufficient (not necessary) condition that $\delta+\rho<1$, which is reasonable, we conclude that the larger is the decay rate δ, the lower the goodwill at the steady state. Finally, the impact of γ is ambiguous and depends on the other parameter values.

Remark 4.3. Similar sensitivity analyses can be obtained in the other cases and are omitted to avoid repetition.

4.2 No Advertising

In the previous subsection, we studied the case where the solution is interior. Now, we look at the case where the charity does not engage in any advertising activities throughout the whole planning horizon. Let

$$\begin{aligned}
\Lambda &= \frac{\left(\theta_1\gamma-(\delta+\rho)\left(\varphi-1-\bar{c}\right)\right)\gamma}{\delta(\delta+\rho)+\gamma^2}, \\
\Theta &= (\rho+2\delta)^2+4\gamma^2 > 0.
\end{aligned}$$

The following proposition gives the optimal trajectories of $G(t), c(t), e(t)$, and $\lambda(t)$, where $a(t)=0$.

Proposition 4.4. *There exists a solution where advertising is equal to zero at each instant of time, and the optimal goodwill, and administrative and program expense trajectories are given by*

$$\begin{aligned} G^{N_0}(t) &= (G_0 - \Lambda)\, e^{\frac{\rho-\sqrt{\Theta}}{2}t} + \Lambda, \quad G^{N_0}(\infty) = \Lambda, & (24)\\ c^{N_0}(t) &= \varphi - 1 - \gamma\lambda^{N_0}(t), & (25)\\ e^{N_0}(t) &= \theta_1 G^{N_0}(t) - \frac{1}{2}\left(G^{N_0}(t)\right)^2 - c^{N_0}(t), & (26) \end{aligned}$$

and the adjoint variable by

$$\lambda^{N_0}(t) = -\frac{2\,(G_0-\Lambda)}{2\delta+\rho+\sqrt{\Theta}} e^{\frac{\rho-\sqrt{\Theta}}{2}t} + \frac{\theta_1-\Lambda}{\delta+\rho}, \qquad \lambda^{N_0}(\infty) = \frac{\theta_1-\Lambda}{\delta+\rho}.$$

Proof. See Appendix. □

As in the previous scenario, the convergence of the goodwill towards its steady-state value is monotone and its direction (increasing or decreasing) depends on the location of the initial value. Indeed, the evolution over time of the goodwill is given by

$$\dot{G}^{N_0}(t) = \frac{\rho-\sqrt{\Theta}}{2}\,(G_0-\Lambda)\, e^{\frac{\rho-\sqrt{\Theta}}{2}t} \text{ is } \begin{cases} \le 0, & \text{if } G_0-\Lambda \ge 0,\\ \ge 0, & \text{if } G_0-\Lambda \le 0. \end{cases}$$

The administrative cost $c^{N_0}(t)$ is also monotone over time and its evolution is as follows:

$$\dot{c}^{N_0}(t) = \gamma\left(\frac{\rho-\sqrt{\Theta}}{2}\right)\left(\frac{2\,(G_0-\Lambda)}{2\delta+\rho+\sqrt{\Theta}} e^{\frac{\rho-\sqrt{\Theta}}{2}t}\right) \text{ is } \begin{cases} \le 0, & \text{if } G_0-\Lambda \ge 0,\\ \ge 0, & \text{if } G_0-\Lambda \le 0, \end{cases}$$

Finally, as in the interior solution case, the evolution of the program expenses, which is given by

$$\dot{e}^{N_0}(t) = \left(\theta_1 - G^{N_0}(t)\right)\dot{G}^{N_0}(t) - \dot{c}^{N_0}(t),$$

is ambiguous because the two right-hand-side terms have opposite signs.

5 Volunteer Managers

In this section, we show, under some conditions, that $c(t) = 0$ for all t, that is, that the charity is managed by volunteer workers, and that there is no administrative cost. We have two subcases: in the first one, $a(t)$ is positive for all t,and in the second case, we show that $a(t) = 0$.

5.1 Positive Advertising

To simplify the presentation of the results in this case, let

$$\begin{aligned} \Gamma &= \frac{(\delta+\rho)(\gamma\bar{c}-\omega_1)+\theta_1}{\delta(\delta+\rho)+1}.\\ \Delta &= (\rho+2\delta)^2 + 4 > 0. \end{aligned}$$

We have the following result:

Proposition 5.1. *There exists a solution where the administrative expenses are zero and the optimal goodwill, advertising, and program expenses are given by*

$$
\begin{aligned}
G^V(t) &= e^{\frac{\rho-\sqrt{\Delta}}{2}t}\left(G_0-\Gamma\right)+\Gamma, \qquad G\left(\infty\right)=\Gamma, && (27)\\
a^V(t) &= \lambda^V(t)-\omega_1, && (28)\\
e^V(t) &= \theta_1 G^V(t)-\frac{1}{2}\left(G^V(t)\right)^2-\omega_1 a^V(t)-\frac{1}{2}\left(a^V(t)\right)^2, && (29)
\end{aligned}
$$

and the adjoint variable by

$$
\lambda^V(t)=\frac{2\left(\Gamma-G_0\right)}{2\delta+\rho+\sqrt{\Delta}}e^{\frac{\rho-\sqrt{\Delta}}{2}t}+\frac{\theta_1-\Gamma}{\delta+\rho}, \quad \lambda^V(\infty)=\frac{\theta_1-\Gamma}{\delta+\rho}. \tag{30}
$$

Proof. See Appendix. □

We make two comments. First, we note that the evolution over time of the goodwill, given by

$$
\dot{G}^V(t)=\frac{\rho-\sqrt{\Delta}}{2}e^{\frac{\rho-\sqrt{\Delta}}{2}t}\left(G_0-G^V\left(\infty\right)\right),
$$

is decreasing over time if $G_0>G^V\left(\infty\right)$, and increasing otherwise. As before, $G^V\left(t\right)$ is monotone, and convergence to the steady state is from above when $G_0>G^V\left(\infty\right)$, and from below when $G_0<G^V\left(\infty\right)$. Second, the evolution over time of advertising is as follows:

$$
\dot{a}^V(t)=-\frac{\rho-\sqrt{\Delta}}{2}\left(\frac{2\left(G_0-\Gamma\right)}{2\delta+\rho+\sqrt{\Delta}}e^{\frac{\rho-\sqrt{\Delta}}{2}t}\right) \quad \text{is} \quad \begin{cases} \geq 0 & \text{if } G_0\geq G^V\left(\infty\right), \\ \leq 0 & \text{if } G_0\leq G^V\left(\infty\right), \end{cases}
$$

which shows that the sign of $\dot{a}^V(t)$ depends on where the initial goodwill value is located with respect to the steady state. This is the same result we obtained in the scenario where the workers were not volunteers. The advertising trajectory is monotone over time and converges to the steady-state value given by

$$
a^V(\infty)=\frac{\delta\left(\theta_1-\omega_1(\delta+\rho)\right)-\gamma\bar{c}}{\delta(\delta+\rho)+1}.
$$

When $G_0\geq G^V\left(\infty\right)$, while advertising increases over time and its trajectory converges to the above steady-state value, the charity's goodwill diminishes, which is due to the fact that the forgetting effect is stronger than the positive effect stemming from advertising. Note that the initial advertising, which is given by

$$
a^V(0)=\frac{\theta_1-G^V\left(\infty\right)}{\delta+\rho}+2\frac{\left(G^V\left(\infty\right)-G_0\right)}{2\delta+\rho+\sqrt{\Delta}}-\omega_1
$$

is lower when $G_0\geq G^V\left(\infty\right)$ than when $G_0\leq G^V\left(\infty\right)$. Again, the evolution over time of the program expenses, given by

$$
\dot{e}^V(t)=\left(\theta_1-G^V(t)\right)\dot{G}^V(t)-\left(\omega_1-a^V(t)\right)\dot{a}^V(t),
$$

is ambiguous.

5.2 No Advertising

The following proposition provides the results for the case where advertising and administrative expenses are zero.

Proposition 5.2. *There exists a solution where* $a(t) = c(t) = 0$ *with the optimal goodwill and program expenses given by*

$$\begin{aligned} G^{V_0}(t) &= \left(G_0 - \frac{\gamma\bar{c}}{\delta}\right) e^{-\delta t} + \frac{\gamma\bar{c}}{\delta}, \\ e^{V_0}(t) &= \theta_1 G^{V_0}(t) - \frac{\left(G^{V_0}(t)\right)^2}{2}. \end{aligned}$$

Proof. See Appendix. □

The above proposition shows that, under some conditions, it is optimal not to advertise, nor to consume part of the revenues. The goodwill steady-state value is $G^{V_0}(\infty) = \frac{\gamma\bar{c}}{\delta}$. If the initial goodwill is high enough $\left(G_0 > G^{V_0}(\infty)\right)$, then convergence to the steady state is from above, that is, the goodwill decreases over time till it reaches $\frac{\gamma\bar{c}}{\delta}$. If the initial value is low, it is the other way around, that is, convergence to the steady state is from below.

The evolution over time of program expenses is given by

$$\dot{e}^{V_0}(t) = \left(\theta_1 - G^{V_0}(t)\right) \dot{G}^{V_0}(t),$$

which implies that the sign of $\dot{e}^{V_0}(t)$ is the same as $\dot{G}^{V_0}(t)$.

The following corollary characterizes the conditions under which it is not feasible to set $a(t) = c(t) = 0$.

Corollary 5.3. *If* $\frac{\varphi-1}{\gamma} > \omega_1$, *then it is impossible to have* $a(t) = c(t) = 0$.

Proof. The condition directly follows from (65). □

This corollary allows us to understand the gist of the above proposition. The condition in the corollary is satisfied when (ceteris paribus) φ is high or when γ is low. Indeed, if φ, which is the marginal utility of consumption when consumption is nil, is high enough, then it surely never pays not to consume. And when the effect of frugal spending is low, that is, when γ is low, then there is no cost to choosing a positive consumption value. Likewise, the above condition is satisfied (ceteris paribus) when ω_1 is low. Recall that ω_1 is the marginal cost of advertising at zero. The lower is the marginal cost of advertising, the higher, then, is the incentive to advertise.

Notice that when ω_1 is nil and all the other parameters are positive, the condition in the corollary becomes $\frac{\varphi-1}{\gamma} > 0$, and is *always satisfied*. That is, adopting $a(t) = c(t) = 0$ is *impossible* under the assumption that is often made in the literature about advertising cost, namely, that this cost is purely quadratic, i.e., $g(a(t)) = \frac{a^2(t)}{2}$ (see the surveys in Huang et al. (2012) and Jørgensen et al. (2014)). Consequently, assuming $\omega_1 = 0$ is not neutral.

6 The Impact of the Administrative Norm

In this section, we do essentially two things. First, we assess the impact of varying the norm $\bar{c}$ on the results at the steady state. Second, we derive conditions under which the steady-state administrative costs are below this norm in the two cases where the managers are not volunteers. The importance of $\bar{c}$ stems from the fact that donors want to see their contributions mainly dedicated to funding the causes in which they believe, and not to paying unduly high salaries to the charity's employees or for inefficiently managing (high administrative cost) the organization.

Table 2 gives the impact of varying $\bar{c}$ on the results in the different cases. To save on space, we only give the sign of these variations, the derivatives being easy to compute.

Table 2: Impact of varying $\bar{c}$

	N	N_0	V	V_0
$\frac{\partial G(\infty)}{\partial \bar{c}}$	$+$	$+$	$+$	$+$
$\frac{\partial \lambda(\infty)}{\partial \bar{c}}$	$-$	$-$	$-$	$-$
$\frac{\partial a(\infty)}{\partial \bar{c}}$	$-$	NA	$-$	NA
$\frac{\partial c(\infty)}{\partial \bar{c}}$	$+$	$+$	NA	NA
$\frac{\partial e(\infty)}{\partial \bar{c}}$	$?$	$?$	$+$	$+$

The following comments can be made:

1. Recalling the dynamics of goodwill

$$\dot{G}(t) = a(t) - \gamma(c(t) - \bar{c}) - \delta G(t), \qquad G(0) = G_0,$$

we clearly see that the norm $\bar{c}$ acts as free advertising for the charity. Then, obviously, increasing $\bar{c}$ leads to an increase in the steady-state value of G.

2. The steady-state value of the adjoint variable is negatively related to the steady-state value of the goodwill. Indeed, we have

$$\begin{aligned} \lambda^N(\infty) &= \frac{\theta_1 - G^N(\infty)}{\delta + \rho}; \quad \lambda^{N_0}(\infty) = \frac{\theta_1 - G^{N_0}(\infty)}{\delta + \rho}; \\ \lambda^V(\infty) &= \frac{\theta_1 - G^V(\infty)}{\delta + \rho}; \quad \lambda^{V_0}(\infty) = \frac{\theta_1 - G^{V_0}(\infty)}{\delta + \rho}. \end{aligned}$$

Therefore, we have sign $\left(\frac{\partial G(\infty)}{\partial \bar{c}}\right) = -$ sign$\left(\frac{\partial \lambda(\infty)}{\partial \bar{c}}\right)$. Economically speaking, the shadow price of the goodwill is lower when the goodwill is higher, which is a consequence of a higher $\bar{c}$.

3. The impact of $\bar{c}$ on advertising is negative, which is intuitive. Again, interpreting $\bar{c}$ as free advertising means the charity is less in need of paid-for advertising to increase its reputation.

4. Increasing $\bar{c}$ leads to an increase in $c(\infty)$. One interpretation is that the less the donors hold the charity's managers accountable, then the higher are their salaries and other administrative expenses.

5. In the two cases where the charity is managed by volunteer workers, we obtain that increasing $\bar{c}$ leads to an increase in the program expenses. This positive impact is a consequence of the above results. Indeed, as $\bar{c}$ boosts the goodwill, and consequently the donations, the charity has more funds for its programs. In the non-volunteer cases, we could not definitively sign the impact of $\bar{c}$ on the program expenses, because when $\bar{c}$ increases, both the donations *and* consumption increase. Therefore, the net effect will depend on whether donations increase

more than consumption. Moreover, there is an additional positive effect stemming from the decrease in advertising. Formally, we have

$$\begin{aligned}\frac{\partial e^{N}(\infty)}{\partial\bar{c}} &= \frac{1}{\delta+\rho}\left(\frac{\left(1+(\delta+\rho)^2\right)(\omega_1+\gamma(\varphi-\bar{c}-1)+\delta\theta_1)}{\gamma^2+\delta^2+\rho\delta+1}-\gamma\right)\frac{\partial\Phi}{\partial\bar{c}},\\ &> 0\Leftrightarrow\theta_1>\frac{\gamma}{\delta}\frac{\left(\gamma^2+\delta^2+\rho\delta+1\right)}{(1+(\delta+\rho)^2)}-\frac{1}{\delta}\left(\omega_1+\gamma(\varphi-\bar{c}-1)\right).\end{aligned}$$

Moreover, we have

$$\begin{aligned}\frac{\partial e^{N_0}(\infty)}{\partial\bar{c}} &= \left(\frac{(\delta+\rho)^2\left(\delta\theta_1+\gamma(\varphi-\bar{c}-1)\right)-\gamma\left(\delta(\delta+\rho)+\gamma^2\right)}{(\delta(\delta+\rho)+\gamma^2)(\delta+\rho)}\right)\frac{\partial\Lambda(\infty)}{\partial\bar{c}},\\ &> 0\Leftrightarrow\theta_1>\frac{\gamma}{\delta}\frac{\left(\delta(\delta+\rho)+\gamma^2\right)}{(\delta+\rho)^2}-\frac{\gamma}{\delta}\left(\varphi-\bar{c}-1\right).\end{aligned}$$

We see that in the non-volunteer cases, the impact of $\bar{c}$ on program expenses is positive when γ is relatively low. The reason is that the lower is γ, the lesser consumption reacts to a change in the goodwill shadow price.

The above results show that when the consumption standard becomes less lax, say after a charity scandal, this can result in a decrease in the long-run value of the program expenses. This is unavoidable for all the charities run by volunteer managers. For the other charities, the decrease in the long-run value of program expenses is due to the fact that it is sometimes too costly to rebuild trust through more advertising.

In the next proposition, we give the conditions under which the steady-state administrative expenses are less than or equal to the norm.

Proposition 6.1. *If $\bar{c}\geq\varphi-1-\frac{\gamma(\omega_1+\delta\theta_1)}{\delta(\delta+\rho)+1}$, then $c^{N}(\infty)\leq\bar{c}$.*

If $\bar{c}\geq\varphi-1-\frac{\gamma\theta_1}{\delta+\rho}$, then $c^{N_0}(\infty)\leq\bar{c}$.

Proof. Compute the differences

$$\begin{aligned}\bar{c}-c^{N}(\infty) &= \frac{(\delta(\delta+\rho)+1)(1+\bar{c}-\varphi)+\gamma(\omega_1+\delta\theta_1)}{(\delta(\delta+\rho)+1+\gamma^2)},\\ \bar{c}-c^{N_0}(\infty) &= \frac{\delta(\delta+\rho)(1+\bar{c}-\varphi)+\gamma\delta\theta_1}{(\delta(\delta+\rho)+\gamma^2)}.\end{aligned}$$

Clearly, we have

$$\begin{aligned}\bar{c}-c^{N}(\infty) &\geq 0\Leftrightarrow\bar{c}\geq\varphi-1-\frac{\gamma(\omega_1+\delta\theta_1)}{\delta(\delta+\rho)+1},\\ \bar{c}-c^{N_0}(\infty) &\geq 0\Leftrightarrow\bar{c}\geq\varphi-1-\frac{\gamma\theta_1}{\delta+\rho}.\end{aligned}$$

□

Assumption 1 states that $\bar{c} < \varphi - 1$. Consequently, for the two conditions in the statement of the proposition to hold true, the terms $\frac{\gamma(\omega_1+\delta\theta_1)}{\delta(\delta+\rho)+1}$ and $\frac{\gamma\theta_1}{\delta+\rho}$ must be sufficiently large. If the impact of deviating from the norm, measured by γ, is low, then the conditions would be very difficult to satisfy. One interpretation of this result is as follows: if the public and watchdog organizations are lenient towards the charity when it comes to scrutinizing its operations, then the managers will tend to be generous in terms of wages and not very thrifty with the other costs involved in running the charity.

7 Concluding Remarks

In this paper, we introduced a dynamic model of charity's operations. The dynamic nature of our approach stems from the fact that goodwill (reputation) is the main driver of the donations received by a charity, and that it takes time and investments to build goodwill. To account for donors' preference for well-run charities, the goodwill evolution depends on the charity's relative efficiency with respect to an acceptable norm. Depending on the parameter values, we proved the existence of qualitatively different solutions involving either positive or zero values for advertising and administrative expenses.

Two extensions of our work are worth conducting. First, to account for some inherent uncertainties in the charity's revenues, one can let the donations be given by a stochastic function of the goodwill instead of a deterministic one. Further, it is most likely the case that charitable organizations compete for donations. Starting from the premise that individuals have a fixed budget to allocate to various charities, a second extension would be to generalize our model to a setting where charities compete for the consumer's donation.

8 Appendix

The superscripts of G, a, c, e referring to the different cases will be omitted in the proofs to simplify the notation.

8.1 Proof of Proposition 4.1

Proof. We need to solve the following pair of differential equations:

$$\begin{aligned} \dot{G}(t) &= -\delta G(t) + (1+\gamma^2)\lambda(t) - \omega_1 - \gamma\big(\varphi - 1 - \bar{c}\big), && (31)\\ \dot{\lambda}(t) &= G(t) + (\delta+\rho)\lambda(t) - \theta_1. && (32) \end{aligned}$$

This system can be written as a second-order linear-differential equation given by

$$\ddot{\lambda}(t) - \rho\dot{\lambda}(t) - [\delta(\delta+\rho)) + (1+\gamma^2)]\lambda(t) = -\omega_1 - \gamma\big(\varphi - 1 - \bar{c}\big) - \delta\theta_1.$$

Since $\Omega = \rho^2 + 4[\delta(\delta+\rho) + (1+\gamma^2)] > 0$, the general solution of the differential equation is

$$\lambda(t) = v_1 e^{\frac{\rho+\sqrt{\Omega}}{2}t} + v_2 e^{\frac{\rho-\sqrt{\Omega}}{2}t} + \frac{\omega_1 + \gamma\big(\varphi - 1 - \bar{c}\big) + \delta\theta_1}{[\delta(\delta+\rho) + (1+\gamma^2)]}.$$

Using

$$\dot{\lambda}(t) = v_1\left(\frac{\rho+\sqrt{\Omega}}{2}\right)e^{\frac{\rho+\sqrt{\Omega}}{2}t} + v_2\left(\frac{\rho-\sqrt{\Omega}}{2}\right)e^{\frac{\rho-\sqrt{\Omega}}{2}t},$$

we can compute G

$$
\begin{aligned}
G(t) &= v_1\left(-\delta+\frac{-\rho+\sqrt{\Omega}}{2}\right)e^{\frac{\rho+\sqrt{\Omega}}{2}t}+v_2\left(-\delta+\frac{-\rho-\sqrt{\Omega}}{2}\right)e^{\frac{\rho-\sqrt{\Omega}}{2}t} \\
&\quad +\frac{\theta_1\left(1+\gamma^2\right)-(\delta+\rho)\left(\omega_1+\gamma\left(\varphi-1-\bar{c}\right)\right)}{\delta(\delta+\rho)+1+\gamma^2}.
\end{aligned}
$$

Set

$$
\Phi=\frac{\theta_1\left(1+\gamma^2\right)-(\delta+\rho)\left(\omega_1+\gamma\left(\varphi-1-\bar{c}\right)\right)}{\delta(\delta+\rho)+1+\gamma^2}.
$$

Using the initial condition and looking for bounded solutions, we get $v_1=0$ and

$$
v_2=\frac{-2(G_0-\Phi)}{(2\delta+\rho+\sqrt{\Omega})}.
$$

Consequently,

$$
G(t)=G_0e^{\frac{\rho-\sqrt{\Omega}}{2}t}+\Phi\left(1-e^{\frac{\rho-\sqrt{\Omega}}{2}t}\right).
$$

Notice that

$$
\lambda(t)=-\frac{2(G_0-\Phi)}{(2\delta+\rho+\sqrt{\Omega})}e^{\frac{\rho-\sqrt{\Omega}}{2}t}+\frac{\theta_1-\Phi}{\delta+\rho}.
$$

$$
\lim_{t\longrightarrow+\infty}G(t)=\Phi=-(\delta+\rho)\lim_{t\longrightarrow+\infty}\lambda(t)+\theta_1.
$$

Conditions for an interior solution

We have $G(t)>0,\ a(t)>0,\ c(t)>0,\ e(t)>0$ under the following conditions:

$$
\omega_1<\min\{\ \frac{2\left(\Phi-G_0\right)}{2\delta+\rho+\sqrt{\Omega}}+\frac{\theta_1-\Phi}{\delta+\rho},\quad \frac{\theta_1-\Phi}{\delta+\rho},\ \frac{\theta_1\left(1+\gamma^2\right)}{\delta+\rho}-\gamma\left(\varphi-1-\bar{c}\right)\ \},\tag{33}
$$

$$
\frac{\varphi-1}{\gamma}>\max\left\{\frac{2\left(\Phi-G_0\right)}{2\delta+\rho+\sqrt{\Omega}}+\frac{\theta_1-\Phi}{\delta+\rho},\quad \frac{\theta_1-\Phi}{\delta+\rho}\ \right\},\tag{34}
$$

$$
\theta_1\ \min\{G_0,\Phi\}-\frac{\min\{G_0,\Phi\}^2}{2}>\omega_1\ \max\{a(\infty),a(0)\}+\frac{1}{2}\left(\max\{a(\infty),a(0)\}\right)^2+\max\{c(\infty),c(0)\}.\tag{35}
$$

The first two conditions can be rewritten as

$$
\begin{aligned}
\omega_1 &< \min\{\lambda(\infty),\lambda(0),\ \frac{\theta_1\left(1+\gamma^2\right)}{\delta+\rho}-\gamma\left(\varphi-1-\bar{c}\right)\ \}, \\
\frac{\varphi-1}{\gamma} &> \max\{\lambda(\infty),\lambda(0)\}.
\end{aligned}
$$

To show that $G(t)>0$, notice that

$$
\begin{aligned}
\Phi &> 0\Leftrightarrow\theta_1\left(1+\gamma^2\right)-(\delta+\rho)\left(\omega_1+\gamma\left(\varphi-1-\bar{c}\right)\right)>0, \\
&\Leftrightarrow\ \omega_1<\frac{\theta_1\left(1+\gamma^2\right)}{\delta+\rho}-\gamma\left(\varphi-1-\bar{c}\right),
\end{aligned}
$$

which is given by (33). And this implies $G(t)>0$.

We indicated in the text that the condition

$$\frac{\varphi-1}{\gamma}>\lambda\left(t\right)>\omega_1 \tag{36}$$

is equivalent to $a(t)>0,\ c(t)>0$.

Note that conditions (33) and (34) imply the above condition using the monotonicity of λ.

Indeed, when $G_0<\Phi$, λ is decreasing. Condition (33) implies $\omega_1<\frac{\theta_1-\Phi}{\delta+\rho}=\lambda(\infty)<\lambda(t)$ so $a(t)>0$.

Condition (34) implies $\frac{\varphi-1}{\gamma}>\frac{2(\Phi-G_0)}{2\delta+\rho+\sqrt{\Omega}}+\frac{\theta_1-\Phi}{\delta+\rho}=\lambda(0)>\lambda(t)$ so $c(t)>0$.

Finally to show that $e(t)>0$ for all t, i.e.,

$$\theta_1 G(t)-\frac{G(t)^2}{2}>\omega_1 a(t)+\frac{1}{2}\left(a\left(t\right)\right)^2+c\left(t\right),$$

it suffices to have

$$\inf_t\left\{\theta_1 G(t)-\frac{G(t)^2}{2}\right\}>\sup_t\left\{\omega_1 a(t)+\frac{1}{2}\left(a\left(t\right)\right)^2+c\left(t\right)\right\}.$$

Under the assumptions that $\varphi-1>\bar{c}$, we have $\lim\limits_{t\longrightarrow+\infty}\lambda(t)>0$. Since $G_0<\Phi$, $G(t)$ is increasing over time. Moreover, $G(t)$ goes to $\Phi=\theta_1-(\delta+\rho)\lim\limits_{t\longrightarrow+\infty}\lambda(t)<\theta_1$. But then, it is clear that $\theta_1 G(t)-\frac{G(t)^2}{2}$ is minimized when $G(t)=G(0)$.

In addition, since $a(t)$ is decreasing and $c(t)$ is increasing, the result follows.

When $G_0>\Phi$, the above results are obtained in an analogous way.

Finally, since H is concave with respect to (G,a,c,e) and $\lim\limits_{t\longrightarrow+\infty}e^{-\rho t}\lambda(t)G(t)=0$, we have that the solution of the first-order conditions, ($G(t),a(t),c(t),e(t)$) is the solution of the problem. □

8.2 Proof of Proposition 4.4

Proof. We shall show that under the following conditions, $(G(t),a(t),c(t),e(t))$, as defined in Proposition 4.4, is an optimal solution:

$$\varphi-1<\frac{\theta_1\gamma}{\delta+\rho}+\bar{c}, \tag{37}$$

$$\min\left\{\omega_1,\frac{\varphi-1}{\gamma}\right\}>\max\ \left\{\frac{\delta\theta_1+\gamma\left(\varphi-1-\bar{c}\right)}{\delta(\delta+\rho)+\gamma^2},\ \frac{2\left(-G_0+\Lambda\right)}{(2\delta+\rho+\sqrt{\Theta})}+\frac{\delta\theta_1+\gamma\left(\varphi-1-\bar{c}\right)}{\delta(\delta+\rho)+\gamma^2}\right\}, \tag{38}$$

$$\theta_1\ \min\{\ G(0),\ G(\infty)\ \}-\frac{1}{2}(\min\ \{G(0),\ G(\infty)\ \})^2>\varphi-1. \tag{39}$$

Notice that (38) can be written as

$$\min\left\{\omega_1,\frac{\varphi-1}{\gamma}\right\}>\max\ \{\ \lambda(\infty),\ \lambda(0)\ \}. \tag{40}$$

We shall first show that $(G(t),a(t),c(t),e(t))$, as defined in the proposition, is admissible and satisfies the first-order conditions. Sufficient conditions of optimality will then guarantee that it is optimal.

When

$$a(t)=0,$$

$$G(t)=\left(G_0-\Lambda\right)e^{\frac{\rho-\sqrt{\Theta}}{2}t}+\Lambda,$$

$$c\left(t\right)=\varphi-1-\gamma\big(-\frac{2\left(G_0-\Lambda\right)}{2\delta+\rho+\sqrt{\Theta}}e^{\frac{\rho-\sqrt{\Theta}}{2}t}+\frac{\theta_1-\Lambda}{\delta+\rho},\big),$$

$$e(t)=\theta_1 G(t)-\frac{1}{2}\left(G(t)\right)^2-c(t),$$

for all t, the state equation is satisfied. Indeed,

$$\begin{aligned}
\dot{G}\left(t\right) &= \frac{\rho-\sqrt{\Theta}}{2}\left(G_0-\Lambda\right)e^{\frac{\rho-\sqrt{\Theta}}{2}t}, && (41)\\
&= \left(-\delta-\frac{2\gamma^2}{2\delta+\rho+\sqrt{\Theta}}\right)\left(G_0-\Lambda\right)e^{\frac{\rho-\sqrt{\Theta}}{2}t}+\frac{\left(\theta_1\gamma-\left(\delta+\rho\right)\left(\varphi-1-\bar{c}\right)\right)\gamma}{\left(\delta+\rho\right)} && (42)\\
&- \frac{\Lambda(\delta(\delta+\rho)+\gamma^2)}{\left(\delta+\rho\right)}, && (43)
\end{aligned}$$

since $\left(-\delta-\frac{2\gamma^2}{2\delta+\rho+\sqrt{\Theta}}\right)=\frac{\rho-\sqrt{\Theta}}{2}$ and $\Lambda=\frac{(\theta_1\gamma-(\delta+\rho)(\varphi-1-\bar{c}))\gamma}{\delta(\delta+\rho)+\gamma^2}$. So

$$\begin{aligned}
\dot{G}\left(t\right) &= -\delta\left(G_0-\Lambda\right)e^{\frac{\rho-\sqrt{\Theta}}{2}t}-\frac{2\gamma^2}{2\delta+\rho+\sqrt{\Theta}}\left(G_0-\Lambda\right)e^{\frac{\rho-\sqrt{\Theta}}{2}t}+\frac{(\theta_1-\Lambda)\gamma^2}{\delta+\rho}, && (44)\\
&- \gamma(\varphi-1)+\gamma\bar{c}-\delta\Lambda, && (45)\\
&= -\delta G(t)-\gamma\big(\varphi-1-\gamma\big(-\frac{2\left(G_0-\Lambda\right)}{2\delta+\rho+\sqrt{\Theta}}e^{\frac{\rho-\sqrt{\Theta}}{2}t}+\frac{\theta_1-\Lambda}{\delta+\rho}\big)-\bar{c}\big), && (46)
\end{aligned}$$

which is exactly

$$\dot{G}\left(t\right)=-\gamma\big(c\left(t\right)-\bar{c}\big)-\delta G\left(t\right).$$

We also have $G\left(0\right)=G_0$.

Set

$$\lambda(t)=-\frac{2\left(G_0-\Lambda\right)}{2\delta+\rho+\sqrt{\Theta}}e^{\frac{\rho-\sqrt{\Theta}}{2}t}+\frac{\theta_1-\Lambda}{\delta+\rho},$$

and

$$\begin{aligned}
&x(t)=y(t)=0, && (47)\\
&\psi(t)=1, && (48)\\
&s(t)=\omega_1-\lambda(t). && (49)
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
c(t) &= \varphi-1-\gamma\lambda(t), && (50)\\
0 &= a(t)=-\omega_1+\lambda(t)+s(t), && (51)\\
\dot{\lambda}(t) &= G(t)+(\delta+\rho)\lambda(t)-\theta_1. && (52)
\end{aligned}$$

To show that $G(t)>0$, we need to have $\Lambda>0$, which is equivalent to condition (37).

We must verify that $c(t)>0$ and the condition $s(t)\geq 0$, which are respectively equivalent to $\frac{(\varphi-1)}{\gamma}>\lambda(t)$ and $\omega_1\geq\lambda(t)$. Hypothesis (38) will guarantee these conditions.

Indeed, when $G_0 > \Lambda$, $\lambda(t)$ is increasing. So $\lambda(t) \leq \lim_{t\longrightarrow+\infty} \lambda(t)$. Under the hypothesis (38) we have

$$\omega_1 > \frac{\delta\theta_1 + \gamma\left(\varphi - 1 - \bar{c}\right)}{\delta(\delta+\rho)+\gamma^2} = \lim_{t\longrightarrow+\infty} \lambda(t) \geq \lambda(t),$$

and so, $s(t) > 0$. We also have

$$\frac{(\varphi-1)}{\gamma} > \frac{\delta\theta_1 + \gamma\left(\varphi - 1 - \bar{c}\right)}{\delta(\delta+\rho)+\gamma^2} = \lim_{t\longrightarrow+\infty} \lambda(t) \geq \lambda(t),$$

so, $c(t) > 0$.

Now, we can show, as in the proof of Proposition 1, that under condition (39), we have $e(t) > 0$ for all t.

Similar reasoning can be used when $G_0 < \Lambda$.

Finally, since H is concave with respect to (G, a, c, e) and $\lim_{t\longrightarrow+\infty} e^{-\rho t}\lambda(t)G(t) = 0$, we have that $(G(t), a(t), c(t), e(t))$ is an optimal solution of the problem. □

8.3 Proof of Proposition 5.1

Proof. We shall show that under the following conditions, $(G(t), a(t), c(t), e(t))$, as defined in Proposition 5.1, is an optimal solution:

$$\omega_1 < \gamma\bar{c} + \frac{\theta_1}{(\delta+\rho)}, \tag{53}$$

$$\min\left\{-\frac{2\left(G_0 - \Gamma\right)}{2\delta+\rho+\sqrt{\Delta}} + \frac{\theta_1 - \Gamma}{\delta+\rho}, \; \frac{\theta_1 - \Gamma}{\delta+\rho}\right\} > \max\left\{\omega_1, \frac{\varphi-1}{\gamma}\right\}, \tag{54}$$

$$\theta_1 \; \min\left\{\, \Gamma, \; G_0 \,\right\} - \frac{1}{2}(\min\left\{\, \Gamma, \; G_0 \,\right\})^2 > \omega_1 \max\left\{a(0), \; a(\infty)\,\right\} + \frac{1}{2}(\max\left\{a(0), \; a(\infty)\,\right\})^2. \tag{55}$$

Notice that condition (54) can be written as

$$\min\left\{\lambda(0), \; \lambda(\infty)\,\right\} > \max\left\{\omega_1, \frac{\varphi-1}{\gamma}\right\}.$$

We shall first show that $(G(t), a(t), c(t), e(t))$, as defined in the proposition, is admissible and satisfies the first-order conditions. Sufficient conditions of optimality will then guarantee that it is optimal.

When

$$a(t) = -\omega_1 - \left(\frac{2\left(G_0 - \Gamma\right)}{2\delta+\rho+\sqrt{\Delta}} e^{\frac{\rho-\sqrt{\Delta}}{2}t} - \frac{\theta_1 - \Gamma}{\delta+\rho}\right),$$

$c(t) = 0$, $e(t)$ given by (29) and

$$G(t) = e^{\frac{\rho-\sqrt{\Delta}}{2}t} G_0 + \left(1 - e^{\frac{\rho-\sqrt{\Delta}}{2}t}\right) G\left(\infty\right),$$

for all t, the state equation is satisfied. Indeed,

$$\begin{aligned} \dot{G}\left(t\right) &= \frac{\rho-\sqrt{\Delta}}{2} e^{\frac{\rho-\sqrt{\Delta}}{2}t} G_0 - \frac{\rho-\sqrt{\Delta}}{2} e^{\frac{\rho-\sqrt{\Delta}}{2}t} G\left(\infty\right), && (56) \\ &= \frac{\rho-\sqrt{\Delta}}{2} e^{\frac{\rho-\sqrt{\Delta}}{2}t} (G_0 - \Gamma) - \omega_1 + \frac{\theta_1 - \Gamma}{\delta+\rho} - \delta\Gamma + \gamma\bar{c}, && (57) \end{aligned}$$

since $G(\infty) = \Gamma$ and $-\omega_1 + \frac{\theta_1 - \Gamma}{\delta+\rho} - \delta\Gamma + \gamma\bar{c} = 0$ from the definition of Γ. Consequently,

$$\dot{G}(t) = e^{\frac{\rho-\sqrt{\Delta}}{2}t}(G_0 - \Gamma)[-\delta - \frac{2}{2\delta+\rho+\sqrt{\Delta}}] - \omega_1 + \frac{\theta_1 - \Gamma}{\delta+\rho} - \delta\Gamma + \gamma\bar{c},$$

since $\left(-\delta - \frac{2}{2\delta+\rho+\sqrt{\Delta}}\right) = \frac{\rho-\sqrt{\Delta}}{2}$. So

$$\dot{G}(t) = -\omega_1 + \frac{\theta_1 - \Gamma}{\delta+\rho} - \frac{2}{2\delta+\rho+\sqrt{\Delta}}e^{\frac{\rho-\sqrt{\Delta}}{2}t}(G_0 - \Gamma) + e^{\frac{\rho-\sqrt{\Delta}}{2}t}(G_0 - \Gamma)(-\delta) - \delta\Gamma + \gamma\bar{c},$$

which is exactly

$$\dot{G}(t) = a(t) - \delta G(t) + \gamma\bar{c},$$

and the initial condition $G(0) = G_0$ is also satisfied.

Set

$$\lambda(t) = -\frac{2(G_0 - \Gamma)}{2\delta+\rho+\sqrt{\Delta}}e^{\frac{\rho-\sqrt{\Delta}}{2}t} + \frac{\theta_1 - \Gamma}{\delta+\rho}, \tag{58}$$

and

$$s(t) = y(t) = 0, \tag{59}$$
$$\psi(t) = 1, \tag{60}$$
$$x(t) = \gamma\lambda(t) - (\varphi - 1). \tag{61}$$

Then, we have

$$a(t) = -\omega_1 + \lambda(t) = -\omega_1 + \lambda(t) + s(t), \tag{62}$$
$$c(t) = 0 = (\varphi - 1) - \gamma\lambda(t) + x(t), \tag{63}$$
$$\dot{\lambda}(t) = G(t) + (\delta+\rho)\lambda(t) - \theta_1. \tag{64}$$

To show that $G(t) > 0$, we need to have $\Gamma > 0$, which is equivalent to condition (53). It remains to check that $a(t) > 0$, $e(t) > 0$ and $x(t) \geq 0$.

When $G_0 > \Gamma$, $\lambda(t)$ is increasing. So, $\lambda(0) \leq \lambda(t)$. Under hypothesis (54), we have

$$\omega_1 < -\frac{2(G_0 - \Gamma)}{2\delta+\rho+\sqrt{\Delta}} + \frac{\theta_1 - \Gamma}{\delta+\rho} = \lambda(0) \leq \lambda(t),$$

and $\omega_1 < \lambda(t)$ is equivalent to $a(t) > 0$. We also have

$$\frac{(\varphi - 1)}{\gamma} < -\frac{2(G_0 - \Gamma)}{2\delta+\rho+\sqrt{\Delta}} + \frac{\theta_1 - \Gamma}{\delta+\rho} = \lambda(0) \leq \lambda(t),$$

and $\frac{(\varphi-1)}{\gamma} < \lambda(t)$ is equivalent to $x(t) > 0$.

Moreover, since $G(t)$ decreases, and $a(t)$ increases, a sufficient condition for $e(t) > 0$ for all t is therefore

$$\inf_t \left\{\theta_1 G(t) - \frac{1}{2}G(t)^2\right\} > \sup_t \left\{\omega_1 a(t) + \frac{1}{2}a(t)^2\right\},$$

which is obtained due to condition (55).

When $G_0 > \Gamma$ a similar proof applies.

Finally, since H is concave with respect to (G, a, c, e) and $\lim_{t\longrightarrow+\infty} e^{-\rho t}\lambda(t)G(t) = 0$, we have that $(G(t), a(t), c(t), e(t))$ is an optimal solution of the problem. □

8.4 Proof of Proposition 5.2

Proof. We shall show that $(G(t), a(t), c(t), e(t)) = (\ (G_0 - \frac{\gamma\bar{c}}{\delta})e^{-\delta t} + \frac{\gamma\bar{c}}{\delta},\ 0,\ 0,\ \theta_1 G(t) - \frac{G(t)^2}{2}\ )$ is optimal under the following condition:

$$\min\{\ \omega_1,\ \omega_1 + \frac{G_0 - \frac{\gamma\bar{c}}{\delta}}{(2\delta+\rho)}\ \} \geq \frac{(\theta_1 - \frac{\gamma\bar{c}}{\delta})}{(\delta+\rho)} \geq \max\{\frac{\varphi-1}{\gamma} + \frac{G_0 - \frac{\gamma\bar{c}}{\delta}}{(2\delta+\rho)},\ \frac{\varphi-1}{\gamma}\}. \tag{65}$$

For this, we shall first show that it is admissible and that it satisfies the first-order conditions. Sufficient conditions of optimality will then guarantee that it is optimal.

It is clear that $G(t) \geq 0$. Now since $G(t) = (G_0 - \frac{\gamma\bar{c}}{\delta})e^{-\delta t} + \frac{\gamma\bar{c}}{\delta}$, we have

$$\begin{aligned} \dot{G}(t) &= -\delta(G_0 - \frac{\gamma\bar{c}}{\delta})e^{-\delta t}, &(66)\\ &= -\delta G(t) + \gamma\bar{c}. &(67) \end{aligned}$$

So $(G(t), a(t), c(t), e(t))$, where $G(t) = (G_0 - \frac{\gamma\bar{c}}{\delta})e^{-\delta t} + \frac{\gamma\bar{c}}{\delta}$, $a(t) = 0, c(t) = 0$ for all t, satisfies the state equation and $G(t)$ satisfies the initial condition $G(0) = G_0$.

Set $y(t) = 0$, $\psi(t) = 1$ and $\lambda(t) = -\frac{G_0 - \frac{\gamma\bar{c}}{\delta}}{(2\delta+\rho)}e^{-\delta t} + \frac{(\theta_1 - \frac{\gamma\bar{c}}{\delta})}{(\delta+\rho)}$. Therefore, $\lambda(t)$ satisfies the adjoint equation

$$\dot{\lambda}(t) = G(t) + (\delta+\rho)\lambda(t) - \theta_1.$$

Indeed

$$\begin{aligned} \dot{\lambda}(t) &= \delta\frac{G_0 - \frac{\gamma\bar{c}}{\delta}}{(2\delta+\rho)}e^{-\delta t} = -\delta\lambda(t) + \delta\frac{(\theta_1 - \frac{\gamma\bar{c}}{\delta})}{(\delta+\rho)},\\ &= -\delta\lambda(t) + \delta\frac{(\theta_1 - \frac{\gamma\bar{c}}{\delta})}{(\delta+\rho)} + (2\delta+\rho)\lambda(t) - (2\delta+\rho)\lambda(t),\\ &= (\delta+\rho)\lambda(t) + \delta\frac{(\theta_1 - \frac{\gamma\bar{c}}{\delta})}{(\delta+\rho)} - (2\delta+\rho)\left(-\frac{G_0 - \frac{\gamma\bar{c}}{\delta}}{(2\delta+\rho)}e^{-\delta t} + \frac{(\theta_1 - \frac{\gamma\bar{c}}{\delta})}{(\delta+\rho)}\right),\\ &= (\delta+\rho)\lambda(t) - \theta_1 + (G_0 - \frac{\gamma\bar{c}}{\delta})e^{-\delta t} + \frac{\gamma\bar{c}}{\delta}. \end{aligned}$$

Set $s(t) = \omega_1 - \lambda(t)$ and $x(t) = -(\varphi-1) + \gamma\lambda(t)$. Then, we have

$$\begin{aligned} -\omega_1 + \lambda(t) + s(t) = 0, &\quad (68)\\ (\varphi-1) - \gamma\lambda(t) + x(t) = 0, &\quad (69) \end{aligned}$$

with $s(t) \geq 0$ and $x(t) \geq 0$ as shown below.

Indeed, when $G_0 > \frac{\gamma\bar{c}}{\delta}$, $\lambda(t)$ is increasing. So $\lambda(0) \leq \lambda(t) \leq \lim_{t\to+\infty}\lambda(t)$.

Under condition (65), we have

$$\omega_1 \geq \frac{(\theta_1 - \frac{\gamma\bar{c}}{\delta})}{(\delta+\rho)} = \lim_{t\to+\infty}\lambda(t) \geq \lambda(t),$$

and $\omega_1 \geq \lambda(t)$ is equivalent to $s(t) \geq 0$.

We also have

$$\lambda(t) \geq \lambda(0) = -\frac{G_0 - \frac{\gamma\bar{c}}{\delta}}{(2\delta+\rho)} + \frac{(\theta_1 - \frac{\gamma\bar{c}}{\delta})}{(\delta+\rho)} \geq \frac{(\varphi-1)}{\gamma},$$

and $\gamma\lambda(t) \geq (\varphi - 1)$ is equivalent to $x(t) \geq 0$.

To show that $e(t) = \theta_1 G(t) - \frac{G(t)^2}{2} \geq 0$, notice that $G_0 < \theta_1$ by Assumption 3 and that G is decreasing when $G_0 > \frac{\gamma\bar{c}}{\delta}$.

When $G_0 < \frac{\gamma\bar{c}}{\delta}$, the proof is analogous.

Hence $(G(t), a(t), c(t), e(t))$ is a candidate for optimality.

As H is concave with respect to (G, a, c, e) and $\lim_{t \longrightarrow +\infty} e^{-\rho t}\lambda(t)G(t) = 0$, we have that $(G(t), a(t), c(t), e(t))$ is an optimal solution of the problem. □

References

[1] Aldashev, Gani and Thierry Verdier (2010), Goodwill bazaar: NGO competition and giving to development, *Journal of Development Economics*, 91, 48-63.

[2] Aldashev, Gani, Marini Marco and Thierry Verdier (2014), Brothers in alms's: coordination between nonprofits on markets for donations, *Journal of Public Economics*,Volume 117, September 2014, 182-200.

[3] André, Francisco, Abderrahmane, Sokri and Georges. Zaccour (2011), Public Disclosure Programs vs. traditional approaches for environmental regulation, green goodwill and the policies of the firm, *European Journal of Operational Research*, Elsevier, vol. 212 (1), pages 199-212, July

[4] Andreoni, James., Payne, Abigail A. (2013) Charitable giving, Chapter 1 in *Handbook of Public Economics*, Volume 5, (Amsterdam: North Holland), Alan Auerbach, Raj Chetty, Martin Feldstein, Emmanuel Saez, editors, 1-50.

[5] Bekkers, René and Pamela Wiepking (2011), A literature review of empirical studies of philanthropy: Eight mechanisms that drive charitable giving, *Nonprofit and Voluntary Sector Quarterly*, 40 (5) 924- 973.

[6] Calabresi, Guido (2016), The future of Law and Economics, Yale University Press, New Heaven.

[7] Fabsikova, Nina and Stranak, Benedikt (2018), Charitable donations in crisis: Is trust a part of the story? https://devcommslab.org/blog/charitable-donations-in-crisis-is-trust-a-part-of-the-story/

[8] Feng Qi and J. George Shanthikumar (2017), Not-for-profit Operations Management, Chapter 27, *Routledge Companion for Production and Operations Management*, Edited by Sushil Gupta and Martin Starr, Taylor & Francis.

[9] Frumkin, Peter and Mark T. Kim (2001), Strategic positioning and the financing of nonprofit organizations: Is efficiency rewarded in the contributions marketplace? *Public Administration Review*, 61, 3, 266-275.

[10] Glaeser, Edward (2003), The governance of not-for-profit firms, in Edward Glaeser, ed., *The Governance of Nonprofit Firms*, Chicago, University of Chicago Press.

[11] Huang, Jian, Lemg, Mingming., Liang, Liping, (2012), Recent developments in dynamic advertising research, *European Journal of Operational Research*, 220, 591-609.

[12] Jørgensen, Steffen., Zaccour, Georges (2014), Cooperative advertising in marketing channels: Game theoretic analyses, *European Journal of Operational Research*, 237, 1-14.

[13] Meer, Jonathan (2017), Does fundraising create new giving? *Journal of Public Economics*, 145, 82-93.

[14] Monnet, Nathalie and Ugo Panizza (2017), A Note on the economics of philanthropy, The Graduate Institute Geneva.

[15] Okten, Cagla and Burton A. Weisbrod (2000), Determinants of donations in private nonprofit markets, *Journal of Public Economics*, 2000, 75 (2), 255-272.

[16] Privett, Nathalie, Feryal Erhun (2011), Efficient funding: Auditing in the nonprofit sector, *Manufacturing and Service Operations Management*, 13 (4) 471-488.

[17] Rose-Ackerman, Susan (1982), Charitable giving and excessive fundraising, *Quarterly Journal of Economics*, 97 (2). pp. 193-212.

[18] Shang, Jen and Rachel Croson (2009), A Field experiment in charitable contribution: The impact of social information on the voluntary provision of public Ggods, *The Economic Journal*, 119 (October), 1422-1439.

[19] Yörük, Bariş (2016), Charity Rating, *Journal of Economics & Management Strategy*, Wiley Blackwell, 25 (1), pp 195-219.