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On the existence of pairwise stable weighted networks
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Abstract

We prove the existence of pairwise stable weighted networks under assumptions on payoffs which are similar to those in Nash's theorem (quasiconcavity and continuity). Then, we extend our result, allowing payoffs to depend not only on the network, but also on some game-theoretic strategies. The proof is not a standard application of tools from game theory, the difficulty coming from the fact that pairwise stability notion shares both cooperative and non cooperative features. Last, some examples are given, and illustrate how our result may open new paths in the literature on network formation.

JEL codes: C72, D85.
Keywords: Pairwise Stable Network, Weighted Network.

1. Introduction

In recent years, there has been a growing interest in networks for modelling social and economic interactions. A network can be defined as a set N of nodes (representing for example economic agents, families, web sites, scholarly publications, etc.) and a set of links, measuring some relationships between the nodes (for example, friendship or familial relationships, co-author relationships, hyperlinks between web pages, transactions between financial institutions, etc.).

An important contribution of network literature has been to propose strategic models of network formation (e.g., Aumann and Myerson [1], Slikker and van den Nouweland [24] or Myerson [21]),[1] explaining how, and why, agents establish links, based on payoff maximization. In particular, such models have helped to understand the emergence of specific forms of networks.

A key concept involved in network formation theory, introduced in their seminal paper by Jackson and Wolinsky [18], is *pairwise stability.*[2] Roughly, a network is pairwise stable if "no two agents could gain from linking and no single agent could gain by severing one of his or her link" (see [18]). Its main distinctive feature, compared with Nash equilibrium, is its "cooperative" aspect, since it also takes *bilateral deviations* into account (when links are created), whereas Nash equilibrium concept only considers *unilateral deviations.* Beyond this difference, it is also a natural predictive solution concept.

Since the seminal paper of Jackson and Wolinsky [18], pairwise stability has been one of the most popular stability concepts in the network literature (see, for example, Jackson and Watts [16], Goyal and Joshi [12], Calvó-Armengol and İlkılıç [8], Hellmann [13] [14], Miyauchi [20], Bloch and Dutta [3], and for surveys Jackson [15], Mauleon and Vannetelbosch [19]). One of its remarkable features is that it allows to endogenize the formation of networks. Yet, there is no general existence result of weighted pairwise stable networks, in the spirit of Nash equilibrium existence result (see [11], [22] or [23]).

[1] For a recent survey on this subject, see Chapter 6 of "The Oxford Handbook on the Economics of Networks", by Ana Mauleon and Vincent Vannetelbosch [19].

[2] This is not the only stability concept used in network formation theory. For example, for networks whose weights of links are separable functions of the efforts of the agents implied in the link, Nash equilibrium is a good candidate: see [3] or [25]. See also Section 3.3.

The few existing results in the literature consider *unweighted*[3] networks. Unfortunately, in general, there may be no unweighted pairwise stable network, which is due to the possibility of the existence of improving cycles (a closed path of networks for which at each step, the utility of one or two agents is improved by deleting or adding a link). Indeed, Jackson and Watts (see [16] and [17]) have proved that for every profile of payoff functions, either there exists a pairwise stable network, or there exists a closed improving cycle. As a byproduct, under the assumption that there exists some network potential function, closed cycles are ruled out, and the existence of pairwise stable networks is obtained (Jackson and Watts [16]). This general approach has been extended or refined (see Chakrabarti and Gilles [9] or Hellmann [13]), but in practice, it remains difficult to know if there exists or not some potential function.

There are several reasons for being interested in *weighted networks*, i.e. networks for which the relationships are measured by reals. First, in many situations, it is natural to quantify the weight of a relationship in a continuous way (it can measure a debt, some level of confidence, some capacity, a geographical distance...). A second reason is the simplicity of continuous models: for example, in game theory, it is well known that passing to continuous strategy spaces (through mixed strategies for example) allows to get the existence of a Nash equilibrium almost for free. We will see that this is also true for network theory: passing from unweighted networks to weighted ones allows to get the existence of a pairwise stable network under weak assumptions. This should allow for many developments (e.g., structure of the set of pairwise stable networks, refinement notions, conditions for uniqueness, ...).

Aim and main results of our paper

The aim of this paper is to prove the existence of a pairwise stable *weighted* network under conditions similar to Nash theorem existence result: some geometric condition (quasiconcavity of payoffs with respect to each link), and some topological condition (continuity). In particular, as opposed to the case of unweighted networks, the existence of closed improving cycles does not prevent the existence of pairwise stable weighted networks.

Our method of proof (1) is specific to weighted networks (2) does not use a direct game theory argument. Indeed, first, previous arguments for unweighted networks cannot be extended to our framework, because our main assumptions (continuity and quasiconcavity of payoffs) do not guarantee the existence of a potential function, a crucial assumption for proving the existence of unweighted pairwise stable networks. Second, applying directly tools from game theory to our problem is problematic: if we think about the weights of the links as strategic variables, then it is unclear how the players and the associated payoff functions can be defined, because at each link are attached two agents (thus two distinct payoff functions), and also because the rules of pairwise stability mix cooperative and non cooperative behaviours. To solve this issue, we first compute the two sets of optimal weights (in terms of payoff maximization) of each player at every link (all the other weights being fixed). Then, the main idea of the proof is to "merge" these optimal sets, in order to (1) take into account the structure of Pairwise stability concept, (2) keep enough regularity to be able to apply Kakutani's fixed-point theorem to the multivalued function thus defined[4]. This will provide the existence of a pairwise stable weighted network.

More generally, we prove the existence of a *Nash-pairwise stable pair* (g, y), which incorporates a network g and some strategy profile $y = (y_1, ..., y_N)$ of an additional normal form game between

[3]A network is unweighted if a link can be in two possible states (connection or no connection). In particular, in this case, the space of links is discrete.

[4]This is the multivalued function which associates to each weighted network the product of the "merged" optimal sets at each link.

the agents involved in the network. Here, the strategy y_i may have no direct relationship with the network.[5] For example, we can think that some players $i = 1, ..., N$ exert some efforts (these are the strategies y_i of the game) for the provision of a public good, and that at the same time, an endogenous network g of relationships is formed between these agents. Here, the weights of g are not functions of the y_i, but the payoff function of each player depends on (g, y), so that at equilibrium, the network will depend on the choices of efforts (and conversely).

Roughly, a pair (g, y) is Nash-pairwise stable if:

1. The network g being fixed, no player can improve strictly his payoff by changing his individual strategy y_i.
2. The profile of strategies y being fixed, no player can improve strictly his payoff by decreasing his link with another player.
3. Finally, the profile of strategies y being fixed, no two agents could gain from increasing the weight of their common link.

The case where y plays no role gives the standard pairwise stability concept (thus in particular, our general existence result entails the existence of pairwise stable networks), and the case where g plays no role provides a standard Nash existence result. Our formalism encompasses many existing models, and provides some endogenous explanation of network formation in most of them, for example models on public goods provision on networks (e.g., [6]).

Organization of the paper

The paper is organized as follows. After preliminaries, where pairwise stability definition is recalled (Section 2), we state our main existence result and provide some examples (Section 3). The proof of our main result is given in Section 4.

2. Preliminaries

There are N agents[6] which interact in a network of relationships. The strength[7] of the relationship involving two agents $i \in N$ and $j \in N$ is $g_{ij} \in [0, 1]$. For example, g_{ij} can measure information exchange, or time spent together. We let $\mathcal{L} := \{(i, j) \in N \times N : i \neq j\}$ be the set of directed links, and $\mathcal{L}' := \{\{i, j\} \in N \times N : i \neq j\}$ be the set of undirected links. For simplicity, and when there is no risk of confusion, we will denote a link ij, with $ij = ji$ when $ij \in \mathcal{L}'$ and $ij \neq ji$ when $ij \in \mathcal{L}$. Formally:

Definition 2.1.— *A weighted network*[8] *(on N) is a mapping g from $\mathcal{L}$ to $[0, 1]$ such that $g(ij) = g(ji)$ for every $ij \in \mathcal{L}$. The network g is unweighted if for every $ij \in \mathcal{L}$, $g(ij) \in \{0, 1\}$.*

For convenience, if i and j are two distinct elements of N, $g(ij)$ will be simply denoted g_{ij}. Throughout this paper, we let $\mathcal{G}$ be the set of weighted networks. For every $ij \in \mathcal{L}$, δ^{ij} is the network with only one link (of weight 1) between i and j, that is δ^{ij}_{kl} is equal to 0 if $kl \neq ij$ and equal to 1 otherwise. The following definition is the natural adaptation of pairwise stability concept (Jackson

[5] Contrarily to most standard models of network formation using games, as Bloch-Dutta [3], where players exert some efforts (which are the strategies of the game) to create the links, the weights of the links being direct functions of the efforts.

[6] For simplicity, N denotes either the set of agents, or its cardinal, and we assume $N \geq 2$.

[7] The normalization in $[0, 1]$ is without any loss of generality.

[8] Note that what we call "weighted network" is usually called "undirected weighted network" in the literature. The network would be directed if we would authorize $g(ij)$ and $g(ji)$ to be different.

and Wolinsky [18]) to weighted networks. Hereafter, we let $v_i : \mathcal{G} \to \mathbf{R}$ be the payoff function of each agent $i \in N$, and $v = (v_i)_{i \in N}$ be the profile of payoff functions. Moreover, if $g = (g_{ij})_{ij \in \mathcal{L}}$ is a weighted network, then for every $ij \in \mathcal{L}$ and every $x \in [0,1]$, $g' = (x, g_{-ij})$ denotes the weighted network such that $g'_{kl} = g_{kl}$ for every $kl \neq ij$ and $kl \neq ji$, and $g'_{ij} = g'_{ji} = x$.

Definition 2.2.— *A weighted network $g \in \mathcal{G}$ is said to be pairwise stable (resp. weakly pairwise stable) with respect to v if:*

1. *for every $ij \in \mathcal{L}$, for every $x \in [0, g_{ij}[$, $v_i((x, g_{-ij})) \leq v_i(g)$.*
2. *for every $ij \in \mathcal{L}$, for every $x \in]g_{ij}, 1]$, if $v_i((x, g_{-ij})) > v_i(g)$ then $v_j((x, g_{-ij})) < v_j(g)$ (resp. $v_j((x, g_{-ij})) \leq v_j(g)$).*

Thus, g is pairwise stable if no two agents could gain from increasing the weight of their common link (Condition 2) and no single agent could gain by diminishing it (Condition 1). Condition 1 has to be true if we reverse ij into ji, thus it holds for both players i and j. The difference between pairwise stability and weak pairwise stability depends on what is meant by "no two agents could gain": if the mutual gain is required to be strict for the two players, we get weak pairwise stability concept, if it is required to be strict only for one agent, we get the (standard) pairwise stability concept.

A first remark is that pairwise stability notion treats differently an increase and a decrease of a link (the network being weighted or not). In social networks, for example, you can decide *alone* to remove some link with another person, but you should decide *together* to increase the time you spend to write to each other.

A second important remark is that Jackson and Wolinsky's definition of pairwise stability (as well as our extension to weighted networks) requires that each link g_{ij} can be increased independently of the other links. In particular, this does not allow to bound exogenously the sum of weights around every agent, in order to take into account that building a link is costly in terms of time, of energy, etc. Indeed, if such an exogenous bound exists, then when the associated constraint is binded for some agent, increasing the weight of some link around this agent requires to decrease at the same time the weight of another neighbor link in order to respect the constraint, which is forbidden in pairwise stability definition. Fortunately, this is not really a problem, since if we want to model some cost of link creation (if any), we can simply incorporate it into the payoff function,[9] which will bound (endogenously) the weights of links around this agent at every pairwise stable network g.

3. Existence of Nash-pairwise stable profiles

In the following, we provide a general existence result of a solution concept we call Nash-pairwise stable pair, and which involves two parts: first a weighted network g which some agents form

[9] For example, the initial payoff v_i of player i can be modified to incorporate this cost, by defining $\tilde{v}_i(g) = v_i(g) - \lambda(\sum_{j \neq i} g_{ij})$ or $\tilde{v}_i(g) = v_i(g) - \lambda(\sum_{j \neq i} g_{ij}^2)$ (where $\lambda > 0$ is some fixed parameter), depending if the cost is a linear or a quadratic function of the weights $(g_{ij})_{j \neq i}$ around player i. Note that an endogenous modelling of the costs induced by link creation may be often more relevant than an exogenous one. For example, if each agent i has a maximum amount of money to allocate, in order to keep (or to increase) the weights of the links around him, we could assume that i can always borrow some (costly) extra amount of money to increase some link if necessary. Remark that such modelling question is not specific to Network theory, but can be found everywhere in Economic theory (see for example the modelling of transaction costs in General Equilibrium).

endogenously according to pairwise stability rules,[10] and simultaneously, some normal form game that the agents can play. Here, the strategies of this game have no direct relationship with the formation of the weights (as opposed to Bloch [3], where the strategies can be seen as amounts of resources invested to form the weights of the links). An illustration, which is developed hereafter, would be a public good model with N agents exerting some efforts (these are the strategies of the game) for the provision of a public good, the agents forming at the same time some network of relationships. Here, there are two different strategic aspects: (1) the choice of some level of connection with each other agent (depending on its own level of efforts, but also on his location in the network), and (2) the choice of the effort itself, which could open the door to free riding. Remark that even if there are no direct relationship between the effort of each player and the weight of its links, at equilibrium, the two variables should be related, so that finally each effort influences indirectly the formation of the network. For example, we could imagine situations where free riders choose very low efforts, and connect to all agent providing some high level of provision of public good.

3.1 The general existence result

We now describe the model considered in this paper, which encompasses several existing models of networks and games. Each player $i \in N$ has to choose some strategy y_i in some strategy space Y_i (for example, in the public model example discussed above, y_i is the level of effort of player i for the provision of a public good). We denote by $Y = \Pi_{i \in N} Y_i$ the set of strategy profiles of all players. We assume that the payoff function of each player $i \in N$ is a function $v_i : \mathcal{G} \times Y \to \mathbf{R}$. We denote by $v = (v_1, ..., v_N)$ the profile of payoff functions. For example, in the public goods model above, the payoff of player i depends not only on the profile of efforts $y = (y_1, ..., y_N)$, but also on the network g, which incorporates the possibility of externalities. In particular, in this example, the notion of Nash-pairwise stability notion allows to endogenize at the same time the effort of each player and the formation of the network.

Definition 3.1.— *A network-game is a pair (Y, v), where Y is the set of strategy profiles, and $v = (v_1, ..., v_N)$ the profile of payoff functions defined on $\mathcal{G} \times Y$.*

Hereafter, we recall the following standard convention: if $y = (y_i)_{i \in N}$ is a strategy profile, then for every $i \in N$, y_{-i} denotes the profile of all strategies except strategy y_i, that is $y_{-i} = (y_j)_{j \in N, j \neq i}$, and the usual abuse of notation $y = (y_i, y_{-i})$ will be done. Our main result will require some of the following assumptions:

(A1) Compactness and Convexity Assumption. For every $i \in N$, Y_i is a nonempty compact and convex subset of some finite dimensional vector space.

(A2) Continuity Assumption. For every $i \in N$, the function $v_i : \mathcal{G} \times Y \to \mathbf{R}$ is continuous.[11]

(A3) Quasiconcavity Assumption (resp. strict Quasiconcavity Assumption).

(i) For every $(g, y) \in \mathcal{G} \times Y$ and every player $i \in N$, $v_i(g, (d_i, y_{-i}))$ is assumed to be quasiconcave (resp. strictly quasiconcave) with respect to $d_i \in Y_i$.

[10] In particular, this concept differs from a simple Nash equilibrium: two players at a same node will have to choose a same action (a weight), according to rules that give them some power to increase or decrease this common weight.

[11] The set of weighted networks $\mathcal{G}$ is a convex and compact subset of the set $\mathcal{F}(\mathcal{L}, \mathbf{R})$ of functions from $\mathcal{L}$ to $\mathbf{R}$, where $\mathcal{F}(\mathcal{L}, \mathbf{R})$ is endowed with the natural multiplication and addition, and with its natural Euclidean topology. The set $\mathcal{G}$ is endowed with the induced topology.

(ii) For every $(g, y) \in \mathcal{G} \times Y$ and every $ij \in \mathcal{L}$, $v_i((x, g_{-ij}), y)$ is assumed to be quasiconcave (resp. strictly quasiconcave) with respect to $x \in [0, 1]$.

Remark 3.1. Convexity, Compactness, Continuity and Quasiconcavity assumptions are standard in Game theory, in particular to get the existence of a Nash equilibrium in strategic games.

We now define the main stability notion of the paper. Hereafter, (Y, v) is a network-game:

DEFINITION 3.2.— *The pair $(g, y) \in \mathcal{G} \times Y$ is Nash-pairwise stable (resp. weakly Nash-pairwise stable) with respect to v if:*

1. $\forall i \in N$, $\forall d_i \in Y_i$, $v_i(g, (d_i, y_{-i})) \leq v_i(g, y)$,

2. $\forall ij \in \mathcal{L}$, $\forall x \in [0, g_{ij}[$, $v_i((x, g_{-ij}), y) \leq v_i(g, y)$.

3. $\forall ij \in \mathcal{L}$, $\forall x \in]g_{ij}, 1]$, $v_i((x, g_{-ij}), y) > v_i(g, y) \Rightarrow v_j((x, g_{-ij}), y) < v_j(g, y)$ *(resp.* $v_j((x, g_{-ij}), y) \leq v_j(g, y)$*).*

If $(g, y) \in \mathcal{G} \times Y$ is Nash-pairwise stable (resp. weakly Nash-pairwise stable), we say that g is a pairwise stable network (resp. a weakly pairwise stable network) associated to the strategy profile y.

Remark 3.2. Assertion 1. says that no player can improve his payoff by modifying his strategy y_i. Assertion 2. says that no player can improve his payoff by decreasing the weight of his relationship with another player. Assertion 3. says that there is no pair of players who can both improve their payoffs by increasing the weight of their relationship.

The proof of the following theorem can be found in the appendix.

THEOREM 3.1.— *For every network-game (Y, v) satisfying (A1), (A2) and Quasiconcavity Assumption (A3) (resp. strict Quasiconcavity Assumption (A3)), there exists $(g, y) \in \mathcal{G} \times Y$ which is weakly Nash-pairwise stable (resp. Nash-pairwise stable) with respect to v.*

3.2 EXISTENCE OF WEIGHTED PAIRWISE STABLE NETWORKS

If the payoff function v_i of each player $i \in N$ goes from $\mathcal{G}$ to $\mathbf{R}$ (i.e., there is no more strategy spaces Y_i), then we get, as a corollary of Theorem 3.1:

(A3bis) Quasiconcavity Assumption (resp. strict Quasiconcavity Assumption). For every $g \in \mathcal{G}$ and every $ij \in \mathcal{L}$, $v_i((x, g_{-ij}))$ is assumed to be quasiconcave (resp. strictly quasiconcave) with respect to $x \in [0, 1]$.

THEOREM 3.2.— *Let $v = (v_n)_{n \in N}$ be a profile of continuous payoff functions from $\mathcal{G}$ to $\mathbf{R}$ satisfying Quasiconcavity Assumption (A3bis) (resp. strict Quasiconcavity Assumption (A3bis)), then there exists some network $g \in \mathcal{G}$ which is weakly pairwise stable (resp. pairwise stable) with respect to v.*

Proof. Define $Y_i = \{0\}$ for every $i \in N$, and $\tilde{v}_i(g, 0) = v_i(g)$. If v is continuous and satisfies quasiconcavity assumption (A3bis) (resp. strict quasiconcavity assumption (A3bis)), then we can apply Theorem 3.1 to $(Y, \tilde{v})$, which gives the existence of a weakly Nash-pairwise stable (resp. Nash-pairwise stable) pair $(g, \{0\})$. Then g is weakly pairwise (resp. pairwise) stable with respect to v.

EXAMPLE 3.1.— *(Continuity is necessary for the existence of a weakly pairwise stable network)*

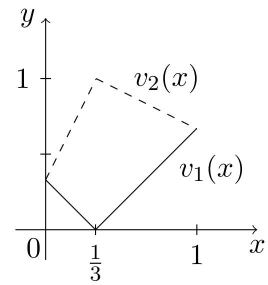
Consider two players, $x \in [0,1]$ being the weight of the link between them. The payoffs are defined for $i = 1,2$ by $v_i(x) = -x$ if $x \neq 0$ and $v_i(0) = -1$. Then $x > 0$ is not pairwise stable, since some player (in fact both) can increase his payoff by decreasing the weight $x > 0$ of the link. But $x = 0$ is not pairwise stable, since both players can increase strictly their payoffs by choosing $x = \frac{1}{2}$. Thus, there is no weakly pairwise stable network, although the payoffs are quasiconcave.

EXAMPLE 3.2.— *(Quasiconcavity is necessary for the existence of a weakly pairwise stable network)*

The following example illustrate why quasiconcavity is, in general necessary. Again, consider two players, $x \in [0,1]$ being the weight of the link between them. Let

$$v_1(x) = \begin{cases} \frac{1}{3} - x & \text{if } x \in [0, \frac{1}{3}], \\ x - \frac{1}{3} & \text{if } x \in [\frac{1}{3}, 1] \end{cases}$$

$$v_2(x) = \begin{cases} \frac{1}{3} + 2x & \text{if } x \in [0, \frac{1}{3}], \\ \frac{7}{6} - \frac{x}{2} & \text{if } x \in [\frac{1}{3}, 1] \end{cases}$$



The network $x = 0$ is not weakly pairwise stable, because $x = 1$ is strictly better for both players. Similarly, any $x \in]0, \frac{1}{3}]$ is not weakly pairwise stable because $x = 0$ is strictly better for player 1. Last, $x \in]\frac{1}{3}, 1]$ is not weakly pairwise stable because $x = \frac{1}{3}$ is strictly better for player 2.

EXAMPLE 3.3.— *(Strict quasiconcavity is necessary for the existence of a pairwise stable network)*

In Example 3.8, it is proved that $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the only weakly pairwise stable network. It is not pairwise stable, because if player 1 and 2 both increase $x = \frac{1}{2}$, player 1's payoff is unchanged, but player 2's payoff increases strictly. In this example, strict quasiconcavity does not hold, for example when $x = \frac{1}{2}$.

3.3 Link with other stability concepts of the literature

Pairwise stability notion of Jackson and Wolinsky is not the only stability concept applied to networks in the literature (see for example pairwise Nash stability of Bloch and Jackson [4], pairwise stable networks with transfers (Bloch and Jackson [4] [5]), or strongly pairwise stablility concept of Bloch and Dutta [3]). Thus, a natural question is whether our technique of proof can be applied to these concepts.[12] In this subsection, we prove that among all these concepts, Pairwise stability is the only one which allows some general existence result similar to Theorem 3.1.

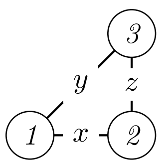
Before, we briefly examine the relationship beween Theorem 3.2 and the possible existence of unweighted pairwise stable networks. First, if a weighted pairwise stable network happens to be unweighted (that is if all its weights are equal to zero or one), then it is also pairwise stable in the sense of Jackson and Wolinsky. But in general, the converse is false.[13] Second, there are many situations for which our result can be applied although there does not exist any unweighted pairwise stable network (see, for example, Example 3.8 below).

[12] We thank an anonymous referee for suggesting us to develop the results of this subsection.

[13] More precisely, consider some payoff functions defined on the set of weighted networks. It is possible to restrict them to unweighted networks. Then, in general, there could exist some unweighted pairwise stable network in the sense of Jackson and Wolinsky (for the restricted payoff functions) which does not satisfy the definition of weighted pairwise stability.

In the following example, we now focus on strongly pairwise stability concept (see Dutta and Mutuswami [10] or Bloch and Dutta [3]).[14]

Example 3.4.— *There are three agents (the variables x, y, z on the links indicate the weights of the links):*



The payoffs are given by $v_1(x,y,z) = x(\frac{3}{4} - z) + y$, $v_2(x,y,z) = z(\frac{3}{4} - y) + x$ *and* $v_3(x,y,z) = y(\frac{3}{4} - x) + z$. *As in [3], for every link ij, we assume that agent i exerts some effort* $x_i^j \in [0, \frac{1}{2}]$ *which contributes to the final weight of this link. More precisely, given* x_i^j *and* x_j^i*, the weight of link ij is assumed to be equal to* $x_i^j + x_j^i$ *(this is a particular example of the additive case treated in [3]).*

This defines a game, where the strategy of each player i is $(x_i^j, x_i^k) \in [0, \frac{1}{2}]^2$, $j \neq k$ *in* $\{1,2,3\} - \{i\}$*, and where the payoff of player i is* $v_i(x,y,z)$*, the weights x, y and z being defined additively from all the efforts, as described above.*

By definition, given some profile of efforts $(x_1^2, x_1^3, x_2^1, x_2^3, x_3^1, x_3^2)$*, the network* $g = (x,y,z) = (x_1^2 + x_2^1, x_1^3 + x_3^1, x_2^3 + x_3^2)$ *is strongly pairwise stable if the profile of efforts is a Nash equilibrium of the non cooperative game defined above, and if there is no pair (i,j) of players that can simultaneously increase their payoffs by modifying their strategies (see [3]).*

We now prove by contradiction that there is no strongly pairwise stable network $(x_1^2, x_1^3, x_2^1, x_2^3, x_3^1, x_3^2)$ *(although the payoff functions are multiaffine, and thus satisfies our set of assumptions). First, from Nash equilibrium condition, player 1, who wants to increase y as high as possible, should choose the maximum level of effort* $x_1^3 = \frac{1}{2}$*; similarly, player 2 chooses* $x_2^1 = \frac{1}{2}$ *and player 3 chooses* $x_3^2 = \frac{1}{2}$.

Now, we prove that all the other efforts should be equal to $\frac{1}{4}$. *By contradiction, if* $x_1^2 > \frac{1}{4}$*, then* $x = \frac{1}{2} + x_1^2 > \frac{3}{4}$*, thus player 3 would like to decrease the weight y of his link with player 1, thus he chooses the minimal level* $x_3^1 = 0$*, which entails* $y = x_1^3 + x_3^1 = \frac{1}{2}$. *Then, Players 2 should choose the maximal level* $x_2^3 = \frac{1}{2}$ *to maximize the weight z, consequently* $z = x_2^3 + x_3^2 = 1$. *Now, for $z = 1$, player 1 should choose a minimal level* $x_1^2 = 0$ *to minimize the weight x, a contradiction with* $x_1^2 > \frac{3}{4}$. *Hence, by contradiction, we get* $x_1^2 \leq \frac{1}{4}$*, and by symmetry,* $x_3^1 \leq \frac{1}{4}$ *and* $x_2^3 \leq \frac{1}{4}$. *In particular, x, y and z are less or equal to* $\frac{3}{4}$. *Let us prove that these inequalities are in fact equalities: indeed, by contradiction, if for example* $x < \frac{3}{4}$ *(that is if* $x_1^2 < \frac{1}{4}$*), player 3 would choose a maximal level* $x_3^1 = \frac{1}{2}$ *to maximize y, a contradiction with* $x_3^1 \leq \frac{1}{4}$. *Thus finally, we should have* $x = y = z = \frac{3}{4}$ *and* $x_1^2 = x_3^1 = x_2^3 = \frac{1}{4}$.

To finish the contradiction proof, let us prove that the bilateral condition in the definition of strong pairwise stability is false. Indeed, if player 2 modifies $x_2^3 = \frac{1}{4}$ *for* $x_2^3 = \frac{1}{2}$*, and player 3 modifies* $x_3^1 = \frac{1}{4}$ *for* $x_3^1 = 0$*, these two modifications being simultaneous, then it induces a new network* $g = (x,y,z) = (\frac{3}{4}, \frac{1}{2}, 1)$*, which gives a new payoff of 1 for players 2 and a new payoff of 1 for player 3, instead of* $\frac{3}{4}$ *previously. Thus* $x = y = z = \frac{3}{4}$ *is not strongly Nash pairwise stable, and finally, there is no strongly Nash pairwise stable profile of strategies.*

[14] Bloch and Jackson [4] concept of pairwise Nash stability is very close to strongly pairwise stability, but it can be applied only to unweighted networks. This explain why we do not discuss this concept here.

Another important concept in the literature is pairwise stability with transfers, introduced by Bloch and Jackson ([4],[5]). It has been defined for unweighted networks, but it can be generalized for weighted networks in the following natural way:

Definition 3.3.— *The network g is pairwise stable with transfers if:*

for every $ij \in \mathcal{L}$, for every $x \in [0,1]$, $v_i((x, g_{-ij})) + v_j((x, g_{-ij})) \leq v_i(g) + v_j(g)$.

Remark that we could define for each link ij a fictive player, with strategy space $[0,1]$ and some payoff function $v_i(g) + v_j(g)$. Then, g is pairwise stable with transfers if and only if it is a Nash equilibrium of this non cooperative game.

The following example proves that under our main assumptions (continuity and quasiconcavity) the situation is similar to the previous example: a pairwise stable network with transfers may not exist.

Example 3.5.— *Consider the network in Example 3.4, with the same notations. The payoffs are now given by $v_1(x,y,z) = (1-z)(1-2x)-y$ if $x \leq \frac{1}{2}$ and $v_1(x,y,z) = -y$ otherwise, $v_2(x,y,z) = -y$ if $x \leq \frac{1}{2}$ and $v_2(x,y,z) = -y+(2x-1)z$ otherwise; last, $v_3(x,y,z) = -(x+z-1)^2 - v_2(x,y,z) - 2y$. It is easy to see that these three payoff functions satisfy quasiconcavity and continuity assumptions.*

Clearly, if (x,y,z) is pairwise stable with transfers, we should have $y = 0$. Then, as remarked above, (x,z) has to be a Nash equilibrium of the non cooperative game where one (fictive) player is link 12, whose strategy is x, and has a payoff $v_1(x,0,z) + v_2(x,0,z)$. The other fictive player is link 23, its strategy is z, and its payoff $v_2(x,0,z) + v_3(x,0,z) = -(x+z-1)^2$. Since player 23 plays optimally given x, we should have $z = 1-x$. Also, player 12 chooses an optimal x given z: thus, we should have $x = 0$ if $z < \frac{1}{2}$, $x = 1$ if $z > \frac{1}{2}$, and $x \in \{0,1\}$ for $z = \frac{1}{2}$, which is contradictory with $x = 1-z$.

3.4 Several applications of our main existence result.

The following subsection presents different existing models (or extensions of existing models), for which our main existence result adds some new network-formation aspect.

Example 3.6.— *(A Public good Provision model)*

We extend a model introduced by Bramoullé and Kranton [6], by introducing some network-formation aspect. Consider N agents. Agent i exerts some effort $e_i \in [0, \overline{e_i}]$, where $\overline{e_i} > 0$. Given the (endogenous) network $g \in \mathcal{G}$, the payoff function of agent i is

$$v_i(g,e) = b(e_i + \sum_{j \neq i} g_{ij} e_j) - c_i e_i - d_i \sum_{j \neq i} g_{ij}$$

where $e = (e_1, ..., e_n)$ is the profile of efforts, $c_i > 0$ is the marginal cost of effort for agent i, $d_i > 0$ is the marginal cost of forming a link for i, and $b : [0, +\infty[\to [0, +\infty[$ is strictly concave and strictly increasing. When $(d_1, ..., d_n) = 0$ and when g is exogenous, this is exactly Bramoullé and Kranton's model [6].

We can apply Theorem 3.1, since each payoff function v_i is continuous, concave with respect to each link and concave with respect to each strategy e_i. Consequently, there exists a weakly Nash-pairwise stable pair $(g, e = (e_1, ..., e_n)) \in \mathcal{G} \times [0, \overline{e_i}]^N$. Remark that we can get the existence of a Nash-pairwise stable pair if we restrict the players to choose efforts in $[\underline{e_i}, \overline{e_i}]$, where $\underline{e_i} > 0$ and $\overline{e_i} > \underline{e_i}$: indeed, in this case, the payoffs are strictly-concave with respect to each link, and the strict version of Theorem 3.1 can be applied.

This extends the analysis of Bramoullé and Kranton [6], since beyond the existence of a Nash equilibrium of a profile of efforts, we get the existence of an endogenous network, associated to e, which satisfies pairwise stability condition.

Example 3.7.— *(Patent race: Goyal and Joshi [12])*

We now apply our main existence result to the following extension of Goyal and Joshi Patent race model [12] to weighted networks. Consider n firms who are competing for obtaining a patent. The firm which wins the race gets the patent of value 1, the others get 0. We let $g_{ij} \in [0,1]$ measure some possible cooperation in R&D between firm i and firm j. Let $\tau(n_i(g))$ denote the (random) time at which firm i is ready to deposit some patent: it is assumed to follow some exponential distribution whose parameter is $n_i(g) = \sum_{j \in N-\{i\}} g_{ij}$. Thus, $P(\tau(n_i(g(s))) \leq t) = 1 - e^{-tn_i(g)}$, that is firm i can get the patent sooner if the relationships with its neighbors increase. Assuming that the distribution of the time of innovation is independent across the firms, we get that the expected payoff of firm i is (see [12]):

$$\pi_i(g) = \frac{n_i(g)}{\rho + 2n_i(g) + \sum\limits_{j \neq i} n_j(g_{-i})}$$

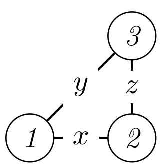
If we fix some $j \neq i$, we can also write

$$\pi_i(g) = \frac{g_{ij} + a_{ij}(g)}{2g_{ij} + b_{ij}(g)}$$

where g_{-i} denotes the network where all links with i are 0, ρ is the common discount factor, $a_{ij}(g) = \sum\limits_{j' \neq j, j' \neq i} g_{ij'}$, $b_{ij}(g) = \rho + 2 \sum\limits_{j' \neq j, j' \neq i} g_{ij'} + \sum\limits_{j \neq i} n_j(g_{-i})$. Since we have $2a_{ij}(g) - b_{ij}(g) < 0$, an easy computation proves that $\pi_i(g)$ is strictly concave with respect to g_{ij}, thus satisfies strict quasiconcavity Assumption, and continuity is straightforward. Thus we can apply Theorem 3.2, and there exists some pairwise stable network.

The following example illustrates that there can be only one weakly pairwise stable network for which the weights are in $]0,1[$ (that is, such network is not unweighted).

Example 3.8.— *There are three players (the variables x, y, z on the links indicate the weights of the links):*



The payoffs are given by $v_1(x,y,z) = x(\frac{1}{2} - z) + y$, $v_2(x,y,z) = z(\frac{1}{2} - y) + x$ and $v_3(x,y,z) = y(\frac{1}{2} - x) + z$. Let us prove that $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the only weakly pairwise stable network. First, by Theorem 3.2, a weakly pairwise stable network $g = (x,y,z)$ exists. If $x > \frac{1}{2}$, then player 3 should decrease the weight y of his link with player 1, i.e. we should have $y = 0$. Then, both players 2 and 3 should increase together the weight z of their common link, i.e. $z = 1$. But then, player 1 should decrease the weight x of his link with 2, i.e. $x = 0$, which contradicts $x > \frac{1}{2}$. Since all the variables play the same roles, we finally have $x \leq \frac{1}{2}$, $y \leq \frac{1}{2}$ and $z \leq \frac{1}{2}$. Now, if $x < \frac{1}{2}$, then both player 1 and *player 3 should increase together the weight y of their common link, i.e. $y = 1$, a contradiction with $y \leq \frac{1}{2}$. By symmetry, we finally have $x = y = z = \frac{1}{2}$.*

Example 3.9.— *(A Two-way flow model)*

We now adapt a model of Bala and Goyal [2] to weighted networks. We define $n_i(g) = \sum_{j \in N - \{i\}} g_{ij}$ the sum of the weights of all direct links from agent i to another agent. For every $j \neq i$, a path from i to j is a finite sequence $x_0 = i, x_1, ..., x_k = j$ of distinct elements of N. Let $\mathcal{P}_j$ be the (finite) set of all paths from i to j, and $n_i'(g)$ be the sum on all paths in some $\mathcal{P}_j$ for all $j \neq i$ of the product of weights along these paths. We can interpret $n_i'(g)$ as the benefit that agent i receives from his links, and $n_i(g)$ as the cost of maintaining his links.

For every agent i, define

$$v_i(g) = n_i'(g) - c_i n_i(g),$$

where $c_i > 0$ is the marginal cost of maintaining the links of player i.

The payoffs are multiaffine, thus there exists some weakly pairwise stable network. For example, assume there are 3 players, and consider the network $g = (x, y, z)$ in Example 3.8. In this case, we find $v_1(x, y, z) = (x+y)(z+1-c_1)$, $v_2(x, y, z) = (x+z)(y+1-c_2)$ and $v_3(x, y, z) = (y+z)(x+1-c_3)$. If $c_1 < 1, c_2 < 1$ anc $c_3 < 1$, then $(1, 1, 1)$ is the only pairwise stable network, which corresponds to the case where everybody fully connect to each other. If $c_1 \in]1, 2[$, $c_2 \in]1, 2[$ and $c_3 \in]1, 2[$, then $(0, 0, 0)$ and $(1, 1, 1)$ are pairwise stable (no connection at all or full connection), but interestingly, $(x = c_3 - 1, y = c_2 - 1, z = c_1 - 1)$ is another pairwise stable network which cannot exist in the unweighted model. It corresponds to the level of weights such that no player has any possibility to modify his payoff. Also, if $c_1 > 2, c_2 > 2$ and $c_3 > 2$, then $(0, 0, 0)$ is the only pairwise stable network.

Example 3.10.— *(Information transmission)*

We extend to weighted networks an information transmission model due to Calvó-Armengol [7]. Hereafter, we follow the presentation of Calvó-Armengol and İlkılıç [8]. There are N agents. If agent i and agent j are in a full relationship ($g_{ij} = 1$), some information can be transmitted from one player to another player, with some probability p_{ij}. We assume that if the relationship is weighted ($g_{ij} \in [0, 1]$), the probability of transmission is $g_{ij} p_{ij}$. The payoff of player $i \in N$ is defined by

$$v_i(g) = 1 - \prod_{j \in N - \{i\}} (1 - p_{ji} g_{ij}) - c n_i(g),$$

where g is a weighted network on N, $c > 0$, $n_i(g) = \sum_{j \in N - \{i\}} g_{ij}$. The first term corresponds to the probability that the message is transmitted to player i, and the second term to the cost of maintaining his links. The payoffs are multiaffine, thus there exists some weakly pairwise stable network.

4. Appendix: proof of Theorem 3.1

First, we present informally the proof, in order to highlight the difference between our main model (involving a generalization of Pairwise stability concept of Jackson and Wolinsky) and a standard normal form game.

A natural idea to prove the existence of a Nash-pairwise stable pair $(g, y) \in \mathcal{G} \times Y$ would be to use the game-theoretic concept of best-response: for every pair of network and strategy profile (g, y), we can try to define a set $\Phi(g, y)$ of pairs $(g', y') \in \mathcal{G} \times Y$ which are profiles of "best-responses" (of all players) against (g, y) (in a sense to be precised), and then look for fixed-points of Φ.

It is easy to define what could mean that $y' = (y'_1, ..., y'_n) \in Y$ is a profile of best-responses against (g, y): g being fixed, we could simply require that for every player $i \in N$, y'_i is an optimal response of player i against (g, y) (this is defined formally in Step 1 below). But defining what would be a best response $g' \in \mathcal{G}$ against (g, y) is more problematic: indeed, for every link ij of g, two agents $k \in \{i, j\}$ are both involved in the link. In particular, each one can compute his own set $PBR^k_{ij}(g, y)$ of optimal weights at ij (where all the other weights g_{-ij} and the strategy profile y are fixed). Then, the problem is to (1) "merge" the two sets $PBR^i_{ij}(g, y)$ and $PBR^j_{ij}(g, y)$ into one coherent set $PBR_{ij}(g, y)$ (a sort of "merged best-response" at link ij), so that (2) a fixed-point of Φ then provides a weakly Nash-pairwise stable pair, and so that (3) the multivalued function Φ satisfies Kakutani's theorem assumptions (and consequently admits a fixed-point). Step 2 below explains how (1) can be solved, Lemma 4.1 below is the main result for (2), and (3) is proved in Step 3 below. In the last Step 4, we prove that a weakly Nash-pairwise stable pair (g, y) is also Nash-pairwise stable when strict quasiconcavity is assumed.

Step 1: Definition of standard best-responses with respect to $y \in Y$. For every player $i \in N$, $BR_i(g, y)$ denotes the standard best-response of player i with respect to y_i, the network g and the other strategies y_{-i} being fixed, that is

$$BR_i(g, y) = \{d_i \in Y_i : \forall d'_i \in Y_i, v_i(g, (d_i, y_{-i})) \geq v_i(g, (d'_i, y_{-i}))\}.$$

Step 2: Definition of merged best-responses with respect to the weights of the links.

For every $(g, y) \in \mathcal{G} \times Y$, and for every non ordered link ij, y and all the weights g_{-ij} being fixed, we define $PBR^i_{ij}(g, y)$ the best response of player i with respect to the weight of the link ij:

$$PBR^i_{ij}(g, y) = \{g'_{ij} \in [0, 1], \forall g''_{ij} \in [0, 1], v_i(g'_{ij}, g_{-ij}, y) \geq v_i(g''_{ij}, g_{-ij}, y)\},$$

and we consider a similar definition for $PBR^j_{ij}(g, y)$.

As discussed above, neither PBR^i_{ij} nor PBR^j_{ij} are completely relevant for our issue, since in the definition of a weakly Nash-pairwise stable pair, player i may have no power alone to impose some weight in $PBR^i_{ij}(g, y)$ (and similarly for j). This is the fundamental difference between our model (as well as Pairwise stability concept) and a standard game where the weights of the links would be strategies of some players. The following multivalued function "merges" the two best-responses PBR^i_{ij} and PBR^j_{ij} of both players i and j, taking into account the rules of Nash-Pairwise stability concept: let us define

$$PBR_{ij}(g, y) = [a_{ij}(g, y), b_{ij}(g, y)] \tag{4.1}$$

where the function a_{ij} is defined by

$$a_{ij}(g, y) = \min\{\min PBR^i_{ij}(g, y), \min PBR^j_{ij}(g, y)\} \tag{4.2}$$

and the function b_{ij} by

$$b_{ij}(g, y) = \min\{\max PBR^i_{ij}(g, y), \max PBR^j_{ij}(g, y)\}. \tag{4.3}$$

Remark that the existence of $PBR^i_{ij}(g, y)$ and $PBR^j_{ij}(g, y)$, and the fact that these sets admit a maximum and a minimum come from assumptions (A1) and (A2).

The following lemma proves that $PBR_{ij}(g, y)$ is a subset of "stable" links (in the sense of weak Pairwise stability) given the other links of g and given y.

Lemma 4.1.— *Under Assumptions (A1), (A2) and (A3), for every undirected link ij, $PBR_{ij}(g,y)$ is included in the set of weights $g'_{ij} \in [0,1]$ such that (g'_{ij}, g_{-ij}) satisfies the definition of weak Pairwise stability for link ij, that is:*

(1) There does not exist $g''_{ij} < g'_{ij}$ such that $v_i(g''_{ij}, g_{-ij}, y) > v_i(g'_{ij}, g_{-ij}, y)$, or such that $v_j(g''_{ij}, g_{-ij}, y) > v_j(g'_{ij}, g_{-ij}, y)$.

(2) There does not exist $g''_{ij} > g'_{ij}$ such that $v_i(g''_{ij}, g_{-ij}, y) > v_i(g'_{ij}, g_{-ij}, y)$ and such that $v_j(g''_{ij}, g_{-ij}, y) > v_j(g'_{ij}, g_{-ij}, y)$.

In the following, by abuse of notation, we will say that such link g'_{ij} is weakly Pairwise stable given (g_{-ij}, y).

Proof. From quasiconcavity and continuity assumptions, $PBR^i_{ij}(g,y)$ and $PBR^j_{ij}(g,y)$ are closed intervals. By definition, for every $g'_{ij} < \min PBR^i_{ij}(g,y)$ or for every $g'_{ij} > \max PBR^i_{ij}(g,y)$, we have $v_i(g'_{ij}, g_{-ij}, y) < v_i(g_{ij}, g_{-ij}, y)$ (and similarly for j). To prove the above lemma, let us consider the following cases:

(a) Case 1: assume $PBR^i_{ij}(g,y)$ and $PBR^j_{ij}(g,y)$ are disjoint. Without any loss of generality (permuting i and j if necessary), we can assume that for every $(g'_{ij}, g''_{ij}) \in PBR^i_{ij}(g,y) \times PBR^j_{ij}(g,y)$, $g'_{ij} < g''_{ij}$, so that $PBR_{ij}(g,y) = PBR^i_{ij}(g,y)$. We only have to prove that every weight $g'_{ij} \in PBR^i_{ij}(g,y)$ is weakly Pairwise stable given (g_{-ij}, y). First, player i has no strict incentive to decrease g'_{ij} since g'_{ij} is optimal for him. Second, player j has no strict incentive to decrease g'_{ij}, otherwise there would exist $g''_{ij} < g'_{ij}$ such that $v_j(g''_{ij}, g_{-ij}, y) > v_j(g'_{ij}, g_{-ij}, y)$. Then, defining $g'''_{ij} = \min PBR^j_{ij}(g,y)$, from the definition of $PBR^j_{ij}(g,y)$, we would get $v_j(g'''_{ij}, g_{-ij}, y) > v_j(g'_{ij}, g_{-ij}, y)$. But from quasiconcavity of v_j with respect to g_{ij}, and since $g'_{ij} \in [g''_{ij}, g'''_{ij}]$, we would get $v_j(g'_{ij}, g_{-ij}, y) \geq \min\{v_j(g''_{ij}, g_{-ij}, y), v_j(g'''_{ij}, g_{-ij}, y)\} > v_j(g'_{ij}, g_{-ij}, y)$ a contradiction. Third, the two players have no strict incentive to increase $g'_{ij} \in PBR^i_{ij}(g,y)$ together (since player i has no such incentive), which proves the lemma in the first case.

(b) Case 2: assume $PBR^i_{ij}(g,y)$ and $PBR^j_{ij}(g,y)$ intersect each other, but that no set is included in the other. Without any loss of generality (permuting i and j if necessary), we can assume for example that $\min PBR^i_{ij}(g,y) < \min PBR^j_{ij}(g,y)$ and $\max PBR^i_{ij}(g,y) < \max PBR^j_{ij}(g,y)$. Thus, as previously, we have $PBR_{ij}(g,y) = PBR^i_{ij}(g,y)$, and we can follow the proof of Case 1 above.

(c) Case 3: if $[PBR^i_{ij}(g,y) \subset PBR^j_{ij}(g,y)]$ or $[PBR^j_{ij}(g,y) \subset PBR^i_{ij}(g,y)]$.

Without any loss of generality, permuting i and j if necessary, we can assume $PBR^i_{ij}(g,y) \subset PBR^j_{ij}(g,y)$. In this case, $PBR_{ij}(g,y)$ is equal to the interval $[\min PBR^j_{ij}(g,y), \max PBR^i_{ij}(g,y)]$ (which is included in $PBR^j_{ij}(g,y)$). Let us prove that every weight g'_{ij} in this interval is weakly Pairwise stable given (g_{-ij}, y). First, player j has no strict incentive to decrease g'_{ij} since g'_{ij} is optimal for him. Second, player i has no strict incentive to decrease g'_{ij}: if $g'_{ij} \in PBR^i_{ij}(g,y)$, this is clear, and if $g'_{ij} \notin PBR^i_{ij}(g,y)$ (so that $g'_{ij} < \min PBR^i_{ij}(g,y)$), we can do a contradiction proof: if there would exist $g''_{ij} < g'_{ij}$ such that $v_i(g''_{ij}, g_{-ij}, y) > v_i(g'_{ij}, g_{-ij}, y)$, simply define $g'''_{ij} = \min PBR^i_{ij}(g,y)$, so that $v_i(g'''_{ij}, g_{-ij}, y) > v_j(g'_{ij}, g_{-ij}, y)$. From quasiconcavity of v_i with respect to g_{ij}, and since $g'_{ij} \in [g''_{ij}, g'''_{ij}]$, we would get $v_j(g'_{ij}, g_{-ij}, y) > v_j(g'_{ij}, g_{-ij}, y)$ a contradiction. Last, as in the previous cases, the two players have no strict incentive to increase $g'_{ij} \in PBR_{ij}(g,y)$ together, because player j has no such incentive since g'_{ij} is optimal for him. This ends the proof of the lemma.

Step 3: Kakutani's theorem assumptions.

By definition, PBR_{ij} has clearly nonempty and convex values. The following lemma will help to prove that the graph of PBR_{ij} is closed:

LEMMA 4.2.— *Let X be a topological space, and $\Phi : X \to \mathbf{R}$ be a multivalued function such that for every $x \in X$, there are two reals $0 \leq a(x) \leq b(x) \leq 1$ such that $\Phi(x) = [a(x), b(x)]$. Then Φ has a closed graph if and only if a is a lower semicontinuous function and b is an upper semicontinuous function.*

Proof. First we prove the implication. Let us assume that Φ has a closed graph. If a is not lower semicontinuous at x, there exists $\varepsilon > 0$ and a sequence $(x^n)_{n\in\mathbf{N}}$ in X converging to x such that for every n large enough, $a(x^n) < a(x) - \varepsilon$. But since $a(x^n) \in \Phi(x^n)$, considering $(a(x^{\phi(n)}))_{n\in\mathbf{N}}$ a subsequence of $a(x^n)$ converging to some $\bar{a}$, and passing to the limit in the inequality $a(x^{\phi(n)}) < a(x) - \varepsilon$, we get (1) $\bar{a} \leq a(x) - \varepsilon$ and (2) $\bar{a} \in \Phi(x) = [a(x), b(x)]$ (from closeness of the graph of Φ), a contradiction. Similarly, we prove that b is upper semicontinuous.

Now, let us prove the converse implication: let us assume that a is lower semicontinuous and that b is upper semicontinuous. Consider a sequence $(x^n)_{n\in\mathbf{N}}$ in X converging to x, and a sequence $y^n \in \Phi(x^n) = [a(x^n), b(x^n)]$ converging to y. Since $a(x^n) \leq y^n \leq b(x^n)$, passing to the limit, and from the assumptions above on a and b, we get $a(x) \leq y \leq b(x)$, that is $y \in \Phi(x)$. This finally proves that Φ has a closed graph. This ends the proof of the lemma.

To finish the proof of Step 3, just remark that PBR^i_{ij} and PBR^j_{ij} have closed graphs with nonempty and convex values (from Berge theorem, because they are best-responses of functions which are continuous, quasiconcave with respect to the maximization variable, and defined on the compact set $[0, 1]$). Thus, from the above lemma applied to the multifunctions PBR^i_{ij} and PBR^j_{ij}, we get that $\min PBR^i_{ij}$ and $\min PBR^j_{ij}$ are lower semicontinuous functions. As a consequence, $\min\{\min PBR^i_{ij}, \min PBR^j_{ij}\}$ is a lower semicontinuous function. Similarly, we get that $\max PBR^i_{ij}$ and $\max PBR^j_{ij}$ are upper semicontinuous functions, and consequently the function $\min\{\max PBR^i_{ij}, \max PBR^j_{ij}\}$ is upper semicontinuous. To finish, just apply the converse part of Lemma 4.2, where a is defined by $a(g, y) = \min\{\min PBR^i_{ij}(g, y), \min PBR^j_{ij}(g, y)\}$ and where b is defined by $b(g, y) = \min\{\max PBR^i_{ij}(g, y), \max PBR^j_{ij}(g, y)\}$. This finally implies that for every ij, PBR_{ij} has a closed graph.

Finally, we can apply Kakutani's theorem to the multivalued function Φ defined by

$$\Phi(g, y) = \Pi_{ij\in\mathcal{L}} PBR_{ij}(g, y) \times \Pi_{i\in N} BR_i(g, y).$$

Thus it admits a fixed-point (g, y). By definition, (g, y) is a weakly Nash-Pairwise Stable pair, since for every link ij, g_{ij} is weakly Pairwise stable given (g_{-ij}, y) (from Lemma 4.1 above) and since each y_i is optimal for every player i, given g and y_{-i}.

Step 4. To finish the proof, we prove that the weakly Nash-pairwise stable pair (g, y) is also Nash-pairwise stable when strict quasiconcavity is assumed. By contradiction, assume that (g, y) is not Nash-pairwise stable. Thus, there exist $ij \in \mathcal{L}$ and $g'_{ij} \in [0, 1]$, such that we have

$$v_i((g'_{ij}, g_{-ij}), y) > v_i(g, y)$$

and

$$v_j((g'_{ij}, g_{-ij}), y) = v_j(g, y),$$

From strict quasiconcavity assumption, we would get

$$v_i((\frac{g'_{ij}+g_{ij}}{2}, g_{-ij}), y) > v_i(g,y)$$

and

$$v_j((\frac{g'_{ij}+g_{ij}}{2}, g_{-ij}), y) > v_j(g,y),$$

a contradiction with the definition of (g, y) being weakly Nash-pairwise stable.

Remark 4.1. When there are only two agents, assuming that each payoff function $v_i : [0,1] \to \mathbf{R}$ is continuous[15] and quasiconcave, the existence of pairwise stable networks can be derived as follows: from continuity, each set $\arg\max_{x\in[0,1]} v_i(x)$ is nonempty and closed, thus it admits some minimum element x_i. Then $\bar{x} = \min\{x_1, x_2\}$ easily defines a pairwise stable network. Unfortunately, passing to three agents, the same idea cannot be applied, because of discontinuity issues. Indeed, for $N = \{1, 2, 3\}$, call x the weight of the link between 1 and 2, y the weight of the link between 1 and 3 and z the weight of the link between 2 and 3. Assume the payoff function of each player $i \in N$ is a function of $(x, y, z) \in [0,1]^3$ satisfying continuity and quasiconcavity assumption. A natural extension of the two-players approach described above should drive us to define $f_{12}(x, y, z) = \min\{\min argmax_{y\in[0,1]} u_2(x, y, z), \min argmax_{x\in[0,1]} u_1(x, y, z)\}$, and similarly for f_{13} and f_{23}, by circular permutation. Then, a fixed-point of the mapping $f = (f_{12}, f_{13}, f_{23})$ would give a pairwise stable network. But here, a standard fixed-point theorem like Brouwer's theorem cannot be invoked, because f can be discontinuous. For example, for $u_1(x, y, z) = x(y - \frac{1}{2})$ and $u_2(x, y, z) = x$, then f_{12} is discontinuous at every (x, y, z) such that $y = \frac{1}{2}$.

References

[1] Robert Aumann and Roger Myerson. Endogenous formation of links between players and coalitions: an application of the shapley value. The Shapley Value, pages 175–191, 1988.

[2] Venkatesh Bala and Sanjeev Goyal. A noncooperative model of network formation. Econometrica, 68(5):1181–1229, 2000.

[3] Francis Bloch and Bhaskar Dutta. Communication networks with endogenous link strength. Games and Economic Behavior, 66(1):39–56, 2009.

[4] Francis Bloch and Matthew O Jackson. Definitions of equilibrium in network formation games. International Journal of Game Theory, 34(3):305–318, 2006.

[5] Francis Bloch and Matthew O Jackson. The formation of networks with transfers among players. Journal of Economic Theory, 133(1):83–110, 2007.

[6] Yann Bramoullé and Rachel Kranton. Public goods in networks. Journal of Economic Theory, 135(1):478–494, 2007.

[7] Antoni Calvó-Armengol. Job contact networks. Journal of economic Theory, 115(1):191–206, 2004.

[8] Antoni Calvó-Armengol and Rahmi İlkılıç. Pairwise-stability and nash equilibria in network formation. International Journal of Game Theory, 38(1):51–79, 2009.

[15] This could even be relaxed into upper semicontinuity.

[9] Subhadip Chakrabarti and Robert P Gilles. Network potentials. Review of Economic Design, 11(1):13–52, 2007.

[10] Bhaskar Dutta and Suresh Mutuswami. Stable networks. Journal of Economic Theory, 76(2):322–344, 1997.

[11] Irving L Glicksberg. A further generalization of the kakutani fixed point theorem, with application to nash equilibrium points. Proceedings of the American Mathematical Society, 3(1):170–174, 1952.

[12] Sanjeev Goyal and Sumit Joshi. Unequal connections. International Journal of Game Theory, 34(3):319–349, 2006.

[13] Tim Hellmann. On the existence and uniqueness of pairwise stable networks. International Journal of Game Theory, pages 1–27, 2013.

[14] Tim Hellmann. The structure of pairwise stable networks in homogeneous societies with link externalities. Working paper, 2018.

[15] Matthew O Jackson. A survey of network formation models: stability and efficiency. Group Formation in Economics: Networks, Clubs, and Coalitions, pages 11–49, 2005.

[16] Matthew O Jackson and Alison Watts. The existence of pairwise stable networks. Seoul Journal of Economics, 14:299–321, 2001.

[17] Matthew O Jackson and Alison Watts. The evolution of social and economic networks. Journal of Economic Theory, 106(2):265–295, 2002.

[18] Matthew O Jackson and Asher Wolinsky. A strategic model of social and economic networks. Journal of economic theory, 71(1):44–74, 1996.

[19] Ana Mauleon and Vincent Vannetelbosch. Network formation games. In Yann Bramoullé, Andrea Galeotti, Brian Rogers, and Brian W Rogers, editors, The Oxford Handbook of the Economics of Networks. Oxford University Press, 2016.

[20] Yuhei Miyauchi. Structural estimation of pairwise stable networks with nonnegative externality. Journal of Econometrics, 195(2):224–235, 2016.

[21] Roger B Myerson. Game theory: analysis of conflict. Cambridge: Harvard University Press USA, 1991.

[22] John F Nash. Equilibrium points in n-person games. Proceedings of the national academy of sciences, 36(1):48–49, 1950.

[23] J. B. Rosen. Existence and uniqueness of equilibrium points for concave n-person games. Econometrica, 33:520–534, 1965.

[24] Marco Slikker and Anne van den Nouweland. A one-stage model of link formation and payoff division. Games and Economic Behavior, 34(1):153–175, 2001.

[25] Chiu Ki So. Network formation with endogenous link strength and decreasing returns to investment. Games, 7(4):40, 2016.