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# Existence of an equilibrium of property rights

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## Abstract

I study the existence of an equilibrium of private property rights in social systems where individual agents decide to make individual or collective gifts according to their individual preferences on the distribution of private consumption expenditures. It is proved that the distributive core is non-empty whenever there exists, at any feasible distribution of wealth, at least one agent in local unsympathetic isolation. The equality of property rights is in the distributive core if the agents have common opinions on the acceptable orientation of wealth transfers implying that redistributive transfers, when they exist, flow down the scale of wealth. © 1998 Elsevier Science B.V.

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## 1. Introduction

This paper studies the determination of the distribution of private property rights by individual or collective voluntary transfers in social systems where individual agents have preferences on the distribution of private consumption expenditures. A strong (weak) equilibrium of gifts is a vector of individual gifts that is unblocked by any coalition (individual agent), *given* an initial distribution of individual property rights. A strong (weak) equilibrium of rights is a distribution of private property rights that is unblocked by any coalition (individual agent): 0 transfers is a strong (weak) equilibrium of gifts for this distribution of rights. The set of strong equilibria of a given social system is its distributive core. A related concept is the distributive liberal social contract [1] that can be defined as a strong equilibrium of rights that is unanimously preferred to an initial distribution of rights [2].

My 1992 and 1994 papers describe a simple social system, with standard quasi-concave Cobb–Douglas utility functions, where there exists no weak equilibrium of

rights: there is, at any feasible distribution of wealth, at least one agent who is both willing and able to transfer individually some of his own wealth to another agent. The non-existence of a social equilibrium [3] follows there from the absence of upper bounds on transfers, gifts feeding gifts in a “war of gifts”. Such a war of gifts bears interesting analogies with the “potlatch” studied by anthropologists since the classic study of Mauss [4].

The paper designs assumptions on individual utility functions, which imply the existence of a weak or strong equilibrium of rights, or, equivalently, of a weak or strong equilibrium of gifts for at least one initial distribution of rights. The paper is organized as follows. Section 2 defines the equilibrium of rights; Section 3 proposes a first existence theorem; Section 4 gives a second and a third existence theorem and analyzes examples of Cobb–Douglas social systems. Appendix A contains a theorem that gathers useful properties of Cobb–Douglas social systems.

## 2. Equilibrium of property rights

We consider social systems made of individual agents owning, consuming and transferring wealth. There are  $n$  such *individuals*, denoted by an index  $i$  running in  $N = \{1, \dots, n\}$ . *Wealth* is private money wealth, defined as that part of wealth that is expressed in money units and owned by individual agents. It is assumed divisible, and its aggregate amount is assumed independent of individual consumption and transfer decisions. The share  $\omega_i \in [0, 1]$  of total wealth owned by individual  $i$  prior consumption or transfer is his *initial endowment* or *right* (possibly = 0). A consumption  $x_i$  of individual  $i$  is the money value of his consumptions of commodities; it is said to be *feasible* when it is non-negative. A gift  $t_{ij}$  from individual  $i$  to individual  $j$  is a non-negative money transfer from individual  $i$ 's estate (his initial ownership plus the gifts he received from others) to individual  $j$ 's. We ignore alternative uses of wealth, such as disposal or production, so that the following accounting identity is verified for all  $i$ :

$$x_i + \sum_j t_{ij} = \omega_i + \sum_j t_{ji},$$

where, conventionally,  $t_{ji} = 0$  for all  $j$ .

We suppose that individuals have well-defined preferences on the final distribution of wealth, that is, on the vector of individual consumption expenditures. Denoting  $x = (x_1, \dots, x_n)$  such vectors, we endow each individual  $i$  with a utility function  $w_i: x \rightarrow w_i(x)$ , defined on the space of consumption distributions  $\mathbb{R}^n$ . We say that individual  $i$  is *benevolent* (*indifferent*, *malevolent*) to individual  $j \neq i$  in a neighborhood  $V(x^*)$  of distribution  $x^*$  in  $\mathbb{R}^n$  whenever  $w_i$  is strictly increasing (constant, strictly decreasing) in its  $j$ th argument  $x_j$  in  $V(x^*)$ . We say, likewise, that individual  $i$  is, to use Edgeworth's excellent words [5], *unsympathetically isolated* in  $V(x^*)$ , when  $w_i$  is both strictly increasing in  $x_i$ , and constant in  $x_j$  for all  $j \neq i$ , in  $V(x^*)$ . An individual is said *unsympathetically isolated* if he is *unsympathetically isolated* in  $\mathbb{R}^n$ .

A distribution of initial rights  $(\omega_1, \dots, \omega_n)$  is denoted  $\omega$ . It is an element of the

unit-simplex  $S_n = \{x \in \mathbb{R}_+^n \mid \sum_i x_i = 1\}$  of  $\mathbb{R}^n$ . The elements of  $S_n$  are the *feasible* distributions of wealth<sup>1</sup>. A *social system*  $w$  is a vector  $(w_1, \dots, w_n)$  of individual utility functions. A *social system of private property* is a pair  $(w, \omega)$ . A distribution of rights  $\omega$  is said to be *weakly (strongly) Pareto-efficient* in social system  $w$  if there is no feasible distribution  $x \in S_n$  such that  $w_i(\omega) < w_i(x)$  for all  $i \in N$  ( $w_i(\omega) \leq w_i(x)$  for all  $i \in N$ , and  $w_i(\omega) < w_i(x)$  for at least one  $i \in N$ ).  $P(w)$  ( $P^*(w)$ ) will denote the set of weakly (strongly) Pareto-efficient distributions of  $w$ .

A *gift-vector* of individual  $i$  is a vector  $t_i = (t_{i1}, \dots, t_{in})$ . A gift-vector  $t$  is a vector  $(t_1, \dots, t_n)$ . A *coalition*  $I$  is a non-empty subset of the set of agents  $N$ .  $t_I$  ( $t_{\setminus I}$ ) is the vector of gifts obtained from  $t$  by deleting the components  $t_i$  such that  $i \notin I$  ( $i \in I$ ).  $(t_{\setminus I}, t_I^*)$  is the gift-vector obtained from  $t$  and  $t^*$  by substituting  $t_i^*$  for  $t_i$  in  $t$  for all  $i \in I$ .  $\Delta_i t$  is the net transfer  $\sum_j (t_{ji} - t_{ij})$  accruing to individual  $i$  when  $t$  is the gift-vector.  $x(\omega, t)$  is the vector of individual consumption expenditures  $(\omega_1 + \Delta_1 t, \dots, \omega_n + \Delta_n t)$ , that is, given the accounting identity above, the unique consumption distribution associated with the distribution of rights  $\omega$  and the gift-vector  $t$ .

We say that gift-vector  $t$  is *blocked (unblocked)* by coalition  $I$  in the social system of private property  $(w, \omega)$  if there exists some (no)  $t_I^*$  such that, for all  $i \in I$ ,  $x_i(\omega, (t_{\setminus I}, t_I^*))$  is feasible and  $w_i(x(\omega, (t_{\setminus I}, t_I^*))) > w_i(x(\omega, t))$ . A gift-vector  $t$  is a *weak equilibrium* of gifts of  $(w, \omega)$  if  $x(\omega, t)$  is feasible and  $t$  is unblocked by coalition  $\{i\}$  for all  $i \in N$ . A gift-vector  $t$  is a *strong equilibrium of gifts* of  $(w, \omega)$  if  $x(\omega, t)$  is feasible and  $t$  is not blocked by any coalition.

We can now set the following definitions of a *weak (strong) equilibrium of property rights*:

**Definition.** A distribution of rights  $\omega$  is a *weak (strong) equilibrium of property rights* of social system  $w$  if gift-vector 0 is a weak (strong) equilibrium of gifts of  $(w, \omega)$ .

The set of weak (strong) equilibria of rights of social system  $w$  is denoted  $M(w)$  ( $C(w)$ ).  $C(w)$  is the *distributive core* of  $w$ . By definition:  $C(w) \subset M(w) \cap P(w)$ .

### 3. Existence as a consequence of indifference

Section 3.1 establishes a useful preliminary result. The first existence theorem is presented in Section 3.2.

#### 3.1. Strongly blocking coalitions

A class of blocking coalitions is of particular interest in the type of distributive problems studied here: the coalitions whose members are all willing and able to transfer some of their individual wealth outside the coalition. Formally, let us say that gift-vector

<sup>1</sup>Remember that disposal and production activities are assumed away, so that wealth must be either consumed or transferred.

$t$  is *strongly blocked* by coalition  $I$  in social system  $(w, \omega)$  if there exists a  $t_I^*$  such that, for all  $i \in I$ ,  $x_i(\omega, t) > x_i(\omega, (t_{\setminus I}, t_I^*)) \geq 0$  and  $w_i(x(\omega, (t_{\setminus I}, t_I^*))) > w_i(x(\omega, t))$ . We then say, equivalently, that  $(t_{\setminus I}, t_I^*)$  *strongly blocks*  $t$ . If a gift-vector is strongly blocked, it is, clearly, blocked. The lemma below states a partial converse, namely that if 0 is blocked by a coalition, then it must be strongly blocked by some subcoalition of the former. It follows from this that the strong equilibrium of rights can be defined, equivalently, as a situation where status quo is unblocked, or as a situation where status quo is not strongly blocked.

**Lemma.** (i) If 0 is strongly blocked by coalition  $I$  in social system  $(w, \omega)$ , then it is blocked by coalition  $I$  in  $(w, \omega)$ . (ii) If 0 is blocked by coalition  $I$  in social system  $(w, \omega)$ , then there exist a coalition  $J \subset I$  and a gift-vector  $t \neq 0$  such that:  $(0_{\setminus J}, t_J)$  strongly blocks 0 in  $(w, \omega)$ ; and  $t_{ij} > 0$  if and only if  $i \in J$  and  $x_j(\omega, (0_{\setminus J}, t_J)) > \omega_j$ .

**Proof.** (i) is an immediate consequence of definitions. Let us establish (ii). Suppose that 0 is blocked by coalition  $I$  in social system  $(w, \omega)$ , i.e. that there exists a gift-vector  $t$  such that, for all  $i \in I$ ,  $x_i(\omega, (0_{\setminus I}, t_I)) \geq 0$ , and  $w_i(x(\omega, (0_{\setminus I}, t_I))) > w_i(\omega)$ . Then, necessarily:  $x(\omega, (0_{\setminus I}, t_I)) \neq \omega$ ;  $x_i(\omega, (0_{\setminus I}, t_I)) = \omega_i + \sum_{j \in I} t_{ji} \geq \omega_i$  whenever  $i \notin I$ ; and therefore there exist  $i \in I$  such that  $x_i(\omega, (0_{\setminus I}, t_I)) < \omega_i$  and  $i \in N$  such that  $x_i(\omega, (0_{\setminus I}, t_I)) > \omega_i$ . Denote:  $J$  the non-empty set of agents  $i$  such that  $x_i(\omega, (0_{\setminus I}, t_I)) < \omega_i$  ( $J \subset I$ ; it is the set of “net givers”);  $K$  the non-empty set of agents  $i$  such that  $x_i(\omega, (0_{\setminus I}, t_I)) > \omega_i$  ( $K$  is the set of “net receivers”);  $\theta = \sum_{i \in J} \omega_i - x_i(\omega, (0_{\setminus I}, t_I)) = \sum_{i \in K} x_i(\omega, (0_{\setminus I}, t_I)) - \omega_i > 0$  the total amount of redistributed wealth;  $\lambda_i$  the share  $\theta^{-1}(\omega_i - x_i(\omega, (0_{\setminus I}, t_I)))$  of agent  $i \in J$  in  $\theta$ ;  $\mu_i$  the share  $\theta^{-1}(x_i(\omega, (0_{\setminus I}, t_I)) - \omega_i)$  of agent  $i \in K$  in  $\theta$ ;  $t^*$  the gift-vector such that  $t_{ij}^* = \lambda_i \mu_j \theta > 0$  whenever  $(i, j) \in J \times K$ ,  $t_{ij}^* = 0$  otherwise. We then have  $x(\omega, (0_{\setminus J}, t_J^*)) = x(\omega, (0_{\setminus I}, t_I))$ , and the proof is completed.  $\square$

### 3.2. First existence theorem

The logical (if not psychological) relevance of unsympathetic isolation as a sufficient condition for the existence of a strong equilibrium of rights appears straightforwardly from the following simple observation: if  $\omega^i$  denotes the feasible distribution such that  $i$ 's wealth is 1 and  $j$ 's wealth is 0 for all  $j \neq i$ , and if  $I$  is the set of unsympathetically isolated agents of social system  $w$ , then the convex hull of family  $\{\omega^i\}_{i \in I}$  is contained in the core of  $w$ . The existence of a single unsympathetically isolated agent is therefore sufficient for the non-emptiness of the distributive core (if  $i$  is this unsympathetically isolated agent, then  $\omega^i$  is in the core). Theorem 1 below establishes existence under the still much weaker assumption that there is, at any distribution of rights  $\omega$ , at least one agent who is unsympathetically isolated in some neighborhood of  $\omega$  in  $\mathbb{R}^n$  (local unsympathetic isolation of at least one agent at any point of the unit-simplex).

**Theorem 1.** Suppose that, for all  $\omega \in S_n$ , there exist an agent  $i$  and a neighborhood  $V(\omega)$  of  $\omega$  in  $\mathbb{R}^n$  such that agent  $i$  is unsympathetically isolated in  $V(\omega)$ , and that, for all

$i \in N$ ,  $w_i$  is continuous in  $\mathbb{R}^n$ ; for all  $(x, x') \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ ,  $w_i(x) > w_i(x')$  and  $0 < \lambda < 1$  imply  $w_i(\lambda x + (1 - \lambda)x') > w_i(x')$ .<sup>2</sup> Then  $C(w)$  is non-empty.

**Proof.** The proof will proceed in two steps.

(i) Let us prove first that: if there exists an agent  $i$  and a neighborhood  $V(\omega)$  of  $\omega$  in  $\mathbb{R}^n$  such that agent  $i$  is unsympathetically isolated in  $V(\omega)$ , and if, for all  $(x, x') \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ ,  $w_i(x) > w_i(x')$  and  $0 < \lambda < 1$  imply  $w_i(\lambda x + (1 - \lambda)x') > w_i(x')$ , then agent  $i$  belongs to none of the coalitions that strongly block 0 in  $(w, \omega)$ .

Suppose that  $i$  is locally unsympathetically isolated at  $\omega$ , and belongs to a coalition  $I$  that strongly blocks 0 in  $(w, \omega)$ . There exists then a  $t$  such that, for all  $j \in I$ ,  $\omega_j > x_j(\omega, (0_{\setminus I}, t)) \geq 0$ , and  $w_j(x(\omega, (0_{\setminus I}, t))) > w_j(x(\omega, 0)) = w_j(\omega)$ . And we have, in particular,  $w_i(x(\omega, (0_{\setminus I}, t))) > w_i(\omega)$ . Let  $x^* = x(\omega, (0_{\setminus I}, t))$ . We have  $x^* \geq 0$ . The convexity assumption on  $i$ 's distributive preferences implies  $w_i(\lambda x^* + (1 - \lambda)\omega) > w_i(\omega)$  for all real numbers  $\lambda \in ]0, 1[$ . But this contradicts the local unsympathetic isolation of agent  $i$ .

(ii) It will suffice to prove, now, that: if, for all  $\omega \in S_n$ , there exists an agent that belongs to none of the coalitions that strongly block 0 in  $(w, \omega)$ , and if  $w_i$  is continuous in  $\mathbb{R}^n$  for all  $i \in N$ , then  $C(w)$  is non-empty.

Define, for any  $\omega \in S_n$ , set  $\Phi(\omega) = \{x \in S_n \mid x_i = 0 \text{ whenever } i \text{ belongs to some coalition that strongly blocks 0 in social system } (w, \omega)\}$ . And suppose that, for all  $\omega \in S_n$ , there exists an agent that belongs to none of the coalitions that strongly block 0 in  $(w, \omega)$ . We want to establish first that  $\omega \rightarrow \Phi(\omega)$  has a fixed point, and second that this fixed point lies in the distributive core.

It follows readily from definitions that  $\Phi(\omega)$  is compact and convex for all  $\omega \in S_n$ . By assumption, for any  $\omega \in S_n$ , there exists an agent  $i$  who belongs to none of the coalitions that strongly block 0 in  $(w, \omega)$ ; but then  $\omega^i \in \Phi(\omega)$  for such an  $i$ ; therefore,  $\Phi(\omega) \neq \emptyset$  for all  $\omega \in S_n$ , and  $\Phi: \omega \rightarrow \Phi(\omega)$  is a compact and convex-valued correspondence on  $S_n$ . A sufficient condition for the existence of a fixed point of  $\Phi$  is then the upper-hemi continuity of this correspondence (Kakutani fixed-point theorem). Consider two converging sequences  $(x^q)_{q \in \mathbb{N}}$  and  $(z^q)_{q \in \mathbb{N}}$  of elements of  $S_n$  such that  $x^q \in \Phi(z^q)$  for all  $q \in \mathbb{N}$ , and denote  $x^*$  and  $z^*$  their respective limits. We have to prove that  $x^* \in \Phi(z^*)$ . Suppose that agent  $i$  belongs to a coalition  $I$  that strongly blocks 0 in  $(w, z^*)$ , i.e. that there exists  $I \supset \{i\}$  and a gift-vector  $t$  such that, for all  $j \in I$ ,  $z_j^* > x_j(z^*, (0_{\setminus I}, t)) \geq 0$ , and  $w_j(x(z^*, (0_{\setminus I}, t))) > w_j(x(z^*, 0)) = w_j(z^*)$ .  $z_j^* > x_j(z^*, (0_{\setminus I}, t))$  implies  $t_j > 0$ . Using the continuity of functions  $t_1 \rightarrow x_j(z, (0_{\setminus I}, t_1))$  and  $t_1 \rightarrow w_j(x(z, (0_{\setminus I}, t_1)))$ , we can assume therefore that  $x_j(z^*, (0_{\setminus I}, t)) > 0$  for all  $j \in I$ . Continuity of functions  $z \rightarrow x_j(z, (0_{\setminus I}, t))$  and  $z \rightarrow w_j(x(z, (0_{\setminus I}, t)))$  implies then that there exists  $q_0 \in \mathbb{N}$  such that  $z_j^q > x_j(z^q, (0_{\setminus I}, t)) > 0$ , and  $w_j(x(z^q, (0_{\setminus I}, t))) > w_j(x(z^q, 0)) = w_j(z^q)$  for all  $q \geq q_0$  and all  $j \in I$ , which means that coalition  $I$  strongly blocks 0 in  $(w, z^q)$  for all  $q \geq q_0$ . The

<sup>2</sup>The convexity assumption on distributive preferences that is made in this Theorem and in Theorem 3 below is of the same type as the convexity assumption on consumers' preferences used by Arrow and Debreu in their proof of the existence of a competitive equilibrium [8]. It is stronger than quasi-concavity and weaker than strict quasi-concavity (op. cit.: comment on IIIc, including the footnote).

definition of  $\Phi$  implies then  $x_i^q = 0$  for all  $q \geq q^0$ , and therefore  $x_i^* = 0$ , so that  $x^* \in \Phi(z^*)$ . This establishes the upper-hemi continuity of  $\Phi$ , and, therefore, the existence of a fixed point for this correspondence.

Denote  $\omega^*$  a fixed point, and let us prove that it lies in  $C(w)$ . By the lemma above, it suffices to establish that 0 is not strongly blocked in  $(w, \omega^*)$ . Suppose the contrary, i.e. that there exist a coalition  $I$  and a gift-vector  $t$  such that, for all  $i \in I$ ,  $\omega_i^* > x_i(\omega^*, (0_{\setminus I}, t_I)) \geq 0$ , and  $w_i(x(\omega^*, (0_{\setminus I}, t_I))) > w_i(\omega^*)$ . By definition of  $\Phi$ , we must then have  $\omega_i^* = 0$  for all  $i \in I$ , a contradiction.  $\square$

#### 4. Stability of the equality of rights

Section 3 explored a first class of sufficient conditions for the existence of a stable distribution of rights, founded on unsympathetic isolation and in particular on the seemingly weak assumption of the existence, at any feasible distribution of rights of a locally unsympathetically isolated agent. Existence relied there on the stabilizing influence of unsympathetic isolation on voluntary redistribution, and on the continuity and convexity properties of distributive preferences.

The type of assumptions on distributive preferences that lie at the heart of the present section is quite different in spirit, since existence is founded, now, on some kind of common views, embodied in distributive preferences, on a natural or acceptable orientation for redistribution. These common views imply that redistributive transfers, when they exist, flow down the scale of wealth. We establish below that the equality in property rights is then a strong equilibrium.

##### 4.1. Second existence theorem

Denote:  $\omega^E$  the *equal distribution of rights*  $((1/n), \dots, (1/n))$ ;  $e_{ij}$  the vector of  $\mathbb{R}^n$  whose components are all equal to 0 except the  $i$ th, equal to  $-1$ , and the  $j$ th, equal to  $+1$ .

The second existence theorem relies on the following assumptions:

**Assumption 1.**  $w_i(x + \tau e_{ij}) \leq w_i(x)$  for all  $\tau \in \mathbb{R}_+$  whenever  $x_j \geq x_i$ .

**Assumption 2.**  $w_i(x + \tau e_{jk}) \geq w_i(x)$  for all  $\tau \in [0, \frac{1}{2}(x_j - x_k)]$  whenever  $j$  and  $k$  are distinct from  $i$  and  $x_j \geq x_k$ .

Assumption 1 says that an agent does not desire to transfer his own wealth to wealthier agents. Assumption 2 means that an agent does not object to a redistribution of wealth between two other agents as long as the giver is at least as rich as the beneficiary of the transfer. Taken together, Assumptions 1 and 2 say that the agents share the common opinion that, if there are wealth transfers, they should flow down the scale of wealth. Since disposal has been assumed away, Assumptions 1 and 2 imply  $P^*(w) \subset C(w)$  ([2] Theorem 1; a variant is established below in Theorem 4 of Appendix A). A sufficient

condition for the existence of a strong equilibrium of rights is then the continuity of utility functions, which implies, as is well known, the non-emptiness of  $P^*(w)$ . Theorem 2 establishes, somewhat unexpectedly, that this condition of continuity can be dispensed with.

**Theorem 2.** (i) *If social system  $w$  verifies Assumption 1, then  $\omega^E$  is a weak equilibrium of property rights of  $w$ .* (ii) *If  $w$  verifies Assumptions 1 and 2, then  $\omega^E$  is a strong equilibrium of property rights of  $w$ .*

**Proof.** (i) Suppose that  $w$  verifies Assumption 1 and that  $\omega^E$  is not a weak equilibrium. There exist  $I = \{i\}$  and  $t_I \neq 0$  such that  $\omega_i^E > x_i(\omega^E, (0_{\setminus I}, t_I)) \geq 0$  and  $w_i(x(\omega^E, (0_{\setminus I}, t_I))) > w_i(\omega^E)$ . Let:  $x^* = x(\omega^E, (0_{\setminus I}, t_I))$ ;  $J = \{j \in N | t_{ij} > 0\}$ . Suppose, without loss of generality, that  $i = 1$  and  $J = \{2, \dots, m\}$ , with  $m \leq n$ . Define recursively the following sequence:  $x^1 = \omega^E$ ; for all  $j \in J$ ,  $x^j = x^{j-1} + t_{ij}e_{ij}$ . Observe that, for all  $j \in J$ , all  $x \in [x^{j-1}, x^j]$  and all  $k \neq i$ :  $x_i \leq \omega_i^E \leq x_k$ . Assumption 1 implies then that  $w_i$  is non-increasing along  $[x^{j-1}, x^j]$  for all  $j \in J$ . But  $x^m = x^*$ , and we have a contradiction.

(ii) Suppose that  $w$  verifies Assumptions 1 and 2 and that  $\omega^E$  is not a strong equilibrium. By the lemma, (ii), there exist  $I$  and  $t_I$  such that, for all  $i \in I$ :  $\omega_i^E > x_i(\omega^E, (0_{\setminus I}, t_I)) \geq 0$ ;  $w_i(x(\omega^E, (0_{\setminus I}, t_I))) > w_i(\omega^E)$ ; and  $t_{ij} > 0$  if and only if  $\omega_j^E < x_j(\omega^E, (0_{\setminus I}, t_I))$ . Let:  $t^* = (0_{\setminus I}, t_I)$ ;  $x^* = x(\omega^E, t^*)$ ;  $J = \{j \in N | t_{ij}^* > 0\}$ . Suppose, without loss of generality, that  $I = \{1, \dots, m\}$ , with  $m < n$  and that  $J = \{m + 1, \dots, m + p\}$ , with  $m + p \leq n$ . Define recursively the following sequence:  $x^0 = \omega^E$ ; given  $(i, j) \in I \times J$ ,  $x^{(i-1)p+j-m} = x^{(i-1)p+j-m-1} + t_{ij+1}^*e_{ij+1}$  if  $j < m + p$ ,  $x^{(i-1)p+j-m} = x^{(i-1)p+j-m-1} + t_{i+1m+1}^*e_{i+1m+1}$  if  $j = m + p$ . Observe that, for all  $(i, j) \in I \times J \setminus \{m + p\}$ , all  $x \in [x^{(i-1)p+j-m-1}, x^{(i-1)p+j-m}]$ :  $x_i < \omega_i^E < x_j$ . And  $x_i^{(i-1)p} = \omega_i^E$  for all  $i$ . Assumption 1 implies then that  $w_i$  is non-increasing along  $[x^{(i-1)p+j-m-1}, x^{(i-1)p+j-m}]$  for all  $(i, j) \in I \times J$ . Assumption 2 implies that  $w_k$  is non-increasing along  $[x^{(i-1)p+j-m-1}, x^{(i-1)p+j-m}]$  for all  $k \in I \setminus \{i\}$  and all  $(i, j) \in I \times J$ . But  $x^{mp} = x^*$ , and we have a contradiction.  $\square$

*4.2. Third existence theorem*

**Assumption 3.** For all  $i$  and all  $x \in S_n$ , there is a neighborhood  $V_i(x)$  of  $x$  in  $\mathbb{R}^n$  such that  $w_i$  is, in  $V_i(x)$ : strictly increasing in  $x_i$ ; and constant in  $x_j$  whenever  $x_j \geq x_i$ .

Assumption 3 says that the agents are locally unsympathetically isolated from the individuals who are at least as rich as themselves. As Assumptions 1 and 2 above, it implies that redistributive transfers, when they exist, flow down the scale of wealth. Assumption 3 implies moreover that the least wealthy agent at  $x$  is locally unsympathetically isolated at  $x$ : the existence of a strong equilibrium of rights will then follow from the additional assumptions of continuity and convexity of distributive preferences (Theorem 1 above). Theorem 3 implies that continuity can be dispensed with.

**Theorem 3.** *Suppose that  $w$  verifies Assumption 3 and that, for all  $(x, x') \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ ,*

$w_i(x) > w_i(x')$  and  $0 < \lambda < 1$  imply  $w_i(\lambda x + (1 - \lambda)x') > w_i(x')$ . Then  $\omega^E$  is a strong equilibrium of rights of  $w$ .

**Proof.** Assumption 3 implies that all agents are locally unsympathetically isolated at  $\omega^E$ . The convexity assumption on distributive preferences implies then that 0 is not strongly blocked in  $(w, \omega^E)$  (proof of Theorem 1: (i)). One concludes with the lemma, (ii).  $\square$

### 4.3. Examples

The examples presented below are Cobb–Douglas social systems, defined as social systems  $w$  such that, for all  $i$ , there exists  $\alpha^i = (\alpha_{i1}, \dots, \alpha_{in}) \in S_n$ , with  $\alpha_{ii} > 0$ , such that:  $w_i(x) = 0$  if  $\alpha_{ij} > 0 \ni x_j$  for some  $j$ ;  $w_i(x) = \prod_{j=1}^n x_j^{\alpha_{ij}}$  otherwise (with the convention  $0^0 = 1$ ). Some of their useful properties are gathered in Theorem 5 of Appendix A.

**Example 1.** Let  $n = 3$  and suppose that  $\alpha_{ij} = \alpha > \alpha_{ii}$  for all  $i$  and all  $j \neq i$ . This social system verifies the assumptions of the first and second existence theorems, except local unsympathetic isolation and Assumption 1 (Theorem 5: (vi) and (vii)). We have

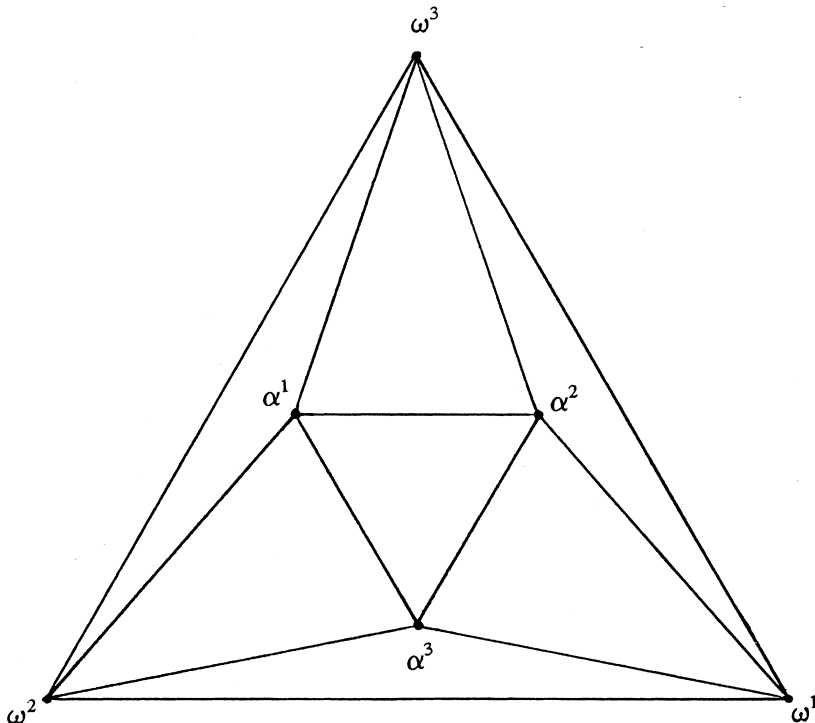
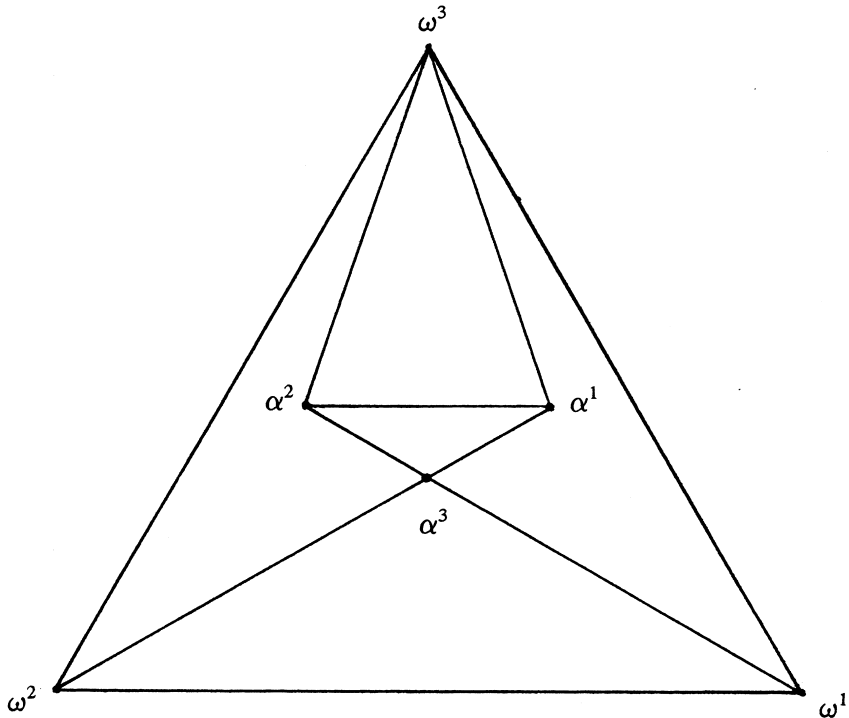


Fig. 1.



$M(w) = \cap_{i \in N} M_{ii}$ , where  $M_{ii} = \text{co}\{\alpha^i, \omega^j, \omega^k\}$ , the convex hull of  $\{\alpha^i, \omega^j, \omega^k\}$ , with  $j \neq i$  and  $k \neq i, j$  (Theorem 5: (ii) and (iii)). This implies  $C(w) = M(w) = \emptyset$  (cf. Fig. 1). This means that there is, at any  $\omega \in S_n$ , an agent who is both willing and able to transfer some of his own wealth to another agent.

**Example 2.** Let  $n=3$ , and suppose that:  $\alpha_{11} = \alpha_{13} = \alpha_{22} = \alpha_{23} > \frac{1}{3}$ ;  $\alpha^3 = \omega^E$ . This social system verifies the assumptions of the first and second existence theorems, except local unsympathetic isolation and Assumption 2 (Theorem 5: (vi) and (viii)).  $M(w) = \{\omega^E\}$  (Theorem 5: (ii) and (iii), and Fig. 2). And  $\partial w(\omega^E)(-\epsilon/2)$



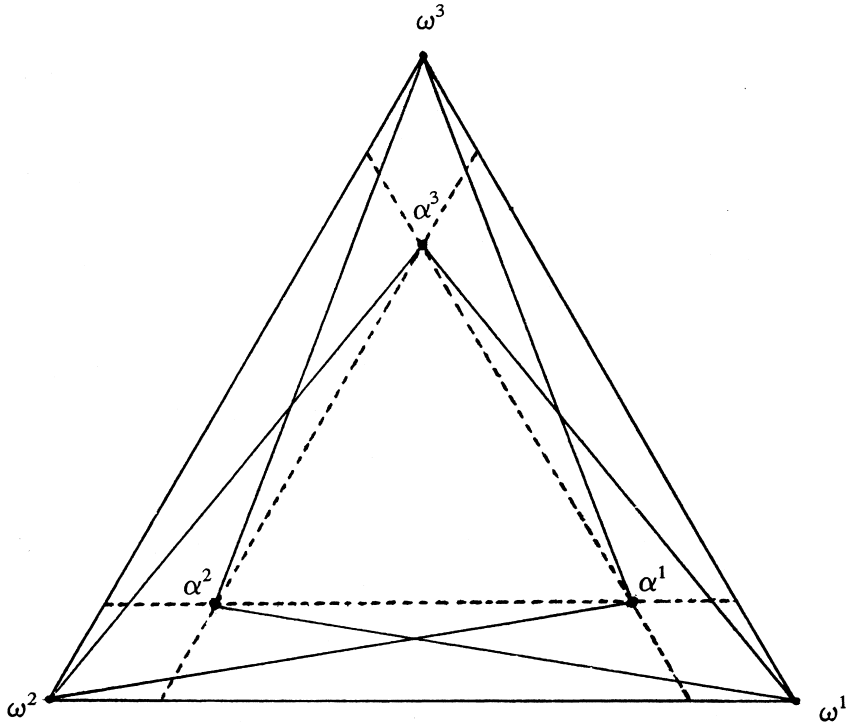


Fig. 3.

equilibrium of gifts, and the necessary and sufficient conditions for the maximization of  $w_i(x(\omega, (t_{\setminus\{i\}}, t_i)))$  with respect to  $t_i$  in  $\{t_i \mid x_i(\omega, (t_{\setminus\{i\}}, t_i)) \geq 0\}$  imply that  $\Omega(x) = \{\omega \in S_n \mid \exists t \text{ such that: } x = x(\omega, t); \text{ and } t_{ij} = 0 \text{ whenever } x \notin M_{ij}\}$ . One verifies in Fig. 3 that:  $\Omega(\alpha^i) = \{\omega \in S_n \mid \omega_j \leq \alpha_j^i \ \forall j \neq i\}$ ;  $\Omega(x) = \{x\}$  for all  $x$  of the relative interior of  $M(w)$  in  $S_n$ ;  $\cup_{x \in M(w)} \Omega(x) = S_n$  (existence of a weak equilibrium of gifts for all initial distribution of rights);  $\Omega(x) \cap \Omega(x') = \emptyset$  for all pair  $(x, x')$  of distinct distributions (unicity of the equilibrium distribution). Because  $P(w) \setminus \{\alpha^1, \alpha^2, \alpha^3\}$  is contained in the relative interior of  $M(w)$  in  $S_n$  (cf. Fig. 3), we must have then that: if  $\omega \in P(w) \cup (\cup_{i \in N} \Omega(\alpha^i))$ , the weak and strong equilibria of gifts coincide, and therefore a strong equilibrium exists; if  $\omega \notin P(w) \cup (\cup_{i \in N} \Omega(\alpha^i))$ , the unique Nash equilibrium distribution is Pareto-inefficient, so that there is then no strong equilibrium of gifts. The non-existence of a strong equilibrium of gifts for some initial distributions of rights is here a consequence of a general feature of this type of construct: the Pareto-inefficiency of some Nash equilibria. The recognition of this feature is an interesting contribution of these models to the explanation of collective voluntary transfers [1,6]. The search for conditions on utility functions that would imply the existence of a strong equilibrium of gifts for all initial distributions of rights therefore appears pointless.

## 5. Conclusion

We studied the existence of an equilibrium of property rights, or equivalently the existence of an equilibrium of gifts for one initial distribution of rights at least. The first existence theorem states that a strong equilibrium of rights exists when distributive preferences are continuous and verify a relevant convexity assumption, and when there is, at any distribution of rights, at least one agent in local unsympathetic isolation. The second existence theorem states that the equality of property rights is: a weak equilibrium of rights if no agent wants to transfer wealth to richer individuals (Assumption 1); a strong equilibrium of rights if the agents share the common opinion that transfers, when they exist, should flow down the scale of wealth (Assumptions 1 and 2). The third existence theorem states that the equality of property rights is a strong equilibrium of rights if the agents are locally indifferent to individuals at least as rich as themselves, and if distributive preferences verify the convexity assumption.

The second and third existence theorems can be strengthened considerably with mild additional assumptions on utility functions. If utility functions are strictly increasing in own consumption, continuous and quasi-concave, Assumption 1 or Assumption 3 implies the existence of a weak equilibrium of gifts for *all* initial distributions of rights [7] (Corollary 3). If utility functions are continuous, Assumptions 1 and 2 imply the existence of a distributive liberal social contract for *all* initial distributions of wealth [2] (Theorem 1).

The definition of conditions on utility functions that would imply the existence of a strong equilibrium of gifts for *all* initial distribution of rights is, on the contrary, hopeless and in some sense pointless: non-existence follows here from the possible distributive inefficiency of Nash equilibria of gifts (Example 3), a general property of this type of construct, and one of their distinctive contributions to the explanation of collective voluntary transfers.

## Acknowledgements

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## Appendix A

**Theorem 4.** *If  $w$  verifies Assumptions 1 and 2, then  $P^*(w) \subset C(w)$ .*

**Proof.** Suppose that  $\omega \notin C(w)$ , and let us prove that  $\omega \notin P^*(w)$ .

By the lemma, (ii), there exist  $I$  and  $t_I$  such that, for all  $i \in I$ :  $\omega_i > x_i(\omega, (0_{\setminus I}, t_I)) \geq 0$ ;  $w_i(x(\omega, (0_{\setminus I}, t_I))) > w_i(\omega)$ ; and  $t_{ij} > 0$  if and only if  $\omega_j < x_j(\omega, (0_{\setminus I}, t_I))$ . Let:  $t^* = (0_{\setminus I}, t_I)$ ;  $x^* = x(\omega, t^*)$ ;  $J = \{j \in N \mid t_{ij}^* > 0\}$ . Suppose, without loss of generality, that  $I = \{1, \dots, m\}$ , with  $m < n$  and that  $J = \{m + 1, \dots, m + p\}$ , with  $m + p \leq n$ .

Suppose that  $x_i^* < x_j^*$  for an  $(i, j) \in I \times J$ . Assumption 1 implies then that  $w_i$  is

non-decreasing along  $[x^*, x^* + \frac{1}{2}(x_j^* - x_i^*)e_{ji}]$ . Assumption 2 implies that  $w_k$  is non-decreasing along  $[x^*, x^* + \frac{1}{2}(x_j^* - x_i^*)e_{ji}]$  for all  $k \in I \setminus \{i\}$ . We can assume, therefore, without loss of generality, that  $x_i^* \geq x_j^*$  for all  $(i, j) \in I \times J$ .

Define recursively the following sequence:  $x^0 = \omega$ ; given  $(i, j) \in I \times J$ ,  $x^{(i-1)p+j-m} = x^{(i-1)p+j-m-1} + t_{ij+1}^* e_{ij+1}$  if  $j < m + p$ ,  $x^{(i-1)p+j-m} = x^{(i-1)p+j-m-1} + t_{i+1m+1}^* e_{i+1m+1}$  if  $j = m + p$ . Observe that, for all  $(i, j) \in I \times J \setminus \{m + p\}$ , all  $x \in [x^{(i-1)p+j-m-1}, x^{(i-1)p+j-m}]$ :  $x_i > x_j$ . And  $x_i^{ip} \geq x_j^{ip}$  for all  $(i, j) \in I \times J$ . Assumption 1 implies that  $w_j$  is non-decreasing along  $[x^{(i-1)p+j-m-1}, x^{(i-1)p+j-m}]$  for all  $(i, j) \in I \times J$ . Assumption 2 implies that  $w_k$  is non-decreasing along  $[x^{(i-1)p+j-m-1}, x^{(i-1)p+j-m}]$  for all  $k \in N \setminus \{i, j\}$  and all  $(i, j) \in I \times J$ . But  $x^{mp} = x^*$  by construction, and  $w_i(x^*) > w_i(\omega)$  for all  $i \in I$  by assumption, so that  $\omega \notin P^*(w)$ .  $\square$

**Theorem 5.** *Let  $w$  be a Cobb–Douglas social system. (i) The maximum of  $w_i$  in  $S_n$  is  $\alpha^i$  for all  $i$ . (ii) Set  $M_{ij}(w) = \{\omega \in S_n \mid \text{either } \omega_j = 0, \text{ or } w_i \text{ is differentiable at } \omega \text{ and } \partial_{x_j} w_i(\omega) \geq \partial_{x_k} w_i(\omega) \text{ for all } k\}$  is the convex hull of family  $\{\alpha^i\} \cup \{\omega^k\}_{k \neq j}$ . (iii)  $M(w) = \bigcap_{i \in N} M_{ii}(w)$ . (iv)  $P(w)$  is the convex hull of family  $\{\alpha^i\}_{i \in N}$ . (v) If  $\alpha_{ij} > 0$  for all  $(i, j)$ , then  $P(w) = P^*(w)$ . (vi)  $w$  verifies local unsympathetic isolation if and only if  $\alpha_{ii} = 1$  for at least one  $i$ . (vii)  $w$  verifies Assumption 1 if and only if  $\alpha_{ii} \geq \alpha_{ij}$ , for all  $(i, j)$ . (viii)  $w$  verifies Assumption 2 if and only if  $\alpha_{ij} = \alpha_{ik}$  for all  $i$ , all  $j \neq i$  and all  $k \neq i, j$ . (ix) If  $\alpha_{ii} \geq \alpha_{ij} = \alpha_{ik} > 0$  for all  $i$ , all  $j \neq i$  and all  $k \neq i, j$ , then  $C(w) = P(w)$ .*

The proof of this theorem is available from the author upon request.

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