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On history-dependent optimization models: a unified framework to analyze models with habits, satiation and optimal growth

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Abstract

We provide first a framework for history-dependent utility models. We further consider discrete infinite horizon dynamic optimization programs in which the instantaneous payoff presents such history-dependence. An issue about models with habits is their lack of general framework. Our framework allows to study habit models that are either additive or multiplicative or neither, as well as satiation models. Moreover, with this unified setting one can treat the usual optimal growth models with or without habit formation and with or without satiation effects. As the way the history dependence is formalized allows us to use dynamic programming tools. We show that the value function is the unique fixed point of the Bellman operator. Such history-dependent modelizations have their motivations and applications in many areas among which decision theory, psychology, behavioral and environmental economics.
Keywords: History-dependent, Dynamic programming, Optimal growth, Habits, Satiation.

1. Introduction

"How terribly important habit is. It may largely determine the characteristics or the nature of the brain itself. Habits influence or perhaps can largely determine the choice of trains of thought in one's work." says Ulam[24]. Whether it is habit and virtue as in Aristotle[1] (Nicomachean Ethics, Book II, Chapter 1), whether it is called habits, customs, satiation, or any other ways, history-dependent behavior issues have a long history in diverse and broad areas, as philosophy (see Wright[27], Carlisle[5]), physiology (see Wright[28]), sociology (see Camic 1986[4]), psychology (see Gardner and Rebar[9]), etc.

In Economics, these questions have been raised from both theoretical as well as empirical viewpoints. As the work of Havranek et al.[10] shows the literature about the estimates of habit formation with respect to data analysis techniques and models specifications is vast. Some authors (as Fuhrer[8]) shows that habit formation models are crucial in order to fit the data and some others like Dynan[7] finds no evidence of habits working with panel household data. Refer also to Tserenjigmid[23] and Wathieu[25].

Several optimal growth models with history-dependent utilities have been proposed in the literature since the fundamental Ryder and Heal[20] paper, mostly taking habits into account, with various utility and habit stocks evolution specifications (see Carroll Overland and Weil[6], Wendner[26], Hiraguchi[12, 13]).

We rely on axiomatic foundations on preferences as developed for habits in Rozen[18], and further in Rustichini and Siconolfi[19] and He Dyer and Butler[11] for habits and satiation to propose a general framework for modelling history-dependent utilites in order to study optimal growth models, thus attempting clarifying existing models' differences and similarities.

The paper is organized as follows. In Section 2, we introduce the general adjusted consumption framework as well as the general optimal growth model with adjusted consumption. In Section 3, we show that such frameworks allow to encompass the usual models in the habit and satiation literature as well as the usual optimal growth models with habit previously treated in the literature. Section 4 is reserved for the general results of the existence of a solution and caracterisation of optimal paths given by dynamic programming tools.

2. A general framework

2.1 Introduction to the general adjusted consumption framework

In this model, we consider a representative consumer consuming a single good on periods $t = 0, 1, 2, ...$ with preference over the consumption $\tilde{c} = (c_0, c_1, ...)$ from the set of consumption streams which is a subset of l^∞

$$l_+^\infty = \{\tilde{x} = (x_t)_{t=0}^\infty \in (\mathbb{R}^+)^{\mathbb{N}}, \|\tilde{x}\|_\infty := \sup_{t\in\mathbb{N}} x_t < +\infty\}$$

The preference depends on her consumption history which is modelized in the following way. At time $t = 0$, she has a *time*-0 *history*[1], denoted $\tilde{h}^{(0)} := (h_0^{(0)}, h_1^{(0)}, ...) = (h_j^{(0)})_{j=1}^\infty$ lying in l_+^∞. Having consumed c_0 at date 0, she has then *time*-1 *history* equal to $\tilde{h}^{(1)} := (c_0, \tilde{h}^{(0)})$ at date $t = 1$. And so forth, for any date $t \geq 1$, *time-t history* will be

$$\forall t \geq 1, \tilde{h}^{(t)} = (h_j^{(t)})_{j=1}^\infty := (c_{t-1}, \tilde{h}^{(t-1)}) = (c_{t-1}, ..., c_0, \tilde{h}^{(0)})$$

where $h_j^{(t)}$ is the consumption j periods prior to time t for j such that $1 \leq j \leq t$, $h_j^{(t)} = c_{t-j}$ and $(h_j^{(t)})_{j=t+1}^\infty = \tilde{h}^{(0)}$.
The consumer knows that her date t-preferences are influenced both by her consumption at date t and by her time-t consumption history. Indeed, her date t-preferences change endogenously with respect to a reference point (called *date t-adjustment level*) that is generated from her time-t consumption history $\tilde{h}^{(t)}$ through an *adjustment level function*) denoted by φ, where φ is defined on l_+^∞ with values on $R = (\mathbb{R}^+)^n$ (where $n \geq 1$), i.e.

$$\varphi : l_+^\infty \to R$$

Note that $n = 1$ is considered in order to recover the Ryder and Heal[20] model and $n = 2$ when we wish to modelize habits and satiation effects as in He Dyer and Butler[11] (see Section 3). Thus this framework allows to consider other models with different possible effects. The consumer then faces her *adjusted consumption* stream as follows

$$\left((c_t, \varphi(\tilde{h}^{(t)}))\right)_{t=0}^\infty$$

The consumer's instantaneous utility function u is defined on a subset $\mathcal{D}_u$ of $\mathbb{R}^+ \times R$, i.e.

$$u : \mathcal{D}_u \subseteq (\mathbb{R}^+ \times R) \to \mathbb{R}$$

and for $\beta \in (0, 1)$ the (given) discount factor, the intertemporal utility will be

$$\sum_{t=0}^\infty \beta^t u(c_t, \varphi(\tilde{h}^{(t)}))$$

We now discuss several interpretations and forms of φ that have been introduced and usually dealt with in the related literature. We refer more specifically to the papers by Rozen[18], Ryder and Heal[20], He Dyer and Butler[11], Rustichini and Siconolfi[19], Baucells and Sarin[3].

[1] We use the term *time t-history* where Rozen[18] uses *time t-habit* and the term *reference point at time t* (as Rozen[18]) where Ryder and Heal[20] use the term *customary level of consumption or expected level of consumption.* We use the notation $\tilde{h}^{(0)} = (h_1, h_2, h_3, ...) =: \tilde{h}$ instead of $h^{(0)} = (...., h_3, h_2, h_1) =: h$ in Rozen[18], and $\tilde{h}^{(t)} = (c_{t-1}, ..., c, c_0, \tilde{h})$ instead of $h^{(t)} = (h, c_0, c_1, ..., c_{t-1})$ in Rozen[18].

2.2 The adjustment level function φ

In Rozen[18], consumption histories are habits' streams. Rozen[18] provided habit formation preference axioms that lead to a representation through the instantaneous utility $u(c_t - \varphi(\tilde{h}^{(t)})$ so that φ is a linear function defined by a unique sequence of coefficients $(\alpha_k)_{k\geq 1}$ by

$$\forall \tilde{h} = (h_k)_{k\geq 1}, \varphi(\tilde{h}) = \sum_{k=1}^{+\infty} \alpha_k h_k$$

with the coefficients satisfying $\forall k \geq 1, \alpha_k \in (0,1)$.

When initial history $\tilde{h}$ is given, φ can be defined through a law of motion and if the consumption stream is $\tilde{c} = (c_t)_{t=0}^{+\infty}$, for any $t \geq 1$, $\tilde{h}^{(t)}$ being defined as before, i.e. $\tilde{h}^{(t)} = (c_{t-1}, c_{t-2}, ..., c_1, c_0, \tilde{h})$, one has in our notations as follows:

$$\forall t \geq 1, \varphi(\tilde{h}^{(t+1)}) = \lambda_1 \varphi(\tilde{h}^{(t)}) + \lambda_2 c_t$$

for $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 \leq 1$. Rozen [18] distinguishes between two types of habits whether they have geometric coefficients with $\lambda_1 + \lambda_2 = 1$ or if $\lambda_1 + \lambda_2 < 1$.

Habit formation in continuous time is considered in the seminal paper of Ryder and Heal[20]. Rewriting the discrete time version, c_t denotes the consumption at date t and the habit variable z_t is a weighted average of past consumption levels, so that the law of motion is given by

$$z_{t+1} = \rho c_t + (1-\rho) z_t, \text{ with } \rho \in (0,1)$$

This model can be written with our notations defining φ by the law of motion

$$\forall t \geq 0, \varphi(\tilde{h}^{(t+1)}) = \rho c_t + (1-\rho)\varphi(\tilde{h}^{(t)})$$

so that the instantaneous utility $u(c_t, z_t)$ is $u(c_t, \varphi(\tilde{h}^{(t)}))$.

In both previous models, the adjustment level function can be defined by a recurrence relation. In Rozen, φ is a linear function that can be defined by the following recurrence relation

$$\forall t \geq 0, \varphi(\tilde{h}^{(t+1)}) = G(c_t, \varphi(\tilde{h}^{(t)}))$$

where

$$G(x,y) = \lambda_1 y + \lambda_2 x$$

Similarly, in Ryder and Heal [20], as well as in many of the usual cases, φ is a linear function that can be defined by the following recurrence relation

$$\forall t \geq 0, \varphi(\tilde{h}^{(t+1)}) = G(c_t, \varphi(\tilde{h}^{(t)}))$$

where

$$G(x,y) = \rho x + (1-\rho) y$$

Even though linear formation is important as widely used and prominent in applied literature, Rustichini Siconolfi[19][2] underline that habit stock formation can be generalized to any form of G.

[2] cf p.3

In order to consider such generalizations, we propose a framework that allows not only to cover all the linear cases but also to consider nonlinear dynamics as suggested in Rustichini and Siconolfi[19].

We allow the adjusted level function φ to be defined more generally by

$$\forall t \geq 0, \varphi(\tilde{h}^{(t+1)}) = G(c_t, \varphi(\tilde{h}^{(t)}))$$

with the function G not being necessarily linear.

Moreover, this framework allows to encompass models with both habits and satiation effects. Let us consider such a model as the one described in He Dyer and Butler[11] (as well as in Beaucells and Sarin[3]). Recall that in these models, the instantaneous utility function is given by $[v(c_t + s_t - z_t) - v(s_t)]$ with the habit level z_t satisfying the law of motion (with $\lambda \in (0, 1)$)

$$z_{t+1} = (1 - \lambda)c_t + \lambda z_t$$

and the satiation level s_t is recursively defined by (with $\gamma \in (0, 1]$)

$$s_{t+1} = \gamma(s_t + c_t)$$

We can reformulate it in our framework by defining $\varphi : l^\infty_+ \to R$, with $R = \mathbb{R}^2$ such that for any t, $\varphi(\tilde{h}^{(t)}) = (s_t, z_t)$, imposing that the function φ satisfies the following law of motion

$$\varphi(\tilde{h}^{(t+1)}) = (\gamma(c_t + \pi_1 \circ \varphi(\tilde{h}^{(t)})), (1 - \lambda)c_t + \lambda\pi_2 \circ \varphi(\tilde{h}^{(t)}))$$

where π_i is the projection on the i-th coordinate ($i = 1, 2$).

The instantaneous utility $u : (\mathbb{R}^+ \times \mathbb{R}^2) \to \mathbb{R}$ is defined for any $c \in l^\infty_+$, any $s \in \mathbb{R}$ and $z \in \mathbb{R}$ by

$$u(c, s, z) := v(c + s - z) - v(s)$$

so that

$$u(c_t, \varphi(\tilde{h}^{(t)})) = v(c_t + \pi_1 \circ \varphi(\tilde{h}^{(t)}) - \pi_2 \circ \varphi(\tilde{h}^{(t)})) - v(\pi_1 \circ \varphi(\tilde{h}^{(t)}))$$

2.3 On some utility u usual cases

We recall some of the main usual forms of utility. In particular, in the habit literature, the most common specifications are either additive or multiplicative. Rozen[18] considers an additive specification whereas Caroll Overland and Weil[6] and Hiraguchi[12] consider a multiplicative specification.

For any $c \in \mathbb{R}^+$ and any history level $\phi \in R$, let us define the adjusted instantaneous consumption $a(c, \phi) \in \mathbb{R}$. Habits models correspond to the particular case in which φ is real-valued (ie $R = \mathbb{R}$) and the instantaneaous utility function u is defined by $u(c, \phi) := v(a(c, \phi))$ with v an utility function defined on $\mathbb{R}$. The additive habits models is the particular case (with the notation as in Havranek) in which

$$a(c, \phi) = c - \gamma\phi$$

and multiplicative habits models is the particular case in which

$$a(c, \phi) = c\phi^\gamma$$

Implications of the choice of additive or multiplicative formulations have been discussed in detail (see Wendner [26]). Nevertheless, also neither additive nor multiplicative specifications need to be taken into account as Ryder and Heal[20] remark. The examples given by Ryder and Heal[20] are $u(c,\phi) = -e^{(c-\phi)} - c^{-\frac{1}{2}}$ (then $\mathcal{D}_u = \mathbb{R}_+^* \times \mathbb{R}$), $u(c,\phi) = \sqrt{\frac{c}{\phi}}$ (then $\mathcal{D}_u = \mathbb{R}_+ \times \mathbb{R}_+^*$) and $u(c,\phi) = (-\phi)^3 - c^{-\frac{1}{2}}$ (then $\mathcal{D}_u = \mathbb{R}_+^* \times \mathbb{R}$).

One goal of this paper is to propose a framework allowing to encompass all such utility forms: either additive, multiplicative or neither.

2.4 A framework for optimal growth models with adjusted consumptions

Let us consider an agent facing her utility as described in the previous section, i.e. her instantaneous utility function u is defined on a subset $\mathcal{D}_u$ of $\mathbb{R}^+ \times R$, with $R = (\mathbb{R}^+)^n, n \geq 1$,

$$u : \mathcal{D}_u \subseteq (\mathbb{R}^+ \times R) \to \mathbb{R}$$

and let a production function f be defined on $\mathbb{R}^+$:

$$f : \mathbb{R}^+ \to \mathbb{R}^+$$

Let the discount factor $\beta \in (0,1)$ be fixed. For initial given capital stock $k_0 > 0$ and initial given time-0 history $\tilde{h}^{(0)} \in l_+^\infty$, the following general framework and optimization problem, with f the production function and k_t the capital stock at date t, is given as follows:

$$\mathcal{P}(k_0, \tilde{h}^{(0)}) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(c_t, \varphi(\tilde{h}^{(t)})) \\ \text{s.t.} & \forall t \geq 0, k_{t+1} = f(k_t) - c_t \\ & \forall t \geq 0, c_t \geq 0 \text{ and } k_t \geq 0, \text{ where } k_0 > 0 \text{ is given} \\ & \forall t \geq 1, \tilde{h}^{(t)} = (c_{t-1}, ..., c_1, c_0, \tilde{h}^{(0)}) \text{ where } \tilde{h}^{(0)} = (h_1, ...) \in l_+^\infty \text{ is given} \\ & \forall t \geq 0, (c_t, \varphi(\tilde{h}^{(t)})) \in \mathcal{D}_u \subseteq (\mathbb{R}^+ \times R) \end{cases}$$

This framework allows to encompass many of the existing models and provides a general model allowing to treat several habit and satiation effects. We discuss on that more precisely in the next section.

3. Some models and examples

In this section, we will show that our framework allows to recover the models previously considered in the literature, both in the literature on history-dependent utility as well as in the literature on history-dependent optimal growth models.

3.1 The discrete time version of the Ryder and Heal[20] model

The original optimization problem in the seminal paper of Ryder and Heal[20] is in continous time. We can adapt it in discrete time as follows. The instantaneous consumption is denoted c_t, z_t is the habit level, and k_t the capital stock, at date t. Given initial capital stock k_0 and initial habit level z_0, the model is:

$$\mathcal{P}(k_0, z_0) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(c_t, z_t) \\ \text{s.t.} & \forall t \geq 0, k_{t+1} = f(k_t) - c_t \\ & z_{t+1} = \lambda z_t + (1-\lambda) c_t \\ & c_t, k_t \geq 0 \\ & k_0 > 0 \text{ and } z_0 \geq 0 \text{ are given} \end{cases}$$

Initial habit level and capital stock (k_0, z_0) being given, the Ryder and Heal optimization program can be written through the previously defined general framework by setting the initial history sequence $\tilde{h}^{(0)} := (z_0, z_0, ..., z_0, ...)$ in l^∞_+ and $\varphi : l^\infty_+ \to R$, with $R = \mathbb{R}^+$ such that

$$\begin{cases} \forall \tilde{x} = (x_t)_t \in l^\infty \text{ such that } \exists x \in \mathbb{R}^+, \forall t \in \mathbb{N}, x_t = x, \text{ one has } \varphi(\tilde{x}) = x \\ \\ \forall c \in \mathbb{R}^+, \forall \tilde{x} \in l^\infty_+, \varphi(c, \tilde{x}) = (1-\lambda)c + \lambda\varphi(\tilde{x}) \end{cases}$$

This allows to recover the Ryder and Heal model. Note that any intial history $\tilde{h}^{(0)} \in l^\infty_+$ can be considered, provided that $\varphi(\tilde{h}^{(0)}) = z_0$ and for all $c \in \mathbb{R}^+$ and $\tilde{x} \in l^\infty_+$, $\varphi(c, \tilde{x}) = (1-\lambda)c + \lambda\varphi(\tilde{x})$.

3.2 Some multiplicative habit models (Carroll Overland and Weil[6], Hiraguchi[12])

Similarly, the discrete time version of the Carroll Overland and Weil's as well as Hiraguchi's models can be derived from our framework. Carroll, Overland and Weil[6] and Hiraguchi[12] consider a CRRA utility function of the form (where $\sigma > 1$ and $\gamma \in [0, 1]$)

$$u(c, z) = \frac{1}{1-\sigma}\left(\frac{c}{z^\gamma}\right)^{1-\sigma}$$

The adaptation in our framework is straightforward, defining φ as in Section 3.1.

3.3 Environmental economics

History dependence is important in environmental economics' models. For example, habit is a crucial issue in these models. We discuss the specifications, which include nonlinear dynamics, proposed by Löfgren[17]. Note that Ikefuji[14] and the recent model by Safi and Ben Hassen[21] can be reformulated in our framework.

Löfgren[17] proposes a model with environmental quality habit formation and in which a consumption good moreover causes a negative external effect on the environment. The intertemporal utility of the social planner depends on consumption of two goods and the environment. The social planner maximizes the utility given the negative effect of the consumption good on the environment and taking into account that there is habit formation in environmental quality. The model is adapted in discrete time the following way. The instantaneous utility (which takes the particular quadratic form) depends on n_t which is the environment that displays habit formation, x_t the "dirty" consumption good (the environmental bad), z_t the "clean" consumption good and s_t the habit level related to the environment, i.e.

$$\tilde{u}(n_t, x_t, z_t, s_t)$$

with $(\gamma \in (0,1), \beta \in (0,1))$

$$\begin{cases} n_t = n - \gamma x_t \\ z_t = y - x_t \\ s_{t+1} = \beta n_t + (1-\delta) s_t \end{cases}$$

One can consider the model with the utility function u defined by

$$u(x_t, s_t) = \tilde{u}(n - \gamma x_t, x_t, y - x_t, s_t)$$

and s_t satisfies the recurrence relation

$$s_{t+1} = \lambda + \lambda_1 s_t + \lambda_2 x_t$$

with $\lambda = \beta n, \lambda_1 = (1-\delta), \lambda_2 = -\beta\gamma$, since

$$s_{t+1} = \beta n_t + (1-\delta)s_t = s_{t+1} = \beta(n - \gamma x_t) + (1-\delta)s_t = \beta n - \beta\gamma x_t + (1-\delta)s_t$$

Note that this is a nonlinear dynamics case.

The model can be written in our notations with $c_t := x_t$ in the same manner as in Section 3.1 where φ is defined through the following recurrence relation

$$\forall c \in \mathbb{R}^+, \forall \tilde{h} \in l_+^\infty, \varphi(c, \tilde{h}) = \lambda + \lambda_1 \varphi(\tilde{h}) + \lambda_2 c$$

and

$$u(c, s) = \tilde{u}(n - \gamma c, c, y - c, s)$$

so that

$$u(c_t, \varphi(\tilde{h}^{(t)})) = \tilde{u}(n - \gamma c_t, c_t, y - c_t, \varphi(\tilde{h}^{(t)}))$$

We now turn to the second aspect of our paper and study history-dependent optimal growth problems.

4. The optimal growth model with adjusted consumptions

In this section, we consider discrete infinite horizon dynamic optimization programs in which the instantaneous payoff presents history-dependence.

4.1 On the general model

Let us consider the general model with u, φ and β be given as introduced in Section 2.4.

$$\mathcal{P}_{u,\varphi,\beta}(k_0, \tilde{h}^{(0)}) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(c_t, \varphi(\tilde{h}^{(t)})) \\ \text{s.t.} & \forall t \geq 0, k_{t+1} = f(k_t) - c_t \\ & \forall t \geq 0, c_t \geq 0 \text{ and } k_t \geq 0, \text{ where } k_0 > 0 \text{ is given} \\ & \forall t \geq 1, \tilde{h}^{(t)} = (c_{t-1}, ..., c_1, c_0, \tilde{h}^{(0)}) \text{ where } \tilde{h}^{(0)} = (h_1, ...) \in l_+^\infty \text{ is given} \\ & \forall t \geq 0, (c_t, \varphi(\tilde{h}^{(t)})) \in \mathcal{D}_u \subseteq (\mathbb{R}^+ \times R) \end{cases}$$

It is equivalent to

$$\mathcal{P}_{u,\varphi,\beta}(k_0,\tilde{h}^{(0)}) = \begin{cases} \text{Maximize} & \sum_{t=0}^{+\infty} \beta^t u(f(k_t)-k_{t+1},\varphi(\tilde{h}^{(t)})) \\ \text{s.t.} & \forall t \geq 0, k_{t+1} \in [0, f(k_t)] \\ & \forall t \geq 1, \tilde{h}^{(t)} = (f(k_{t-1})-k_t, f(k_{t-2})-k_{t-1}, ..., f(k_1)-k_2, f(k_0)-k_1, \tilde{h}^{(0)}) \\ & \forall t \geq 0, (f(k_t)-k_{t+1},\varphi(\tilde{h}^{(t)})) \in \mathcal{D}_u \subseteq (\mathbb{R}^+ \times R) \\ & k_0 > 0 \text{ and } \tilde{h}^{(0)} = (h_1, ...) \in l^\infty_+ \text{ given} \end{cases}$$

4.1.1 Notations and feasible sets

Notations. Given $(k,\tilde{h}) \in \mathbb{R}^+ \times l^\infty_+$, let us denote

$$\mathcal{D}_u(k,\tilde{h}) = \{k' \in \mathbb{R}^+, (f(k)-k',\varphi(\tilde{h})) \in \mathcal{D}_u\}$$

Definition 4.1.— *For any $k \in \mathbb{R}^+$ and history $\tilde{h} \in l^\infty_+$, the feasible correspondence Γ is given by*

$$\begin{aligned} \Gamma(k,\tilde{h}) &= \{k' \in [0,f(k)], (f(k)-k',\varphi(\tilde{h})) \in \mathcal{D}_u\} \\ &= [0,f(k)] \cap \mathcal{D}_u(k,\tilde{h}) \end{aligned}$$

For any given initial capital stock $k_0 > 0$, and initial time-0 history $\tilde{h}^{(0)} \in l^\infty_+$, the feasible set $\Pi(k_0,\tilde{h}^{(0)})$ is defined by the set of capital sequences feasible from k_0 and $\tilde{h}^{(0)}$, ie for any $k_0 > 0$, for any $\tilde{h}^{(0)} \in l^\infty_+$,

$$\Pi(k_0,\tilde{h}^{(0)}) = \{\tilde{k} = (k_t)_{t=1}^{+\infty} \in (\mathbb{R}^+)^{\mathbb{N}}, \forall t \geq 0, k_{t+1} \in \Gamma(k_t,\tilde{h}^{(t)}), \tilde{h}^{(t+1)} = (f(k_t)-k_{t+1},\tilde{h}^{(t)})\}$$

4.1.2 Assumptions

Let us give a list of assumptions.

The assumptions on the adjustment function φ:
(R1) $\varphi : l^\infty \to R$ is a continuous function
(R2) φ is a linear function[3].

[3]This assumption ensures uniqueness of the problem solution. In the Ryder and Heal model, φ is linear on the subset of l^∞ of the sequences constant from a certain rank T, i.e. on the set $l^\infty_c := \{\tilde{x} \in l^\infty, \exists T \in \mathbb{N}, \exists x \in \mathbb{R}, \forall t \geq T, x_t = x\}$. Indeed, let $\tilde{x} \in l^\infty_c$, then

$$\begin{aligned} \varphi(\tilde{x}) &= \varphi(x_0, x_1, ..., x_T, x_T, ...) \\ &= \lambda x_0 + (1-\lambda)\varphi(x_1, ..., x_T, x_T, ...) \\ &= \lambda x_0 + \lambda(1-\lambda)x_1 + (1-\lambda)^2\varphi(x_2, ..., x_T, x_T, ...) \\ &= \lambda \sum_{i=0}^{T-1} (1-\lambda)^i + \lambda(1-\lambda)^T x_T \end{aligned}$$

The assumptions on the utility u:
(P0) u is continuous on $\mathcal{D}_u$
(P1) u is increasing in the present consumption c, ie for any $c \in \mathbb{R}^+$ and $y \in \mathbb{R}$ such that $(c,y) \in int(\mathcal{D}_u)$,$u_c(c,y) > 0$
(P2) u is nonincreasing in any coordinate of the adjustment level y, ie for any $c \in \mathbb{R}^+$ and $y \in R$ such that $(c,y) \in int(\mathcal{D}_u)$, $u_{y_i}(c,y) \leq 0$
(P3) concave in (c,y) and strictly concave in c
(P4) (i) $\forall y \in R$, $\lim\limits_{c \to 0^+} u_c(c,y) = M(y) \leq +\infty$ and $\lim\limits_{c \to +\infty} u_c(c,y) < 1$
(ii) $\forall y \in R, u(0,y) = 0$

The assumptions on the production function f:
(f1) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing, continuous and concave function, $f(0) = 0$.

4.1.3 The objective is well-defined

Lemma 4.1.— *Assume* $e : \mathbb{R}^+ \to \mathbb{R}^+$ *is an increasing continuous function that satisfies (i) or (ii). Then there exists* $a, a' \in \mathbb{R}^+$ *such that* $a \neq 1$ *and for any* $x > 0, e(x) \leq ax + a'$*, and hence for any* $x > 0, e^+(x) = \max\{0, e(x)\} \leq ax + a'$.
(i) e is strictly concave, differentiable and $\lim\limits_{x \to 0^+} e'(x) = M \leq +\infty$ *and* $\lim\limits_{k \to +\infty} e'(x) < 1$
(ii) $e(0) = 0$ *and e is concave*

Proof. (i)[4] Assume $e : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing, continuous and twice differentiable function and that e is strictly concave, $\lim\limits_{x \to 0+} e'(x) = M \leq +\infty$ and $\lim\limits_{x \to +\infty} e'(x) < 1$.
If $\lim\limits_{x \to 0+} e'(x) \leq 1$, then by the concavity of e, the function e' is decreasing, and for any x in $\mathbb{R}^+$, one has $e'(x) \leq 1$ so one can find $a, a' \in \mathbb{R}^+$ such that $a \neq 1$ and for any $x > 0, e(x) \leq ax + a'$.
If $\lim\limits_{x \to 0+} e'(x) > 1$, since $\lim\limits_{x \to +\infty} e'(x) < 1$, the function e admits a fixed-point $\bar{x}$ (ie $e(\bar{x}) = \bar{x}$). If $x > \bar{x}$, then $e(x) < x \leq (1+\varepsilon)x$ for $\varepsilon > 0$, and if $x \leq \bar{x}$, since e is increasing, then $e(x) \leq e(\bar{x}) = \bar{x}$. So for any x in $\mathbb{R}^+$, $e(x) \leq (1+\varepsilon)x + \bar{x}$ and one can define $a = 1+\varepsilon, a' = \bar{x}$

(ii)[5] Assume $e : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing, continuous and twice differentiable function and that $e(0) = 0$ and e is concave.
Since e is continuous on $\mathbb{R}^+$, e has a maximum on $[0,1]$. Define $m = \max\{y / y \in [0, e(x)], x \in [0,1]\}$.
One obtains
if $y \in [0, e(x)]$ with $x \in [0,1]$, then $y \leq m$.
Now, assume $y \in [0, e(x)]$ with $x > 1$. One has $y \leq e(x)$ and since $e(0) = 0$, using the fact that e is concave, since $\frac{1}{x} \in]0,1[$

$$\left.\begin{array}{l} y \leq e(x) \\ 0 \leq e(0) \end{array}\right\} \Rightarrow \frac{1}{x}y + (1-\frac{1}{x})0 \leq \frac{1}{x}e(x) + (1-\frac{1}{x})e(0) \leq e(\frac{1}{x}x + (1-\frac{1}{x})0)$$

the last inequality coming from the concavity of e. Then $\frac{1}{x}y \leq e(1)$ and one obtains
if $y \in [0, e(x)]$ with $x > 1$, then $y \leq xe(1)$.
Finally, for any $x \in R^+$, if $y \in [0, e(x)]$ then $y \leq x(e(1)+m)$, ie one can define $a = e(1)+m$. QED

[4] the proof is analogous as the proof of proposition 2.2.1 p.17 in Le Van and Dana[15]
[5] the proof is analogous as the explanation in comment (2) on assumption (H2) p.161 in Le Van and Morhaim[16]

LEMMA 4.2.— *Assume (f1), (P3) and either (P4)(i) or (P4)(ii). Let* $k_0 > 0$ *and* $\tilde{h}^{(0)} \in l_+^\infty$ *be given. Then there exist* $a, a_1(k_0), a_2 \in I\!R^+$ *such that for any feasible sequence* $\tilde{k} = (k_t)_{t=1}^{+\infty} \in \Pi(k_0, \tilde{h}^{(0)})$ *and its associated history* $\tilde{h}^{(t)}$*, for any* $t \geq 0$,

$$u^+(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) \leq a_1(k_0)a^t + a_2$$

Proof. By assumption (f1) and lemma 4.1, there exists $a, a' \in \mathbb{R}^+$ such that $a \neq 1$ and

$$\forall k \in \mathbb{R}^+, f(k) \leq ak + a'$$

One can check by induction that for any $k_0 \in \mathbb{R}^+$ and for all feasible sequence $\tilde{k} = (k_t)_t$ ie such that for all $t, k_{t+1} \in \Gamma(k_t, \tilde{h}^{(t)})$

$$\forall t, k_t \leq a^t k_0 + \frac{1-a^t}{1-a}a' = (k_0 - \frac{a'}{1-a})a^t + \frac{a'}{1-a}$$

For any sequence $\tilde{k} = (k_t)_t \in \Pi(k_0, \tilde{h})$, the associated histories are such that for all t, $\tilde{h}^{(t)} = (c_{t-1}, c_t, ..., c_0, \tilde{h}^{(0)})$ such that for all $j = 0, ..., t-1, 0 \leq c_j \leq f(k_{j-1}) \leq (k_0 - \frac{a'}{1-a})a^j + \frac{a'}{1-a}$, so that for all t, $\tilde{h}^{(t)}$ belongs to a compact set Π^h of the product topology in l_∞^+. Since u and φ are continuous, there exists $\overline{\varphi} \in R$ such that

$$\forall c \in \mathbb{R}^+, \forall \tilde{h}^{(t)} \in \Pi^h, u(c, \varphi(\tilde{h}^{(t)})) \leq u(c, \overline{\varphi})$$

By assumption (P3) and either (P4)(i) or (P4)(ii), applying Lemma 4.1, there exist $A, A' \in \mathbb{R}^+$ such that

$$\forall c \in \mathbb{R}^+, u^+(c, \overline{\varphi}) \leq Ac + A'$$

so that along a feasible path $\tilde{k} = (k_t)_{t=1}^{+\infty} \in \Pi(k_0, \tilde{h})$, by $x \to u(x, y)$ is increasing, for any $t \geq 0$,

$$\begin{aligned} u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) &\leq u(f(k_t) - k_{t+1}, \overline{\varphi}) \\ &\leq u(f(k_t), \overline{\varphi}) \\ &\leq u(ak_t + a', \overline{\varphi}) \\ &\leq u^+(ak_t + a', \overline{\varphi}) \\ &\leq A(ak_t + a') + A' \end{aligned}$$

and so for any $t \geq 0$,

$$\begin{aligned} u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) &\leq Aa(a^t k_0 + \tfrac{1-a^t}{1-a}a') + Aa' + A' \\ &\leq Aa((k_0 - \tfrac{a'}{1-a})a^t + \tfrac{a'}{1-a}) + Aa' + A' \end{aligned}$$

ie

$$u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) \leq a_1(k_0)a^t + a_2$$

with $a_1(k_0) = Aa(k_0 - \frac{a'}{1-a})$ and $a_2 = Aa\frac{a'}{1-a} + Aa' + A'$. QED

PROPOSITION 4.1.— *Assume* $a\beta < 1$. *Let* $k_0 \in I\!R^{+*}$ *and* $\tilde{h}^{(0)} \in l_+^\infty$ *be given. Then for any feasible sequence* $\tilde{k} = (k_t)_{t=1}^{+\infty} \in \Pi(k_0, \tilde{h})$*, the limit*[6] $\lim_{T \to +\infty} \sum_{t=0}^{T} \beta^t u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)}))$*, with* $\forall t \geq 1, \tilde{h}^{(t)} = (f(k_{t-1}) - k_t, f(k_{t-2}) - k_{t-1}, ..., f(k_1) - k_2, f(k_0) - k_1, \tilde{h}^{(0)})$*, is well-defined.*

[6] ie the objective

Proof. By Lemma 4.2, along a feasible path $\tilde{k} = (k_t)_{t=1}^{+\infty} \in \Pi(k_0, \tilde{h}^{(0)})$, with associated histories $\tilde{h}^{(t)} = (f(k_{t-1}) - k_t, f(k_{t-2}) - k_{t-1}, ..., f(k_1) - k_2, f(k_0) - k_1, \tilde{h}^{(0)})$,

$$\forall t, u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) \leq a_1(k_0)a^t + a_2$$

Then for all $T \in \mathbb{N}$,

$$\begin{aligned}\sum_{t=0}^{T} \beta^t u(f(k_t) - k_{t+1}, \varphi(h^{(t)})) &\leq \sum_{t=0}^{T} \beta^t (a_1(k_0)a^t + a_2) \\ &= a_1(k_0) \sum_{t=0}^{T} (a\beta)^t + a_2 \sum_{t=0}^{T} \beta^t\end{aligned}$$

since $0 < a\beta < 1$ and $0 < \beta < 1$ the conclusion follows.

QED

For a sequence $\tilde{k} = (k_t)_{t=1}^{+\infty} \in \Pi(k_0, \tilde{h}^{(0)})$, we denote by $\mathcal{U}(\tilde{k})$ the objective ie

$$\mathcal{U}(\tilde{k}) = \sum_{t=0}^{+\infty} \beta^t u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)}))$$

Assumption on the objective not being equal to $-\infty$ at the optimum
(A) $\forall k_0 > 0$ and $\tilde{h}^{(0)} \neq 0, \exists \tilde{k} \in \Pi(k_0, \tilde{h}^{(0)})$ such that $\mathcal{U}(\tilde{k}) > -\infty$.

4.2 Existence and uniqueness of the solution

Proposition 4.2.— *Assume (R1), (f1), (P0), (P3) and either (P4)(i) or (P4)(ii). Assume moreover that $\alpha\beta \in (0,1)$, $\Pi(k_0, \tilde{h}^{(0)}) \neq \emptyset$ and $\mathcal{D}_u$ is a closed subset of $\mathbb{R}^2$. Then there exists an optimal solution. Moreover, if φ is linear, the solution is unique.*

Proof Let us show that the objective is upper semi-continuous and the feasible sequence set is a compact set of the product topology. Recall that one can check by induction that for any given $k_0 \in \mathbb{R}^+$ and $\tilde{h}^{(0)} \in l_+^\infty$, for all feasible sequence $\tilde{k} = (k_t)_t \in \Pi(k_0, \tilde{h}^{(0)})$, for all $t, k_{t+1} \in [0, f(k_t)]$, which implies

$$\forall t, k_t \leq a^t k_0 + \frac{1 - a^t}{1 - a} a' = (k_0 - \frac{a'}{1 - a})a^t + \frac{a'}{1 - a}$$

The feasible set $\Pi(k_0, \tilde{h}^{(0)})$ is included in a compact set for the product topoplogy. Moreover, it is closed (by the continuity of f and φ, and by $\mathcal{D}_u$ being a closed subset[7]). So the feasible set $\Pi(k_0, \tilde{h}^{(0)})$ is a compact set for the product topoplogy.

We next show that the objective $\mathcal{U}$ is upper semi-continuous.

Let us consider a sequence $\tilde{k}^n = \{(k_t^n)_{t=1}^{+\infty}\}_n \subset \Pi(k_0, \tilde{h}^{(0)})$ that converges to $\tilde{k} = (k_t)_{t=1}^{+\infty} \in \Pi(k_0)$. Note that when n converges to $+\infty$, the sequence of associated histories $\forall t \geq 1, \widetilde{h^n}^{(t)} = (f(k_{t-1}^n) - k_t^n, f(k_{t-2}^n) - k_{t-1}^n, ..., f(k_1^n) - k_2^n, f(k_0) - k_1^n, \tilde{h}^{(0)})$ converges to the associated history $\forall t \geq 1, \tilde{h}^{(t)} = (f(k_{t-1}) - k_t, f(k_{t-2}) - k_{t-1}, ..., f(k_1) - k_2, f(k_0) - k_1, \tilde{h}^{(0)})$ and by the continuity of φ, for any $t \geq 1$, the sequence $(\varphi(\widetilde{h^n}^{(t)}))_n$ converges to $\varphi(\tilde{h}^{(t)})$.

[7]Note that this is true in both usual cases with habits only, as either $\mathcal{D}_u = (\mathbb{R}^+)^2$ and $\mathcal{D}_u = \{(x, y) \in \mathbb{R}^2, x - y \geq 0\}$

Let us show that $\overline{\lim}_{n\to+\infty} \mathcal{U}(\tilde{k}^n) \leq \mathcal{U}(\tilde{k})$.

For any $t \geq 0$, by Lemma 4.2, for any $\tilde{k} = (k_t)_{t=1}^{+\infty} \in \Pi(k_0, \tilde{h}^{(0)})$ and with history $\tilde{h}^{(t)}$ associated to $\tilde{k}$

$$u^+(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) \leq a_1(k_0)a^t + a_2$$

and by $0 < a\beta < 1$, for any $\varepsilon > 0$, there exists T_ε such that for any $(k_t)_{t=1}^{+\infty} \in \Pi(k_0, \tilde{h}^{(0)})$, and for any $T \geq T_\varepsilon$,

$$\sum_{t=T}^{+\infty} \beta^t u^+(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) \leq \varepsilon$$

So for any $\varepsilon > 0$, there exists T_ε such that for any $n \in \mathbb{N}$ and for any $T \geq T_\varepsilon$,

$$\sum_{t=T}^{+\infty} \beta^t u^+(f(k_t^n) - k_{t+1}^n, \varphi(\widetilde{h^n}^{(t)})) \leq \varepsilon$$

and for any $n \in \mathbb{N}$ and for any $T \geq T_\varepsilon$,

$$\begin{aligned}\sum_{t=0}^{+\infty} \beta^t u(f(k_t^n) - k_{t+1}^n, \varphi(\widetilde{h^n}^{(t)})) &\leq \sum_{t=0}^{T} \beta^t u(f(k_t^n) - k_{t+1}^n, \varphi(\widetilde{h^n}^{(t)})) + \sum_{t=T}^{+\infty} \beta^t u^+(f(k_t^n) - k_{t+1}^n, \varphi(\widetilde{h^n}^{(t)})) \\ &\leq \sum_{t=0}^{T} \beta^t u(f(k_t^n) - k_{t+1}^n, \varphi(\widetilde{h^n}^{(t)})) + \varepsilon\end{aligned}$$

By taking $n \to +\infty$ (and using the continuity of u, f and φ in the part at the right),

$$\overline{\lim}_{n\to+\infty} \sum_{t=0}^{+\infty} \beta^t u(f(k_t^n) - k_{t+1}^n, \varphi(\widetilde{h^n}^{(t)})) \leq \sum_{t=0}^{T} \beta^t u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) + \varepsilon$$

Since this is true for any $T \geq T_\varepsilon$, by taking $T \to +\infty$,

$$\overline{\lim}_{n\to+\infty} \sum_{t=0}^{+\infty} \beta^t u(f(k_t^n) - k_{t+1}^n, \varphi(\widetilde{h^n}^{(t)})) \leq \sum_{t=0}^{+\infty} \beta^t u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) + \varepsilon$$

Since this is true for any $\varepsilon > 0$, by taking $\varepsilon \to 0$,

$$\overline{\lim}_{n\to+\infty} \sum_{t=0}^{+\infty} \beta^t u(f(k_t^n) - k_{t+1}^n, \varphi(\widetilde{h^n}^{(t)})) \leq \sum_{t=0}^{+\infty} \beta^t u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)}))$$

So $\mathcal{U}$ is upper semi-continuous on $\Pi(k_0, \tilde{h}^{(0)})$.

By Weierstrass Theorem (see Aubin [2], Theorem 5.3.1), since $\mathcal{U}$ is upper semi-continuous and $\Pi(k_0, \tilde{h}^{(0)})$ is a compact set for the product topology, there exists an optimal solution.

The assumption (P3) together with on one hand the fact that u is nondecreasing with respect to the first variable and f is concave, and on the other hand the function φ is linear, ensures that there is a unique optimal consumption sequence, hence there is a unique optimal capital sequence.

4.3 The value function and Bellman equation

Definition 4.2.— *The value function V is defined on $I\!R^{+*} \times l_+^\infty$ by for any $(k_0, \tilde{h}^{(0)}) \in I\!R^+ \times l_+^\infty$*

$$V(k_0,\tilde{h}) = \begin{cases} Max & \sum_{t=0}^{+\infty} \beta^t u(f(k_t)-k_{t+1}, \varphi(\tilde{h}^{(t)})) \\ s.t. & \forall t \geq 0, k_{t+1} \in \Gamma(k_t, \tilde{h}^{(t)}) \\ & \forall t \geq 1, \tilde{h}^{(t)} = (f(k_{t-1})-k_t, f(k_{t-2})-k_{t-1}, \ldots.., f(k_1)-k_2, f(k_0)-k_1, \tilde{h}^{(0)}) \\ & k_0 > 0 \text{ and } \tilde{h}^{(0)} \in l_+^\infty \text{ are given} \end{cases}$$

Proposition 4.3.— *The value function V is upper semi-continuous.*

Proof. A direct proof using lemma 4.1 can be done. Indeed[8], let us consider a sequence $(k_0^n, \widetilde{h^n}^{(0)})_n \subset \mathbb{R}^{+*} \times l_+^\infty$ that converges to $(k_0, \tilde{h}) \in \mathbb{R}^{+*} \times l_+^\infty$ and let us consider a subsequence $(k_0^{n_i}, \widetilde{h^{n_i}}^{(0)})_i$ such that

$$\overline{\lim_{n\to+\infty}} V(k_0^n, \widetilde{h^n}^{(0)}) = \lim_{i\to+\infty} V(k_0^{n_i}, \widetilde{h^{n_i}}^{(0)})$$

Let $\varepsilon > 0$. By Lemma 4.1 and as in Proposition 4.1's proof, there exist i_0 and T_0 such that for any $i \geq i_0$ and for any $T \geq T_0$, and for optimal path $(\tilde{k}^{n_i})_i \in \Pi(k_0^{n_i}, \widetilde{h^{n_i}}^{(0)})$ and its associated history $\widetilde{h^{n_i}}^{(t)}$,

$$V(k_0^{n_i}, \widetilde{h^{n_i}}^{(0)}) = \sum_{t=0}^{+\infty} \beta^t u(f(k_t^{n_i}) - k_{t+1}^{n_i}, \varphi(\widetilde{h^{n_i}}^{(t)})) \leq \sum_{t=0}^{T} \beta^t u(f(k_t^{n_i}) - k_{t+1}^{n_i}, \varphi(\widetilde{h^{n_i}}^{(t)})) + \varepsilon$$

Fix $T \geq T_0$. The subsequence $(\tilde{k}^{n_i})_i$ that belongs[9] to $\Pi(k_0^{n_i}, \widetilde{h^{n_i}}^{(0)})$ can be assumed to converge to some $\tilde{k}$ in $\Pi(k_0, \tilde{h}^{(0)})$. By the definition of the associated history, this implies that $(\widetilde{h^{n_i}}^{(t)})_i$ converges to $\tilde{h}^{(t)}$ the history associated to $\tilde{k}$, and by the continuity of φ, that $(\varphi(\widetilde{h^{n_i}(t)}))_i$ converges to $\varphi(\tilde{h}^{(t)})$. Let $i \to +\infty$,

$$\overline{\lim_{n\to+\infty}} V(k_0^n, \widetilde{h^n}^{(0)}) \leq \sum_{t=0}^{T} \beta^t u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) + \varepsilon$$

Let $T \to +\infty$,

$$\overline{\lim_{n\to+\infty}} V(k_0^n, \widetilde{h^n}^{(0)}) \leq \mathcal{U}(\tilde{k}) + \varepsilon \leq V(k_0, \tilde{h}^{(0)})$$

QED

Lemma 4.3.— *(i) Assume*[10] *that for all $\tilde{h}^{(0)} \in l_+^\infty$ and $k_0 > 0, k_0' > k_0$, $\Gamma(k_0, \tilde{h}^{(0)}) \subset \Gamma(k_0', \tilde{h}^{(0)})$. Then for any $\tilde{h}^{(0)} \in l_+^\infty$, the function $k_0 \to V(k_0, \tilde{h}^{(0)})$ is increasing.*
(ii) Assume (P2) with $R = I\!R^+$ and that for all $k_0 > 0$ and $\tilde{h}^{(0)}, \tilde{h}'^{(0)} \in l_+^\infty$ such that $\varphi(\tilde{h}^{(0)}) \geq \varphi(\tilde{h}'^{(0)})$, $\Gamma(k_0, \tilde{h}^{(0)}) \subset \Gamma(k_0, \tilde{h}'^{(0)})$ and $\varphi(k_0, \tilde{h}^{(0)}) \geq \varphi(k_0, \tilde{h}'^{(0)})$. Then for any $k_0 > 0$, for any $(\tilde{h}^{(0)}, \tilde{h}'^{(0)}) \in (l_+^\infty)^2$ such that $\varphi(\tilde{h}^{(0)}) \geq \varphi(\tilde{h}'^{(0)})$, $V(k_0, \tilde{h}'^{(0)}) \geq V(k_0, \tilde{h}^{(0)})$.

[8] see Theorem 1 (ii) p.7 in Le Van Morhaim[16]), sauf qu'ici il faut s'assurer que avec la suite des associated histories ca marche aussi

[9] by the compactness of $\Pi(k_0^{n_i}, \widetilde{h^{n_i}}^{(0)})$

[10] Note that if for all $k > 0, k' > k$ and for all $\tilde{d} \in l_+^\infty$, $\mathcal{D}_u(k, \tilde{d}) \subset \mathcal{D}_u(k', \tilde{d})$ is satisfied and f is increasing, this assumption is satisfied. In particular, it is satisfied as soon as f is increasing and $\mathcal{D}_u = (\mathbb{R}^+)^{n+1}$ or $\mathcal{D}_u = \{(x,y) \in \mathbb{R}^2, x - y \geq 0\}$ and $R = \mathbb{R}^+$. When $\mathcal{D}_u = \{(x,y) \in \mathbb{R}^2, x-y \geq 0\}$ and $R = \mathbb{R}^+$, take $k < k'$, on has $z \in \mathcal{D}_u(k,\tilde{d}) \Rightarrow f(k) - z \geq \varphi(\tilde{d}) \Rightarrow f(k') - z \geq f(k) - z \geq \varphi(\tilde{d}) \Rightarrow z \in \mathcal{D}_u(k', \tilde{d})$

Proof. (*i*) Let $\tilde{h}^{(0)} \in l_+^\infty$ and $0 < k_0 < k'_0$. Let $(k_t)_{t=1}^{+\infty}$ be a feasible path from $(k_0, \tilde{h}^{(0)})$ and $\tilde{h}^{(t)}$ its associated habit sequence. Then such a path is also feasible from $(k'_0, \tilde{h}^{(0)})$ (since $k_1 \in \Gamma(k_0, \tilde{h}^{(0)}) \subset \Gamma(k'_0, \tilde{h}^{(0)})$)

$$V(k'_0, \tilde{h}) \geq \sum_{t=0}^{=\infty} \beta^t u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)}))$$

Since the inequality is true for any feasible path from $(k_0, \tilde{h})$, it is also true for the sup on the feasible set

$$V(k'_0, \tilde{h}^{(0)}) \geq V(k_0, \tilde{h}^{(0)})$$

(*ii*) (*a*) Let $k_0 > 0$ and $(\tilde{h}^{(0)}, \tilde{h}'^{(0)}) \in (l_+^\infty)^2$ such that $\tilde{h}^{(0)} \geq \tilde{h}'^{(0)}$, and let us show that $V(k_0, \tilde{h}'^{(0)}) \geq V(k_0, \tilde{h}^{(0)})$. Let $(k_t)_{t=1}^{+\infty}$ be any feasible path from $(k_0, \tilde{h}'^{(0)})$ and $\tilde{h}'^{(t)}$ its associated habit sequence. Then $(k_t)_{t=1}^{+\infty}$ is a feasible path from $(k_0, \tilde{h}^{(0)})$ and $\tilde{h}^{(t)}$ its associated habit sequence satisfies
$\forall t, \tilde{h}^{(t)} = (f(k_t) - k_{t+1}, f(k_{t-1}) - k_t, ..., f(k_0) - k_1, \tilde{h}) \geq (f(k_t) - k_{t+1}, f(k_{t-1}) - k_t, ..., f(k_0) - k_1, \tilde{h}'^{(0)}) = \tilde{h}'^{(t)}$.
Since φ is nondecreasing and u is decreasing with respect to its second variable $V(k_0, \tilde{h}'^{(0)}) \geq V(k_0, \tilde{h}^{(0)})$.
(*b*) Let $k_0 > 0$ and $(\tilde{h}^{(0)}, \tilde{h}'^{(0)}) \in (l_+^\infty)^2$ such that $\varphi(\tilde{h}^{(0)}) \geq \varphi(\tilde{h}'^{(0)})$, and let us show that $V(k_0, \tilde{h}'^{(0)}) \geq V(k_0, \tilde{h}^{(0)})$. Let $(k_t)_{t=1}^{+\infty}$ be any feasible path from $(k_0, \tilde{h}'^{(0)})$ and $\tilde{h}'^{(t)}$ its associated habit sequence.
Recall that in this case,

$$\forall \tilde{h}^{(0)} \in l_+^\infty, \forall c \in \mathbb{R}^+, \varphi(c, \tilde{h}^{(0)}) = \lambda_1 \varphi(\tilde{h}^{(0)}) + \lambda_2 c$$

and for any $\tilde{h}^{(0)}, \tilde{h}'^{(0)}$ and $c \in \mathbb{R}^+$, if $\varphi(\tilde{h}^{(0)}) \geq \varphi(\tilde{h}'^{(0)})$ then $\varphi(c, \tilde{h}^{(0)}) \geq \varphi(c, \tilde{h}'^{(0)})$.
Then $(k_t)_{t=1}^{+\infty}$ is a feasible path from $(k_0, \tilde{h}^{(0)})$ and $\tilde{h}^{(t)}$ its associated habit sequence satisfies
$\forall t, \varphi(\tilde{h}^{(t)}) = \varphi(f(k_t) - k_{t+1}, f(k_{t-1}) - k_t, ..., f(k_0) - k_1, \tilde{h}) \geq \varphi(f(k_t) - k_{t+1}, f(k_{t-1}) - k_t, ..., f(k_0) - k_1, \tilde{h}'^{(0)}) = \varphi(\tilde{h}'^{(t)})$. Since u is decreasing with respect to its second variable $V(k_0, \tilde{h}'^{(0)}) \geq V(k_0, \tilde{h}^{(0)})$. QED

(**H**) $\forall k_0 \in R_+, \exists \mathcal{V}(k_0)$ a compact neighborhood of k_0 in R_+, $\forall \varepsilon > 0, \exists T_0$ such that $\forall T \geq T_0, \forall k'_0 \in \mathcal{V}(k_0), \forall \tilde{k}' \in \Pi(k'_0, \tilde{h}^{(0)})$, one has with $\tilde{h}'^{(t)} = (f(k'_{t-1}) - k'_t, f(k'_{t-2}) - k'_{t-1}, ..., f(k'_1) - k'_2, f(k'_0) - k'_1, \tilde{h}^{(0)})$,

$$\sum_{t=T}^{+\infty} \beta^t u^+(f(k'_t) - k'_{t+1}, \varphi(\tilde{h}'^{(t)})) \leq \varepsilon$$

where $u^+(r, r') = \max\{0, u(r, r')\}$.

PROPOSITION 4.4.— *Assume (A) and (H). Then the value function V satisfies*
(i) $\forall k_0, \tilde{h}^{(0)}, \tilde{k} \in \Pi(k_0, \tilde{h}^{(0)}), \overline{\lim}_{t \to +\infty} \beta^t V(k_t, \tilde{h}^{(t)}) \leq 0$.
(ii) $\forall k_0, \tilde{h}^{(0)}$, *and* $\forall \tilde{k} \in \Pi(k_0, \tilde{h}^{(0)})$ *such that* $\mathcal{U}(\tilde{k}) < -\infty$, $\lim_{t \to +\infty} \beta^t V(k_t, \tilde{h}^{(t)}) = 0$.

Proof. (*i*) By (H), $\exists T_0, \forall T > T_0, \forall k'_0 \in \mathcal{V}(k_0), \varepsilon > 0, \forall \tilde{k}' \in \Pi(k_0, \tilde{h}^{(0)})$,

$$\sum_{t=T}^{+\infty} \beta^t u(f(k'_t) - k'_{t+1}, \varphi(\tilde{h}'^{(t)})) \leq \sum_{t=T}^{+\infty} \beta^t u^+(f(k'_t) - k'_{t+1}, \varphi(\tilde{h}'^{(t)})) \leq \varepsilon$$

Let $\tilde{k}' \in \Pi(k'_0, \tilde{h}^{(0)}), T \geq T_0$. For any $\tilde{k}'' = (k''_{T+1}, ...) \in \Pi(k'_T, \tilde{h}'^{(T)})$, one has $(k'_1, ..., k'_T, k''_{T+1}, ...) \in \Pi(k'_0, \tilde{h}^{(0)})$, and

$$\beta^T u(f(k'_T) - k''_{T+1}, \varphi(\tilde{h}'^{(T)})) + \beta^{T+1} u(f(k''_{T+1}) - k''_{T+2}, \varphi(\tilde{h}'^{(T+1)})) + ... \leq \varepsilon$$

so $\beta^T V(k'_T, \tilde{h}^{(T)}) \leq \varepsilon$ which implies (i).

(ii) $\forall \tilde{k} \in \Pi(k_0, \tilde{h}^{(0)})$,

$$-\infty < \mathcal{U}(\tilde{k}) \leq \sum_{t=0}^{T} \beta^t u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)})) + \beta^{T+1} V(k_{T+1}, \tilde{h}^{(T+1)})$$

and

$$0 = \lim_{T \to +\infty} [\mathcal{U}(\tilde{k}) - \sum_{t=0}^{T} \beta^t u(f(k_t) - k_{t+1}, \varphi(\tilde{h}^{(t)}))] \leq \underline{\lim}_{T \to +\infty} \beta^{T+1} V(k_{T+1}, \tilde{h}^{(T+1)})$$

From (i) then $\lim_{T \to +\infty} \beta^{T+1} V(k_t, \tilde{h}^{(T+1)}) = 0$ QED

A standard proof (see Theorem 4.4 p.75 Stokey Lucas and Prescott[22]) allows to show the following result.

Proposition 4.5.— *Assume (R1), (f1), (P0), (P3) and either (P4)(i) or (P4)(ii). Then $\tilde{k}^*$ is an optimal solution if and only if*

$$\forall t \geq 0, V(k^*_t, \tilde{h}^{*(t)}) = u(f(k^*_t) - k^*_{t+1}, \varphi(\tilde{h}^{*(t)})) + \beta V(k^*_{t+1}, \tilde{h}^{*(t+1)})$$

where $\tilde{h}^{*(t)} = (f(k^*_t) - k^*_{t+1}, ..., k^*_1 - k_0, \tilde{h}^{(0)})$

Let B be the Bellman operator, ie $B : \mathcal{F}(\mathbb{R}^+ \times l^\infty_+, \mathbb{R}) \to \mathcal{F}(\mathbb{R}^+ \times l^\infty_+, \mathbb{R})$ be defined by

$$\forall w \in \mathcal{F}(\mathbb{R}^+ \times l^\infty_+, \mathbb{R}), Bw(k, \tilde{h}) = \max_{k' \in \Gamma(k, \tilde{h})} \{u(f(k) - k', \varphi(\tilde{h})) + \beta w(k', (f(k) - k', \tilde{h}))\}$$

where $\Gamma(k, \tilde{h}) = \{k' \in [0, f(k)], (f(k) - k', \varphi(\tilde{h})) \in \mathcal{D}_u\} = [0, f(k)] \cap \{k', (f(k) - k', \varphi(\tilde{h})) \in \mathcal{D}_u\}$.

Definition 4.3.— *Let $\mathcal{F}_b(\mathbb{R}^+ \times l^\infty_+, \mathbb{R})$ the set of upper semi-continuous functions $w \in \mathcal{F}(\mathbb{R}^+ \times l^\infty_+, \mathbb{R})$ such that*
(i) $\forall k_0 \in \mathbb{R}_+, \forall \tilde{k} \in \Pi(k_0, \tilde{h}^{(0)}), \overline{\lim}_{t \to +\infty} \beta^t w(k_t, \tilde{h}^{(t)}) \leq 0$, with $\tilde{h}^{(t)} = (f(k_{t-1}) - k_t, f(k_{t-2}) - k_{t-1}, ..., f(k_1) - k_2, f(k_0) - k_1, \tilde{h}^{(0)})$
(ii) $\forall k_0 \in \mathbb{R}_+, \forall \tilde{k} \in \Pi(k_0, \tilde{h}^{(0)})$ such that[11] $\mathcal{U}(\tilde{k}) > -\infty$, one has $\lim_{t \to +\infty} \beta^t w(k_t, \tilde{h}^{(t)}) = 0$

Proposition 4.6.— *The value function V is the unique fixed-point of the Bellman operator on the set of functions $\mathcal{F}_b(\mathbb{R}^+ \times l^\infty_+, \mathbb{R})$.*

Proof. The proof that V is a fixed-point of the Bellman operator is standard. Uniqueness of the fixed point is shown by contradiction. Indeed, suppose there exists another fixed-point of $\mathcal{B}$ in $\mathcal{F}(\mathbb{R}^+ \times l^\infty_+, \mathbb{R})$ $\mathcal{F}_b(\mathbb{R}^+ \times l^\infty_+, \mathbb{R})$. One can easily check that $W \leq V$. Let us show that $V \leq W$.

[11]Note that assumption (A) ensures that such a $\tilde{k} \in \Pi(k_0, \tilde{h}^{(0)})$ with $\mathcal{U}(\tilde{k}) > -\infty$ exists.

Let $k_0 \in \mathbb{R}^+, \tilde{h}^{(0)}) \in l_+^\infty$. For any $\tilde{k} \in \Pi(k_0, \tilde{h}^{(0)})$ such that $\mathcal{U}(\tilde{k}) > -\infty$. On has, with $\tilde{h}^{(1)}) = (f(k_0) - k_1, \tilde{h}^{(0)})$,

$$\begin{aligned} W(k_0, \tilde{h}^{(0)}) &= BW(k_0, \tilde{h}^{(0)}) \\ &\geq u(f(k_0) - k_1, \tilde{h}^{(0)}) + \beta W(k_1, (f(k_0) - k_1, \tilde{h}^{(0)})) \\ &= u(f(k_0) - k_1, \tilde{h}^{(0)}) + \beta W(k_1, \tilde{h}^{(1)}) \end{aligned}$$

and so by induction, with $\tilde{h}^{(t)}) = (f(k_{t-1}) - k_t, f(k_{t-2}) - k_{t-1}, ..., f(k_1) - k_2, f(k_0) - k_1, \tilde{h}^{(0)})$,

$$\begin{aligned} W(k_0, \tilde{h}^{(0)}) &\geq \sum_{t=0}^{T} u(f(k_t) - k_{t+1}, \tilde{h}^{(t)}) + \beta^{T+1} W(k_{T+1}, \tilde{h}^{(T+1)}) \\ &\geq \lim_{T \to +\infty} \sum_{t=0}^{T} u(f(k_t) - k_{t+1}, \tilde{h}^{(t)}) + \lim_{T \to +\infty} \beta^{T+1} w(k_{T+1}, \tilde{h}^{(T+1)}) \\ &\geq \mathcal{U}(\tilde{k}) \end{aligned}$$

which implies that $W(k_0, \tilde{h}^{(0)}) \geq V(k_0, \tilde{h}^{(0)})$ (since for any $\tilde{k} \in \Pi(k_0, \tilde{h}^{(0)})$, one has $W(k_0, \tilde{h}^{(0)}) \geq \mathcal{U}(\tilde{k})$ and $V(k_0, \tilde{h}^{(0)})$ is the sup of $\mathcal{U}(\tilde{k})$ for $\tilde{k}$ in $\Pi(k_0, \tilde{h}^{(0)})$).

QED

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