Regular Distributive Efficiency and the Distributive Liberal Social Contract

JEAN MERCIER YTHIER
Université de Metz, BETA, Institut d'Economie Publique

Abstract

We consider abstract social systems of private property, made of n individuals endowed with nonpaternalistic interdependent preferences, who interact through exchanges on competitive markets and Pareto-improving lump-sum transfers. The transfers follow from a distributive liberal social contract defined as a redistribution of initial endowments such that the resulting market equilibrium allocation is both: (i) a distributive optimum (i.e., is Pareto-efficient relative to individual interdependent preferences) and (ii) unanimously weakly preferred to the initial market equilibrium. We elicit minimal conditions for meaningful social contract redistribution in this setup, namely, the weighted sums of individual interdependent utility functions, built from arbitrary positive weights, have suitable properties of nonsatiation and inequality aversion; individuals have diverging views on redistribution, in some suitable sense, at (inclusive) distributive optima; and the initial market equilibrium is not a distributive optimum. We show that the relative interior of the set of social contract allocations is then a simply connected smooth manifold of dimension $n - 1$. We also show that the distributive liberal social contract rules out transfer paradoxes in Arrow–Debreu social systems. We

Jean Mercier Ythier, Université de Metz, BETA, Institut d'Economie Publique (jean.mercier-ythier@wanadoo.fr).

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show, finally, that the liberal social contract yields a norm of collective action for the optimal provision of any pure public good.

1. Introduction

This paper derives the global structure of the set of Pareto-efficient distributions of wealth and its subset of distributive liberal social contracts in abstract social systems made of individuals owners endowed with nonpaternalistic interdependent preferences, who interact by means of competitive market exchange and Pareto-improving lump-sum redistribution.

Wealth distribution is formally analogous to a pure public good in the presence of nonpaternalistic utility interdependence (Kolm 1966, Hochman and Rodgers 1969; and the subsequent literature on Pareto optimal redistribution reviewed in Mercier Ythier 2006, 6.1).

Pareto efficiency admits two distinct definitions in this setup, namely, the Pareto efficiency relative to individuals' preferences concerning their own consumption of market commodities, hereafter called *market efficiency*, and the Pareto efficiency relative to individuals' preferences concerning the whole allocation of resources, called *distributive efficiency*. The two definitions articulate consistently in the sense that the latter implies the former, subject to a mild assumption of nonsatiation of the partial preordering of Pareto associated with individual preference relations over allocation. The second fundamental theorem of welfare economics then applies to distributive Pareto optima; that is, distributive optima are Walrasian equilibria relative to suitable vectors of market prices and individual endowments (Winter 1969, Archibald and Donaldson 1976; and the subsequent literature reviewed in Mercier Ythier 2006, 4.1.2).

These facts open the possibility of consistently articulating market exchange and redistribution within a liberal social contract (Kolm 1985, Mercier Ythier 2009).[1] The latter is characterized later as the subset of distributive Pareto optima that are unanimously weakly preferred to some initial Walrasian equilibrium. This notion provides a norm for optimal redistribution, defined in the ideal conditions of perfect contracting in market exchange and social contract redistribution: from a given Walrasian equilibrium that is not a distributive optimum, Pareto-improving lump-sum transfers are performed on the initial distribution of individual endowments so that the resulting Walrasian equilibrium yields a distributive optimum unanimously preferred to the initial Walrasian equilibrium.

The distributive liberal social contract rules out transfer paradoxes in Arrow–Debreu social systems by construction.

[1]See also Kolm 1987a, 1987b, 1996, 5; 2004, Chapter 3; and Mercier Ythier 1998, 2006, 6.1.

While the focus of this paper is the normative analysis of the redistribution of wealth, it must be noted that the formal setup developed below implies, as a special case, an important special case of the standard model of general equilibrium with pure public goods (e.g., Foley 1970, Conley 1994). This formal equivalence obtains with an assumption of weak separability of individuals' allocation preferences relative to their own private consumption and a suitable reinterpretation of commodities (see footnotes 4 and 18 further; also Mercier Ythier 2006, 3.3.3 and 6.1). The distributive liberal social contract, properly reinterpreted, may therefore provide a norm of collective action not only for optimal redistribution but also, more generally, for the optimal provision of any type of pure public good.

This paper is organized as follows: Sections 2 and 3 set and interpret the general equilibrium framework for the analysis of Pareto optimal redistribution. Section 4 discusses the consequences of the public good characteristics of the distribution of wealth in terms of the price-supportability of distributive optima and associate notion of price equilibrium. Section 5 first sets the regularity conditions for a well-behaved set of liberal social contract solutions to optimal redistribution, then examines examples of degenerate solutions to the same problem, and finally elicits sufficient conditions on individual preferences for regular distributive efficiency. Section 6 defines a notion of social contract equilibrium that yields a determinate liberal social contract solution to optimal redistribution. Section 7 is a brief conclusion. The proofs are collected in the Appendix.

2. Pareto Optimal Redistribution in a General Equilibrium Setup[2]

We consider the following simple society of individual owners, consuming, exchanging, and redistributing commodities.

There are n individuals denoted by an index i running in $N = \{1, \ldots, n\}$, and l goods and services, denoted by an index h running in $L = \{1, \ldots, l\}$. We let $n \geq 2$ and $l \geq 1$ in the sequel; that is, we consider social systems with at least two agents and at least one commodity.

The final destination of goods and services is individual consumption. A consumption of individual i is a vector $(x_{i1}, \ldots, x_{il})$ of quantities of his or her consumption of commodities, denoted by x_i. The entries of x_i are nonnegative by convention, corresponding to demands in the abstract exchange economy outlined below. An allocation is a vector $(x_1, \ldots, x_n)$, denoted by x.

Individuals exchange commodities on a complete system of perfectly competitive markets. There is, consequently, for each commodity h, a unique market price, denoted by p_h, which agents take as given (that is, as

[2]This section is an abridged version of the setup developed in Mercier Ythier 2007, 2009.

independent from their consumption or exchange decisions). We let $p = (p_1, \ldots, p_l)$.

We make the following assumptions on commodity quantities: (i) they are perfectly *divisible*; (ii) the total quantity of each commodity is given once and for all (*exchange economy* with *fixed total resources*) and equal to 1 (the latter being a simple choice of units of measurement of commodities); (iii) an allocation x is attainable if it verifies the aggregate resource constraint of the economy, specified as follows: $\sum_{i \in N} x_{ih} \leq 1$ for all h. (This definition of attainability implies *free disposal*.)

The vector of total initial resources of the economy, that is, the diagonal vector $(1, \ldots, 1)$ of $\mathbb{R}^l$, is denoted by ρ. The set of attainable allocations $\{x \in \mathbb{R}^{nl}_+ : \sum_{i \in N} x_i \leq \rho\}$ is denoted by A.

The society is a *society of private property*. In particular, the total resources of the economy are owned by its individual members. The "initial" ownership or endowment of individual i in commodity h, that is, the quantity of commodity h that individual i owns *before market exchange*, is a nonnegative quantity ω_{ih}. The vector $(\omega_{i1}, \ldots, \omega_{in})$ of i's initial endowments is denoted by ω_i. We have $\sum_{i \in N} \omega_i = \rho$ by assumption. The distribution of initial endowments $(\omega_1, \ldots, \omega_n)$ is denoted by ω.

Individuals have preference preorderings concerning allocation, which are well defined (i.e., reflexive and transitive) and complete. The allocation preferences of every individual i are assumed *separable* in his own consumption, that is, i's preference preordering induces a unique preordering on i's consumption set for all i. We suppose that preferences can be represented by utility functions. In particular, the preferences of individual i concerning his own consumption, as induced by his allocation preferences, are represented by the ("private", or "market") utility function $u_i : \mathbb{R}^l_+ \to \mathbb{R}$, which we will sometimes also call *ophelimity function* by reference to Pareto (1913 and 1916). Further, the product function $(u_1 \circ pr_1, \ldots, u_n \circ pr_n) : (x_1, \ldots, x_n) \to (u_1(x_1), \ldots, u_n(x_n))$, where pr_i denotes the ith canonical projection $(x_1, \ldots, x_n) \to x_i$, is denoted by u. Finally, we suppose that individual allocation preferences verify the following hypothesis of *nonpaternalistic utility interdependence*: For all i, there exists a "social" or "distributive" utility function $w_i : u(\mathbb{R}^{nl}_+) \to \mathbb{R}$, increasing in its ith argument, such that the product function $w_i \circ u : (x_1, \ldots, x_n) \to w_i(u_1(x_1), \ldots, u_n(x_n))$ represents i's allocation preferences. Whenever i's distributive utility is increasing in j's ophelimity, it means that individual i endorses j's consumption preferences within his own allocation preferences ("nonpaternalism").[3] For

[3]Note that nonpaternalistic utility interdependence does not imply *distributive benevolence*, in the sense of individual distributive utilities increasing in some others' ophelimities. It is compatible, in particular, with the *distributive indifference* of an individual i relative to any other individual j, that is, the constancy of i's distributive utility in j's ophelimity in some open subset of domain $u(\mathbb{R}^{nl}_+)$ ("local" distributive indifference of i relative to j) or in the whole of it ("global" indifference). It is compatible, also: with local or global

the sake of clarity, we reserve the terms *individual distributive utility function* for functions of the type w_i and *individual social utility function* for functions of the type $w_i \circ u$. The terms *individual distributive preferences* and *individual social preferences*, on the contrary, are used synonymously and designate individual preference relations concerning allocation, in short, individual allocation preferences.

Individual private utilities are normalized so that $u_i(0) = 0$ for all i. Naturally, this can be done without loss of generality, due to the ordinal character of allocation preferences.

We let w denote the product function $(w_1, \ldots, w_n) : \hat{u} \to (w_1(\hat{u}), \ldots, w_n(\hat{u}))$, defined on $u(\mathbb{R}^{nl}_{+})$.

We use as synonymous the following pairs of properties of the preference preordering and its utility representations: *smooth* (C^r, with $r \geq 1$) preordering, and smooth (C^r) utility representations; *monotone* (respectively, strictly monotone, respectively, differentiably strictly monotone) preordering and *increasing* (respectively, strictly increasing, respectively, differentiably strictly increasing) utility representations; *convex* (respectively, strictly convex, respectively differentiably strictly convex) preordering and *quasi-concave* (respectively, strictly quasi-concave, respectively, differentiably strictly quasi-concave) utility representations. Their definitions are recalled for the sole utility representations, in footnote 6.

A social system is a list (w, u, ρ) of social and private utility functions of individuals and aggregate initial resources in consumption commodities. A social system of private property is a list (w, u, ω), that is, a social system where the total resources of society are owned by individuals and distributed between them according to distribution ω.[4]

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distributive malevolence, in the sense of individual distributive utilities decreasing in some others' ophelimities; and, naturally, with any possible combination of local benevolence, indifference or malevolence of any individual relative to any other.

[4]This formal definition of the social system overlaps with an important special case of the standard model of general equilibrium with pure public goods. Partition the set N of individuals into two subsets: the "rich" $\{1, \ldots, m\}$ and the "poor" $\{m+1, \ldots, n\}$, with $0 < m < n$. Suppose that any rich individual is indifferent to the other rich and altruistic toward the poor, that is, $w_i(u(x)) = \mu_i(u_i(x_i), u_{m+1}(x_{m+1}), \ldots, u_n(x_n))$ with a strictly increasing μ_i for all $i \leq m$; the poor are egoistic, that is, $w_i(u(x)) = u_i(x_i)$ for all $i > m$; and the poor have null initial endowments, that is, $\omega_i = 0$ for all $i > m$. Reinterpret, next, the private welfare of poor i as any generic pure public good of type i, private utility function u_i as the production function of public good i, and private consumption x_i as a vector of inputs of "private" commodities $h \in \{1, \ldots, l\}$. We end up with the standard setup for a general equilibrium with public goods produced from private commodities, only distinguished from the most general version of the latter by the assumption, embodied in the specification of individual social utility functions, that preference relations are weakly separable with respect to individual consumption of private commodities. Note that this separability assumption is trivially verified when the private commodity is unique ($l = 1$, the case considered in Mercier Ythier 2006, 3.3.3 and 6, and in Conley 1994).

Finally, the "grand coalition" N can redistribute initial endowments, that is, perform lump sum transfers transforming some "initial" (i.e., pre-transfer) distribution of initial endowments ω into another distribution of initial endowments ω' (with $\sum_{i \in N} \omega'_i = \sum_{i \in N} \omega_i = \rho$).

We now introduce the formal definitions of a *competitive market equilibrium* (Definition 1), and a *distributive liberal social contract* (Definition 4). These notions are complemented by the two concepts of Pareto efficiency naturally associated with them, that is, respectively, the Pareto-efficiency relative to individual private utilities (in short, *market efficiency* or *market optimum,* Definition 2) and the Pareto-efficiency relative to individual social utilities (in short, *distributive efficiency* or *distributive optimum,* Definition 3).

DEFINITION 1: *A pair (p, x) such that $p \geq 0$ is a competitive market equilibrium (also called Walrasian equilibrium) with free disposal of the social system of private property (w, u, ω) if (i) x is attainable, and (ii) $p_h(1 - \sum_{i \in N} x_{ih}) = 0$ for all h; (iii) and x_i maximizes u_i in $\{z_i \in \mathbb{R}^l_+ : \sum_{h \in L} p_h z_{ih} \leq \sum_{h \in L} p_h \omega_{ih}\}$ for all i.*

DEFINITION 2: *An allocation x is a strong (respectively weak) market optimum of the social system (w, u, ρ) if it is attainable and if there exists no attainable allocation x' such that $u_i(x'_i) \geq u_i(x_i)$ for all i, with a strict inequality for at least one i (respectively, $u_i(x'_i) > u_i(x_i)$ for all i). The set of weak (respectively strong) market optima of (w, u, ρ) is denoted by P_u (respectively, $P^*_u \subset P_u$).*

DEFINITION 3: *An allocation x is a strong (respectively weak) distributive optimum of the social system (w, u, ρ) if it is attainable and if there exists no attainable allocation x' such that $w_i(u(x')) \geq w_i(u(x))$ for all i, with a strict inequality for at least one i (respectively $w_i(u(x')) > w_i(u(x))$ for all i). The set of weak (respectively, strong) distributive optima of (w, u, ρ) is denoted by P_w (respectively, $P^*_w \subset P_w$).*

DEFINITION 4: *Let (p, x) be a competitive market equilibrium with free disposal of the social system of private property (w, u, ω). Pair $(\omega', (p', x'))$ is a distributive liberal social contract of (w, u, ω) relative to market equilibrium (p, x) if (p', x') is a competitive market equilibrium with free disposal of (w, u, ω') such that (i) x' is a strong distributive optimum of (w, u, ρ), and (ii) $w_i(u(x')) \geq w_i(u(x))$ for all i.*

For the sake of brevity, the competitive market equilibrium with free disposal of Definition 1 will often be referred to as *Walrasian equilibrium* or even simply as *market equilibrium.* Likewise, we will often refer to the distributive liberal social contract simply as the *social contract.*

Whenever a pair $(\omega', (p', x'))$ is a distributive liberal social contract of (w, u, ω) relative to market equilibrium (p, x), we also refer to ω' as a *distributive liberal social contract* of (w, u, ω) relative to (p, x) and to x' as a *distributive liberal social contract solution* of (w, u, ω) relative to (p, x).

Finally, we introduce two assumptions that will be maintained throughout the main propositions.

Assumption 1 summarizes the working hypotheses of differentiability and convexity. Its contents, notably the second part, relative to distributive preferences are discussed in detail in Mercier Ythier 2009, 3. The definitions of corresponding standard properties of utility functions, such as differentiability, quasi-concavity, strict quasi-concavity, and others, are recalled in the associated footnote, with brief comments on their relations and on some of their elementary consequences.[5]

ASSUMPTION 1[6]: *Differentiable convex social system: (i) For all i, u_i is (a) continuous, increasing, and unbounded above; (b)* C^2 in $\mathbb{R}^l_{++}$; *(c) differentiably*

[5]We use the following standard notations. Let $z = (z_1, \ldots, z_m)$ and $z' = (z'_1, \ldots, z'_m) \in \mathbb{R}^m$, $m \geq 1$: $z \geq z'$ if $z_i \geq z'_i$ for any i; $z > z'$ if $z \geq z'$ and $z \neq z'$; $z \gg z'$ if $z_i > z'_i$ for any i; $z.z'$ is the inner product $\sum_{i=1}^m z_i z'_i$; z^T is the transpose (column-) vector of z; $\mathbb{R}^m_+ = \{z \in \mathbb{R}^m : z \geq 0\}$; ${}^m_{++} = \{z \in \mathbb{R}^m_+ : z\mathbb{R}0\}$. Function $f = (f_1, \ldots, f_q) : V \to \mathbb{R}^q$, which is defined on an open set $V \subset \mathbb{R}^m$, denotes the Cartesian product of the C^2 real-valued functions $f_i : V \to \mathbb{R}$; ∂f and $\partial^2 f$ denote its first and second derivative, respectively; $\partial f(x)$, viewed in matrix form, is the $q \times m$ (Jacobian) matrix whose generic entry $(\partial f_i/\partial x_j)(x)$, also denoted by $\partial_j f_i(x)$ (or sometimes $\partial_{x_j} f_i(x)$), is the first partial derivative of f_i with respect to its jth argument at x; the transpose $[\partial f_i(x)]^T$ of the ith row of $\partial f(x)$ is the gradient vector of f_i at x; finally, $\partial^2 f_i(x)$, viewed in matrix form, is the $m \times m$ (Hessian) matrix whose generic entries $(\partial^2 f_i/\partial x_j \partial x_k)(x)$, also denoted by $\partial^2_{jk} f_i(x)$, are the second partial derivatives of f_i at x.

[6]Recall that u_i is defined on $\mathbb{R}^l_+$, the nonnegative orthant of $\mathbb{R}^l$. We say that such a function is *increasing* (respectively strictly increasing) if $x_i \gg x'_i$ (respectively $x_i > x'_i$) implies $u_i(x_i) > u_i(x'_i)$. It is *quasi-concave* if $u_i(x_i) \geq u_i(x'_i)$ implies $u_i(\alpha x_i + (1-\alpha)x'_i) \geq u_i(x'_i)$ for any $1 \geq \alpha \geq 0$; *strictly quasi-concave* if $u_i(x_i) \geq u_i(x'_i)$, $x_i \neq x'_i$ implies $u_i(\alpha x_i + (1-\alpha)x'_i) > u_i(x'_i)$ for any $1 > \alpha > 0$; *differentiably strictly quasi-concave* in an open, convex set $V \subset \mathbb{R}^l_{++}$ if its restriction to V is C^2 (that is, twice differentiable with continuous second derivatives), strictly quasi-concave, and has a nonzero Gaussian curvature everywhere in V (or equivalently a nonzero determinant of the bordered Hessian $\begin{pmatrix} \partial^2 u_i(x_i) & [\partial u_i(x_i)]^T \\ \partial u_i(x_i) & 0 \end{pmatrix}$ for every x_i in V); *differentiably strictly concave* in an open, convex set $V \subset \mathbb{R}^l_{++}$ if its restriction to V is such that the Hessian matrix $\partial^2 u_i(x_i)$ is negative definite for every x_i in V. Note that the differentiable strict quasi-concavity of u_i in $\mathbb{R}^l_{++}$ implies the existence of a differentiably strictly concave C^2 utility representation of the underlying preference preordering on any compact, convex subset of $\mathbb{R}^l_{++}$ (Mas-Colell 1985, 2.6.4) so that the second part of assumption 1-(i)-(c) does not imply any additional restriction, relative to the first part of the same assumption. Note also that an increasing u_i, which also is differentiably strictly quasi-concave in $\mathbb{R}^l_{++}$, must be *differentiably strictly increasing* in $\mathbb{R}^l_{++}$, that is, such that $\partial u_i(x_i) \gg 0$ everywhere in $\mathbb{R}^l_{++}$ (hence strictly increasing in $\mathbb{R}^l_{++}$). And note, finally, that in the special case of a single market commodity (that is, $l = 1$), we can let $u_i(x_i) = \mathrm{Log}(1 + x_i)$ without loss of generality (as "C^2 differentiable strictly quasi-concave" degenerates, in this simple case, to "C^2 strictly increasing").

Suppose, next, that utility representation u_i is bounded above and verifies all other Assumptions 1-(i). Let $\sup u_i(\mathbb{R}^l_+) = b > a > u_i(\rho)$. Note that $a \in u_i(\mathbb{R}^l_+) = [0, b)$, since u_i is continuous and increasing. Define $\xi : [0, b) \to \mathbb{R}_+$ by: $\xi(t) = t$ if $t \in [0, a)$ and $\xi(t) = t + (t-a)^3 \exp(1/(b-t))$ if $t \in [a, b)$. One verifies by simple calculations that ξ is strictly increasing and that $\xi \circ u_i$ is C^2, unbounded above, and therefore represents the same preordering as u_i and verifies Assumption 1-(i). That is, there is no loss of generality in supposing u_i unbounded above. Assumption 1-(i) notably implies that $u : \mathbb{R}^{nl}_+ \to \mathbb{R}^n_+$ is onto (since u_i is a continuous, increasing, unbounded above function $\mathbb{R}^l_+ \to [0, \infty)$ for all i), so that the domain $u(\mathbb{R}^{nl}_+)$ of individual distributive utility functions coincides with *strictly quasi-concave in* $\mathbb{R}^l_{++}$*, and, in particular, differentiably strictly concave in an open, convex neighbourhood of* $\{x_i \in \mathbb{R}^l_{++} : x_i \leq \rho\}$ *in* $\mathbb{R}^l_{++}$*; (d) such that* $x_i \gg 0$ *whenever* $u_i(x_i) > 0 (= u_i(0))$*. (ii) For all* i*,* w_i *is (a) increasing in its* i*th argument and continuous; (b)* C^2 *in* $\mathbb{R}^l_{++}$*; (c) quasi-concave; (d) such that* $w_i(\hat{u}) > w_i(0)$ *if and only if* $\hat{u} >> 0$*. (iii) For all* i*,* $w_i \circ u$ *is quasi-concave.*

The second assumption is the *differentiable nonsatiation of the weak distributive preordering of Pareto*, which supposes, essentially, that distributive malevolence, if any, is not so intense and/or widespread as to imply the depletion of aggregate resources at distributive optimum. Combined with Assumption 1, it implies the positive aggregate valuation of the private wealth and welfare of all individuals at distributive optimum (Appendix: Theorem 3). That is, we suppose that malevolence, if any, is dominated at social optimum by positive self-valuation, possibly combined with distributive benevolence (if any).

ASSUMPTION 2: *Differentiable nonsatiation of the weak distributive preordering of Pareto: For all* $\mu \in S_n$ *and all* $\hat{u} \in u(A) \cap \mathbb{R}^n_{++}$, $\sum_{i \in N} \mu_i \partial w_i(\hat{u}) \neq 0$.

3. Selfishness in the Marketplace, Altruism in the Society

In this section, we briefly develop an interpretation of some of the key features of the formal model of Section 2 in terms of their implications for individual's market behavior, social contract redistribution, and their articulation.

The separability of individual allocation preferences relative to one's own consumption means, essentially, that the individual behaviour of demand and supply of market commodities can be appropriately described by a stable Walrasian demand function, that is, a function whose variables (the "determinants" of individual demand and supply) are restricted to market prices and individual wealth (the latter reducing, in the setup above, to the value of individual endowment before or after social contract redistribution) and which is homogeneous of degree 0 in these arguments (i.e., individuals are not subject to "monetary illusions") and additive (i.e., individuals spend their whole budget) and verifies the law of demand (i.e., the Slutsky matrix is symmetric semi-definite negative).[7] Notably, the stability of the demand function means, in this context, that individual market demand behaviour is independent of others' consumption.

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the nonnegative orthant of $\mathbb{R}^n$. The definitions above extend readily to functions w_i and $w_i \circ u$.

[7]Standard microeconomic theory establishes the equivalence of maximizing consumption utility subject to linear budget constraint, minimizing expenditure subject to private welfare objectives, behaving according to a Walrasian demand function, and behaving according to a Hicksian demand function, when individual preferences are increasing and differentiably strictly convex.

Nonpaternalistic interdependent utilities consist of individual preferences over the distribution of private welfare. Private welfare is determined by market prices and private wealth through individual (stable) consumption preferences and associate Walrasian demand or, equivalently, through individual indirect private utility functions (see the precise formulation of these notions in Section 4 later). The nonpaternalistic social preferences of individuals, therefore, induce individual preferences over both market prices and wealth distribution,[8] combining individual distributive utilities with indirect private utilities. Accordingly, the associate distributive optima can be characterized equivalently as feasible allocations undominated with respect to individual social preferences or as price-wealth competitive equilibria (i.e., systems of market prices and wealth distribution supporting market optima) undominated with respect to induced individual preferences over prices and wealth (Mercier Ythier 2009, Theorem 3). Therefore, the lump-sum endowment redistribution of the liberal social contract affects individual conditions of optimization through two channels in this construct: endowment redistribution itself and the changes in equilibrium market prices that it induces. These effects of social contract redistribution involve two types of externalities: the public good externalities generated by the changes in the distribution of wealth, whose extent is determined by the extent of actual distributive concerns in society, and the pecuniary externalities generated by induced changes in equilibrium market prices (if any), which necessarily affect *all* individuals in society. If social contract redistribution, as should normally be expected, actually implies changes in equilibrium market prices,[9] wealth distribution then necessarily has the characteristics of a *general* (pure) public good in this setup, if not directly through individual distributive concerns (as may or may not be the case, depending on the extent of the latter), at least indirectly through induced pecuniary externalities.

The condition of unanimous weak preference of social contract equilibrium allocation over initial equilibrium allocation (Definition 4-(ii)) implies an individual right of veto against any change in initial endowment distribution. This essential feature of the notion of distributive social contract developed here interprets as a social contract foundation for individual rights of private property, understood as individuals' shares in aggregate social resources and subsequent individual right of freely allocating one's own share between the alternative uses of own consumption and market

[8]This implies that we concentrate on the distribution aspects of the general notion of liberal social contract of Kolm 1985. We abstract from alternative considerations, such as the treatment of consumption externalities, which are considered in the general notion (see, e.g., Kolm 2004, p. 67, on the latter subject).

[9]Theoretical exceptions are well known and quite specific, essentially, invariance of aggregate demand to redistribution (Bergstrom and Cornes 1983, Bergstrom and Varian 1985) and, in the case of Walrasian production economies, constant returns to scale in firms' production of market commodities.

exchange.[10] In view of the ubiquitous externalities of social contract redistribution, it implies that sizeable redistribution will take place within the distributive liberal social contract only if it receives a wide altruistic support in society (notwithstanding conceivable oddities and complexities briefly evoked in the next paragraph). Obvious circumstances in which such altruistic unanimous agreement can be reached are the cases of individual starvation or social exclusion from extreme poverty. Parts (i)-(d) and (ii)-(d) of Assumption 1 together imply unanimous strict preference for redistribution in situations where the private wealth or welfare of some individual(s) are null. Precisely, they imply that any allocation where all individuals have a positive private wealth and welfare is unanimously strictly preferred to any allocation where some individual's private wealth is equal to 0.

Logically (if not practically) interesting cases of complex redistribution patterns are the so-called *transfer paradoxes*, where, for example, a "donor" transferring (or depleting) part of her endowment ends up better off in terms of her private welfare and/or a "beneficiary" of transfers ends up worse off relative to this welfare criterion, as in the cases of impoverishing transfers discussed in international trade theory.[11] In such cases, the "true" donors are, of course, those whose private welfare diminishes in transfers. The unanimous agreement condition for liberal social contract redistribution implies that all individuals, and notably "true donors," should end up better off in terms of their individual social welfare following the transfers ("paradoxical" or not). That is, true donors should be compensated for their loss in private welfare by some satisfaction from their distributive preferences. It is logically conceivable (if not psychologically plausible) that part of such compensations be derived from the satisfaction of distributive malevolence, some "true donors" enjoying the loss in private welfare of other true donors whom she or he dislikes. Assumption 2, while compatible with such psychological complexities at the individual level, rules them out as possible driving force of redistribution at social contract level by supposing, essentially, that self-appreciation and altruism together dominate malevolence. That is, social contract redistribution, if any, necessarily proceeds from dominant distributive altruism among true donors in this setup, as asserted in the former paragraph. Notably, it can easily be shown that, under Assumptions 1 and 2, the distributive liberal social contract necessarily reduces to status quo if individuals are nonbenevolent.[12] Naturally, nonbenevolence includes

[10]The opportunity of including redistributive gift-giving, whether individual or collective, in this list of alternative uses of private wealth is open to future research. Presumably, private redistributive gift-giving should be crowded out by unanimous social contract redistribution in the multi-commodity setup, as it is in the single-commodity setup, and under essentially the same conditions (Mercier Ythier 1998, Theorem 1).

[11]See, for example, the brief reviews of the transfer problem in Kanbur 2006, 3.1 and in Mercier Ythier 2006, 4.3.

[12]Sketch of proof: Suppose that a market optimum x is not a distributive optimum, and consider a coalition of malevolent true donors at x. Their distributive utilities are jointly

distributive indifference as a special case; that is, the distributive liberal social contract is the status quo, and therefore, in particular, rules out transfer paradoxes in Arrow–Debreu social systems (see Example 3 in Section 5.2 later).

The distributive liberal social contract, so construed, rationally founds a distributive welfare state by providing two rationales for state intervention in distribution matters: the enforcement of the individual rights of private property constitutionally guaranteed in the social contract, including a ban on the nonbenevolent endowment manipulations involved in the transfer problem, and the solution of the social efficiency issues raised by the public good (including pecuniary) externalities of collective redistribution. The same rational foundations extend to the productive public sector (the productive welfare state, so to speak), through the reinterpretation of transfers and individual motives outlined in footnote 4.

4. The Distribution of Wealth and Welfare as Public Goods

This section draws the consequences of the public good characteristics of the distribution of ophelimity or wealth in terms of the latter's valuation by suitably defined supporting prices at distributive optimum.

We denote by v_i the indirect (private) utility function of an individual i in the following discussion. This function is defined in the usual way, as the function $\mathbb{R}^l_{++} \times \mathbb{R}_+ \to \mathbb{R}$, such that $v_i(p, r_i) = \max\{u_i(x_i) : x_i \geq 0 \text{ and } p.x_i \leq r_i\}$ for any price-wealth vector $(p, r_i) \in \mathbb{R}^l_{++} \times \mathbb{R}_+$. Under Assumption 1-(i), indirect private utility functions are >0 and C^1 over $\mathbb{R}^l_{++} \times \mathbb{R}_{++}$; well defined and continuous over $\mathbb{R}^l_{++} \times \mathbb{R}_+$, with $v_i(p, 0) = 0$ for all $p \gg 0$; strictly increasing with respect to wealth; and positively homogeneous of degree 0. We let the distribution of money wealth $(r_1, \ldots, r_n)$ be denoted by r and the product function $(p, r) \to (v_1(p, r_1), \ldots, v_n(p, r_n))$ be denoted by v.

We first recall the definition of *market price equilibrium* and then proceed to the construction, on an analogous pattern, of a notion of *social contract price equilibrium.*

DEFINITION 5: *Attainable allocation x is a market price equilibrium with free disposal of* (w, u, ρ) *if there exists a vector of market prices* $p \geq 0$ *such that* $p.(\rho - \sum_{i \in N} x_i) = 0$ *and* x_i *maximizes* u_i *in* $\{z_i \in \mathbb{R}^l_+ : p.z_i \leq p.x_i\}$ *for all i.*

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decreasing in their private welfare at x by assumption. They are jointly non-increasing in others' private welfare in some neighborhood of x by nonbenevolence. They are jointly increasing in the private welfare of all in some neighborhood of 0 by Assumption 1. There must exist, therefore, a point of satiation of the associate partial preordering at some allocation of segment $[0, x]$, which contradicts Assumption 2.

Under Assumption 1-(i), market price equilibrium is equivalent to market optimum, as a consequence of the first and second theorems of welfare economics.

The Theorem 2 of Mercier Ythier 2009, reproduced in Theorem 3 of the Appendix below, states that, under Assumptions 1 and 2, the weak distributive optima of (w, u, ρ) can be identified with the maxima of $\sum_{i \in N} \mu_i (w_i \circ u)$ in the set of attainable allocations $A = \{x \in \mathbb{R}^{nl}_{+} : \sum_{i \in N} x_i \leq \rho\}$, the vector of weights μ running over the unit-simplex S_n. This fact yields the following definition of a *supported distributive optimum*:

Definition 6: *A weak distributive optimum x of (w, u, ρ) is supported by vector $\mu \neq 0$ of $\mathbb{R}^n_+$ if x maximizes $\sum_{i \in N} \mu_i (w_i \circ u)$ in the set of attainable allocations of the social system.*

The maxima of the "social–social" welfare functions $\sum_{i \in N} \mu_i (w_i \circ u)$ with strictly positive weights are of special interest from a normative perspective because they take into account, to some extent at least, the distributive preferences of *all* individuals. For this reason, we label them *inclusive* distributive optima below, defined formally as follows:

Definition 7: *A weak distributive optimum is inclusive if it is supported by a $\gg 0$ vector μ.*

Supported distributive optima are identical to weak distributive optima by Theorem 3. The set of inclusive distributive optima is contained in the set of strong distributive optima as an immediate consequence of definitions. The latter inclusion is proper in general. (See the remark following Theorem 2 in Section 5 later.) We denote by P^{**}_w the set of inclusive distributive optima. We, therefore, have $P^{**}_w \subset P^{*}_w \subset P_w$, with generally proper inclusions.

Any weak distributive optimum is supported by a strictly positive vector of market prices. A pair $(\mu, p) \in \mathbb{R}^n_+ \times \mathbb{R}^l_{++}$ (with $\mu \neq 0$) supporting any weak distributive optimum x is defined up to a positive multiplicative constant by the first-order conditions of Theorem 3-(ii) and, therefore, can be chosen so that either $\mu \in S_n$ or $p \in S_l$ (but not both, except by coincidence). Note that, in general, μ need not be unique for a given p while p necessarily is unique for any given μ. If $\mu_i > 0$, the term $\mu_i \partial_j w_i(u(x)) \partial_{r_j} v_j(p, p.x_j)$ of the first-order conditions interprets as the marginal valuation, by individual i, of individual j's wealth. The sum $\sum_{i \in N} \mu_i \partial_j w_i(u(x)) \partial_{r_j} v_j(p, p.x_j)$ is the "social–social" marginal valuation of j's wealth at the distributive optimum. It is constant $(=1)$ over j. The distinction of an "individual–social" and a "social–social" marginal valuation of individual wealth is a consequence of the public good character of wealth distribution in this setup. The f.o.c. $\sum_{i \in N} \mu_i \partial_j w_i(u(x)) \partial_{r_j} v_j(p, p.x_j) = 1$ derived in Theorem 3 corresponds, in particular, to the Bowen–Lindahl–Samuelson condition for the optimal provision of "public good" r_j.[13]

"Social–social" marginal valuations of an individual's ophelimities are well defined at any weak distributive optimum while a complete system of individual marginal valuations of his and others' ophelimities is well defined only for inclusive distributive optima (because the definition of a meaningful system of marginal valuations of any individual i supposes a positive supporting μ_i). These facts and the normative reason for a special consideration of inclusive distributive optima justify the introduction of the two additional notions below, which emphasize the inclusive outcomes of social contract redistribution.

Let π_{ij} denote i's marginal valuation of j's wealth, corresponding, according to the former paragraph, to a term of the type $\mu_i \partial_j w_i(u(x)) \partial_{r_j} v_j(p, p.x_j)$. This corresponds to i's *Lindahl price* of j's wealth, in a scheme of Lindahl pricing of wealth distribution as a public good. Note that π_{ii} necessarily is positive at inclusive distributive optimum under Assumption 1, but that π_{ij} could be negative (respectively, $= 0$) for a pair of distinct individuals i and j, if (and only if) i is malevolent (respectively indifferent) to j at this optimum, that is, if $\partial_j w_i(u(x)) < 0$ (respectively, $= 0$). We let $\pi_i = (\pi_{i1}, \ldots, \pi_{in})$ and $\pi = (\pi_1, \ldots, \pi_n)$ in the subsequent discussion. We then define an *inclusive distributive liberal social contract* and a *social contract price equilibrium* as follows:

DEFINITION 8: *Pair $(\omega', (p', x'))$ is an inclusive distributive liberal social contract of (w, u, ω), relative to competitive market equilibrium with free disposal (p, x) of (w, u, ω), if (p', x') is a competitive market equilibrium with free disposal of (w, u, ω') such that (i) $w(u(x')) \geq w(u(x))$ and (ii) x' is an inclusive distributive optimum of (w, u, ρ).*

DEFINITION 9: *Market price equilibrium x' of (w, u, ρ) is a social contract price equilibrium of (w, u, ω), relative to competitive market equilibrium with free disposal (p, x) of (w, u, ω), if (i) $w(u(x')) \geq w(u(x))$ and (ii) there exists (p', π) such that (a) p' supports x'; (b) $\sum_{i \in N} \pi_{ij} = 1$ for all j; and (c) for all i, $r' = (p'.x'_1, \ldots, p'.x'_n)$ maximizes $r \to w_i(v(p', r))$ in $\{r \in \mathbb{R}^n_+ : \pi_i.r \leq \pi_i.r'\}$.*

The next theorem establishes the connections between these last two notions and shows, as a by-product, that the set of ($\gg 0$) social contract price equilibria of a social system of private property, relative to a Walrasian equilibrium x of the latter, is the set of inclusive distributive optima unanimously weakly preferred to x.

[13]The f.o.c. $(\sum_{i \in N} \mu_i \partial_j w_i(u(x))) \partial u_j(x_j) = p$ of Theorem 3-(ii) formally correspond, likewise, to Bowen–Lindahl–Samuelson conditions for "public good" x_j. For a detailed comment of the paradoxes associated with the formal identification of private wealth with a public good, see Mercier Ythier 2006, 6, notably pp. 296–300.

THEOREM 1: *Let* (w, u, ρ) *verify Assumptions 1 and 2, and suppose, moreover, that for all* $p \gg 0$ *and all* $i \in N$*, function* $r \to w_i(v(p, r))$ *is quasi-concave in* $\mathbb{R}^n_{++}$*. The following propositions (i) and (ii) are then equivalent: (i) Allocation* $x^* = \omega^*$ *is a* $\gg 0$ *social contract price equilibrium of* (w, u, ω)*, relative to competitive market equilibrium with free disposal* (p^0, x^0) *of* (w, u, ω)*; (ii) Endowment distribution* $\omega^* = x^*$ *is both (a) an inclusive distributive optimum of* (w, u, ρ) *and (b) an inclusive distributive liberal social contract of* (w, u, ω)*, relative to competitive market equilibrium with free disposal* (p^0, x^0) *of* (w, u, ω)*. In particular, the set of* $\gg 0$ *social contract price equilibria of* (w, u, ω) *relative to* (p^0, x^0) *is equal to* $\{x \in P_w^{**} : w(u(x)) \geq w(u(x^0))\}$.

The assumption that functions $r \to w_i(v(p, r))$ are quasi-concave in $\mathbb{R}^n_{++}$ does not imply significant additional restrictions on individual preferences, relative to the quasi-concavity of distributive utility functions w_i, as established in the following proposition:

PROPOSITION 1: *Suppose that* (w, u) *verifies Assumption 1, and let* $D_{ij}(\hat{u})$ *(respectively,* $D'_{ij}(r)$*) denote the jth principal minor of the bordered Hessian of* w_i *(respectively,* $r \to w_i(v(p, r))$*), evaluated at* $\hat{u} \gg 0$ *(respectively,* $r \gg 0$*). Then,* $D'_{ij}(r) = (\prod_{k \leq j} \partial_{r_k} v_k(p, r_k))^2 D_{ij}(v(p, r))$ *for all* $(p, r) \gg 0$*, and all i and j. In particular, for all i: (i) principal minors* $D'_{ij}(r)$ *verify the necessary condition for the quasi-concavity of* $r \to w_i(v(p, r))$ *in* $\mathbb{R}^n_{++}$*; (ii) and if principal minors* $D_{ij}(\hat{u})$ *verify the sufficient condition for the quasi-concavity of* w_i *in* $\mathbb{R}^n_{++}$*, then* $r \to w_i(v(p, r))$ *is quasi-concave in* $\mathbb{R}^n_{++}$.

Note that, to conclude this section, the concept of social contract price equilibrium introduced above endorses the separation of allocation and distribution as autonomous processes. There is not—and actually there cannot be—in this setup any price system that would simultaneously coordinate the allocation and distribution choices of individuals. The reason for this is quite simple and, indeed, embodied in the basic structure of the construct: for any given endowment distribution, the systems of equilibrium market prices are entirely determined by individual private preferences through the aggregate excess demand function that the latter induce. Symmetrically, the coordination of redistributive transfers by means of Lindahl prices, if any, must be made on the basis of given market prices. We develop an equilibrium concept of this type in Section 6.

5. Global Properties of Regular Distributive Efficiency

This section characterizes the global structure of the sets of inclusive distributive optima and social contract price equilibria, which stems from the characterization of inclusive distributive optima as maxima of positively weighted sums of individual social utilities in the set of attainable allocations. We first elicit, in Section 5.1, the regularity conditions on the system of individual social preferences, ensuring that the sets of inclusive distributive optima and of social contract price equilibria are well behaved in terms of dimension and connectedness. This general property is complemented, in Section 5.2, with the presentation of examples of social systems in which the social contract solution appears degenerate for reasons rooted in their basic structure, that is, in the initial endowment distribution or in the system of individual social preferences. Section 5.3, finally, provides insights on the type of restrictions on individual social preferences required to obtain a well-behaved social contract solution.

5.1. Regular Distributive Efficiency

In this section, we notably concentrate on correspondence $\varphi : S_n \to A$ defined by $\varphi(\mu) = \mathrm{argmax}\{\sum_{i\in N} \mu_i w_i(u(x)) : x \in A\}$. The correspondence is well defined, and its values are contained in P_w, when the social system verifies Assumption 1 and the differentiable nonsatiation of the weak distributive preordering of Pareto (Theorem 3). We summarize some of its elementary properties in the next proposition:

PROPOSITION 2: *Let (w, u, ρ) verify Assumptions 1 and 2. Then, P_w is a nonempty and compact subset of A, and φ is a well-defined, upper hemi-continuous, compact-, and convex-valued correspondence $S_n \to P_w$.*

Correspondence φ will be viewed, consequently, as a correspondence $S_n \to P_w$ from there on. Let $\mathrm{Int}\, S_n$ denote the relative interior of $S_n (= S_n \cap \mathbb{R}^n_{++})$. The restriction of φ to $\mathrm{Int}\, S_n$ appears as a natural candidate for a homeomorphism $\mathrm{Int}\, S_n \to P^{**}_w$, provided notably that $\varphi(\mu)$ and $\varphi^{-1}(x)$ be single valued for all $\mu \in \mathrm{Int}\, S_n$ and all $x \in P^{**}_w$. This need not hold true in general. The following notion of regular distributive efficiency sets minimal sufficient conditions for φ to define such a homeomorphism.

DEFINITION 10: *The differentiable social system (w, u, ρ) is regular with respect to distributive efficiency if (i) $\partial w(u(x))$ is nonsingular for all $x \in P^{**}_w$ and (ii) $\sum_{i\in N} \mu_i (w_i \circ u)$ is differentiably strictly concave at all $x \in \varphi(\mu)$, for all $\mu \in$ $\mathrm{Int}\, S_n$.*

We show in Theorem 2 below that the second regularity condition (differentiable strict concavity) is sufficient for $\varphi(\mu)$ to be single valued for all $\mu \in \mathrm{Int}\, S_n$ and that the first regularity condition of Definition 10 (nonsingularity) is sufficient for $\varphi^{-1}(x)$ to be single valued for all $x \in P^{**}_w$.

The manifold structure of the set of inclusive distributive optima of differentiable social systems and of the set of social contract price equilibria of differentiable social systems of private property then follow from the first regularity condition by means of the Regular Value Theorem.

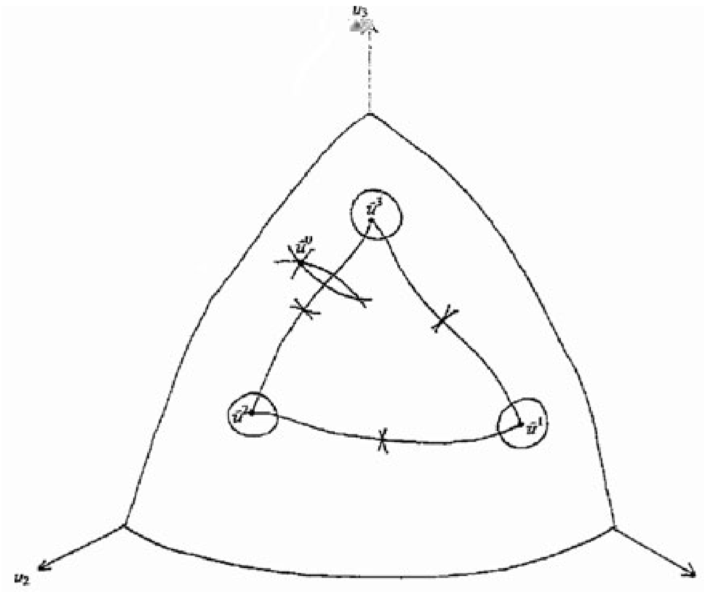

Figure 1: Distributive liberal social contracts in the private utility space.

THEOREM 2: *(i) Let* (w, u, ρ) *verify Assumptions 1 and 2, and suppose that* (w, u, ρ) *is regular with respect to distributive efficiency. Then* P_w^{**} *is a simply connected* C^1 *manifold of dimension* $n-1$*, homeomorphic to* $\operatorname{Int} S_n$*. (ii) Suppose, moreover, that functions* $r \to w_i(v(p, r))$ *are quasi-concave in* $\mathbb{R}^n_{++}$ *for all* $p \gg 0$ *and all* $i \in N$*. Then, for any initial distribution* $\omega \in A$ *and any competitive market equilibrium with free disposal* (p, x) *of* (w, u, ω) *such that* $x \notin P_w$*, the relative interior of the set of social contract price equilibria of* (w, u, ω) *relative to* (p, x) *is a simply connected* C^1 *manifold of dimension* $n-1$*, whose inverse image by* φ *is a simply connected, open subset of* $\operatorname{Int} S_n$*.*

To conclude this first section, note that, as a straightforward consequence of definitions, if $w_i \circ u$ is *strictly* quasi-concave for all i (an assumption that we are not willing to make in general, but that proves useful below for illustrative purposes), then $P_w^* = P_w$. If, moreover, the social system is regular with respect to distributive efficiency, we have $P_w^{**} = \operatorname{Int} P_w$ by Theorem 2, so that, in particular, inclusion $P_w^{**} \subset P_w^*$ is proper in this case (see Proposition 2). Theorem 2 then yields a simple geometric representation of well-behaved social contract solutions for 3-agents social systems, illustrated in Figure 1. The Figure exploits the following consequences of the assumptions of Theorem 2 and the strict quasi-concavity of functions $w_i \circ u$.

From Assumption 1-(i),[14] $u(A)$ is a convex subset of dimension n of $u(\mathbb{R}^{nl}_+) = \mathbb{R}^n_+$, function $x \to u(x)$ is a homeomorphism $P_u \to u(P_u)$ and a C^1 diffeomorphism $\operatorname{Int} P_u \to \operatorname{Int} u(P_u)$, the set of market-efficient ophelimity distributions $u(P_u)\,(= u(P^*_u))$ coincides with the upper frontier of $u(A)$, that is, with set $\{\hat{u} \in \partial u(A) : \hat{u}' > \hat{u} \Rightarrow \hat{u}' \notin \partial u(A)\}$, and its relative interior is a smooth (C^1) hypersurface (that is, $n-1$ dimensional submanifold) of $\mathbb{R}^n$.

These facts and Theorem 2, then, imply that $u(P^{**}_w)$ is a smooth hypersurface of $\mathbb{R}^n$ contained in $\operatorname{Int} u(P_u)$. The same property applies, essentially, to $\operatorname{Int} u(L) = \{\hat{u} \in u(P^{**}_w)) : w(\hat{u}) \gg w(u(x^0))\}$, that is, to the interior of the set of ophelimity distributions of inclusive social contract solutions associated with initial market equilibrium allocation x^0, when the latter is not a distributive optimum: this set is a C^1 hypersurface of $\mathbb{R}^n$ contained in $u(P^{**}_w)$.

Introducing the additional assumption of strict quasi-concavity of functions $w_i \circ u$ yields the following additional properties: the ophelimity distribution that maximizes w_i in P_u is unique and $\operatorname{Int} u(P_w) = u(P^{**}_w)$ (for u is a homeomorphism $P_u \to u(P_u)$, and $\operatorname{Int} P_w = P^{**}_w$ by the strict quasi-concavity assumption).

In Figure 1, we denote by $\hat{u}^i$ the maximum of w_i in P_u and by $\hat{u}^0$ the ophelimity distribution associated with some market equilibrium allocation $x^0 \notin P_w$. From the facts earlier, $u(P_w)$ is the subarea of surface $\operatorname{Int} u(P_u)$, delimited by the continuous curves $\hat{u}^i\hat{u}^j = \operatorname{argmax}\{(w_i(\hat{u}), w_j(\hat{u})) : \hat{u} \in P_u\}$ for all pairs $\{i, j\}$ of distinct individuals of $N = \{1, 2, 3\}$. The set of ophelimity distributions associated with the inclusive distributive optima of the social system is the relative interior of the former surface, that is, surface $u(P_w)\backslash(\hat{u}^1\hat{u}^2 \cup \hat{u}^2\hat{u}^3 \cup \hat{u}^1\hat{u}^3)$. Finally, set $u(L \cap \mathbb{R}^{nl}_{++})$ is the subarea of the former delimited by the indifference curves of w_2 and w_3 through $\hat{u}^0$, and $\operatorname{Int} u(L)$ is its relative interior.

5.2. Examples[15]

The three examples that we develop in this Section exhibit four cases of social systems in which the distributive liberal social contracts, while well defined in the formal sense of Definition 4, nevertheless appear degenerate

[14]The convexity of $u(A)$ is a simple consequence of assumptions 1-(i)-(b) and –(c) and the normalization $u(0) = 0$. Function $x \to u(x)$ is a homeomorphism $P_u \to u(P_u)$ as a consequence of Assumptions 1-(i)-(b) and -(c) (e.g., Mas-Colell 1985, 4.6.2) and a C^1 diffeomorphism $\operatorname{Int} P_u \to \operatorname{Int} u(P_u)$ as consequence of Assumption 1-(i) (Mas-Colell 1985, 4.6.9). Equality $u(P^*_u) = \{\hat{u} \in \partial u(A) : \hat{u}' > \hat{u} \Rightarrow \hat{u}' \notin \partial u(A)\}$ follows from the definition of strong market optimum and the continuity of private preferences (as implied by Assumption 1-(i)-(a)), while equality $u(P_u) = u(P^*_u)$ follows from the strict monotonicity and continuity of private preferences (as implied by Assumptions 1-(i)-(a) and -(c)); its global structure of smooth $n-1$ dimensional manifold follows from Assumption 1-(i) by Mas-Colell 1985, 4.6.9.

[15]This subsection owes much to my lecture notes from Mas-Colell's course on general equilibrium theory at Harvard, notably the part relative to representative consumer theory.

in some important respects. We first briefly summarize their main characteristics and then proceed to the detailed derivation of their salient properties.

The social systems of the first two examples have a representative agent, in the sense that they “behave” as single rational (i.e., preference-maximizing) agents.

In Example 1, all individuals have the same social utility function, although they may differ in their private preferences. These unanimous distributive preferences make a representative agent in the common sense of the notion. They also make a representative agent in the abstract sense above, as its individual optimum is the unique social contract solution, irrespective of the initial distribution. This case of degeneracy stems from a conspicuous violation of the first regularity condition of Definition 10.

In Example 2, we develop two variants of social systems from the same basic Walrasian exchange economy with transferable (quasi-linear) private utility.

The assumption of transferable utility implies the existence of a representative consumer, that is, the invariance of aggregate demand to redistribution.

In the first variant, the social system consists of self-centred utilitarians. Distribution is not a relevant object for the social contract in the sense that, with these assumptions, any market optimum is a distributive optimum. The distributive liberal social contract then translates into the maximization of aggregate wealth on the one hand and the status quo in distribution on the other hand. The social system is ruled, so to speak, according to the views of the representative consumer, which do not coincide with any of the individual views of actual consumers but which, in a literal sense, coincide with their sum. This case of degeneracy involves the violation of the second regularity condition.

In the second variant, the social system is made of a benevolent Sovereign and his egoistic subjects. Individual preferences verify the first and second regularity conditions. The degeneracy of the social contract proceeds from the assumption that the Sovereign has complete control over the numeraire. He implements, consequently, his own optimum, with the effect of precluding the achievement of any inclusive social contract. The representative agent, in this last case, is the Sovereign.

The social system of Example 3 has no representative agent. It is made of unsympathetically isolated individuals, who are concerned only with their own wealth and welfare. It identifies, therefore, with the Walrasian exchange economy that it contains. It verifies all the assumptions of Theorem 2 and nevertheless exhibits, for obvious reasons, the same type of trivial status quo social contracts as the first variant of Example 2 above.

Example 1: Unanimous distributive preferences

Let (w, u, ρ), verifying Assumption 1, be such that all individuals have the same distributive utility function w^*. Distributive utility function w^*, then, is also the unique "social–social" utility function of the social system, that is, $\sum_{i\in N} \mu_i w_i = w^*$ for all $\mu \in S_n$. We suppose, moreover, that w^* is strictly increasing and strictly concave. The social system then verifies all assumptions of Theorem 2, except the first regularity condition that, clearly enough, is violated everywhere in P_w^{**}. Function w^* has a unique maximum in A, which we denote by x^*. One easily verifies that P_w, P_w^* and P_w^{**} then degenerate to the singleton $\{x^*\}$. The latter is also equal to $\varphi(\mu)$ for all $\mu \in S_n$, so that $\varphi^{-1}(x^*) = S_n$. This example, therefore, exhibits a simple (actually, a trivial) case of violation of the properties of Theorem 2 derived from the sole violation of the first regularity condition.

Example 2: Transferable private utility

In this example, it will be convenient to adopt the setup of Balasko 1988; that is, individual private preferences are defined and C^∞ on the whole of $\mathbb{R}^l$, monotone, differentiably strictly convex and bounded from below, and the first commodity is selected as the numeraire (that is, its price is normalized to 1). Walrasian demand and indirect ophelimity functions are then well-defined C^∞ functions on $\{p \in \mathbb{R}^l_{++} : p_1 = 1\} \times \mathbb{R}$, and moreover, we suppose that the restrictions of the latter to $\{p \in \mathbb{R}^l_{++} : p_1 = 1\} \times \mathbb{R}_+$ are of the type $v_i(p, r_i) = r_i + b_i(p)$; that is, we suppose that individuals' private preferences are quasi-linear in the numeraire for nonnegative consumption bundles. In other words, we consider a special case in the general class of exchange economies with transferable utility (Bergstrom and Varian 1985).

Roy's identity and Walras Law readily imply that aggregate demand $\sum_{i\in N} f_i(p, p.\omega_i)$ is invariant to redistribution; that is, $\omega \to \sum_{i\in N} f_i(p, p.\omega_i)$ is constant in the set of nonnegative distributions ω such that $\sum_{i\in N} \omega_i = \rho$. There is, consequently, a unique equilibrium vector of market prices p^* such that $\sum_{i\in N} f_i(p, p.\omega_i) = \rho$ (from Balasko 1988, 3.4.4); that is, this economy has a unique system of equilibrium prices, independent of distribution ω. In addition, aggregate demand $\sum_{i\in N} f_i(p, r_i)$ writes $(r_1 + \cdots + r_n + \sum_{i\in N} \sum_{k\in L, k\geq 2} p_k \partial_{p_k} b_i(p), -\sum_{i\in N} \partial_{p_2} b_i(p), \ldots, -\sum_{i\in N} \partial_{p_n} b_i(p))$, hence is of the general type $G(p, r_1 + \ldots + r_n)$ so that the economy has a representative consumer for nonnegative distributions (Balasko 1988, 7.Ann.3). Finally, the set of market optima associated with nonnegative wealth distributions $(r_1, \ldots, r_n) \in S_n$ reads: $\{(r_1 + \sum_{k\in L, k\geq 2} p_k \partial_{p_k} b_1(p^*), -\partial_p b_1(p^*)),$ $(r_2 + \sum_{k\in L, k\geq 2} p_k \partial_{p_k} b_2(p^*), -\partial_p b_1(p^*)), \ldots, (r_n + \sum_{k\in L, k\geq 2} p_k \partial_{p_k} b_n(p^*),$ $-\partial_p b_1(p^*)) : (r_1, \ldots, r_n) \in S_n\}$, which is identical to S_n up to a simple one-to-one linear transformation. Abusing notations, we denote by P_u the intersection of the latter set with $\mathbb{R}^{nl}_+$, that is, the set of nonnegative market optima.

We now turn to the assumptions on distribution.

In a first variant of the Example, we suppose that all agents are self-centred utilitarians, endowed with linear distributive utility function $w_i : \hat{u} \to \sum_{j \in N} \alpha_{ij} \hat{u}_j$ such that $0 < \alpha_{ij} = \beta < \alpha = \alpha_{ii}$ for all i and all $j \neq i$. Matrix $\partial w(\hat{u})$, then, is constant, positive, symmetric, and has a positive dominant diagonal. The social system verifies the first regularity condition, as $|\partial w(\hat{u})| > 0$ for all $\hat{u}$ by positive diagonal dominance. But it violates the second regularity condition, due to the linearity of $\sum_{i \in N} \mu_i (w_i \circ u)$ in the numeraire. In view of the characterization of the set P_u of nonnegative market optima above, "social-social" utility functions $\sum_{i \in N} \mu_i (w_i \circ u)$ appear essentially as linear functions of the distribution of wealth. In other words, individuals' incomes are perfect substitutes in $\sum_{i \in N} \mu_i (w_i \circ u)$. One easily verifies, in particular, that the set of maxima of $\sum_{i \in N} (1/n)(w_i \circ u)$ in A ($\subset \mathbb{R}^{nl}_+$) is the whole set P_u, as $\sum_{i \in N} (1/n)(w_i \circ u)$ puts the same weight $(1/n)\alpha + ((n-1)/n)\beta$ on all ophelimities. Denoting by P^{**}_w the set of nonnegative market optima, we therefore have $P^{**}_w = P_u$, which contradicts the first property of Theorem 2.

Thus, distribution appears essentially irrelevant as an object of social contract in this social system. The sole basis for unanimous agreement is the concern for market efficiency, that is, to use Marshall's terminology (as this social system exhibits some of the main characteristics of Marshall's static equilibrium), the concern for the maximization of the sum of private surpluses or, equivalently, for the maximization of aggregate wealth (the "wealth of nation"). Moreover, the set of allocations unanimously weakly preferred to any given $x \in P_u$ reduces to $\{x\}$. Therefore, the distributive liberal social contract naturally leads to status quo in this setup, in spite of the existence of distributive concerns in individual preferences.

The second variant of the Example is the macro-social transposition of Becker's theory of family interactions (1974). It is illustrated by Figure 2 for a 3-agents social system. Agent 3 (say, Pharaoh[16]) owns the numeraire (that is, $\omega_{33} = 1$) and has a concave strictly increasing, differentiably strictly concave in $\mathbb{R}^n_{++}$ distributive utility function w_3. All other individuals are egoistic. The determinant of $\partial w(\hat{u})$ reduces to $|\partial w(\hat{u})| = \partial_3 w_3(\hat{u}) \neq 0$. The first regularity condition holds true, therefore, in this social system. The second regularity condition is also verified by Proposition 3 of Section 5.3 below. We denote by x^* the unique maximum of Pharaoh's

[16]From Ramsey to Ramses II, so to speak: Barro's companion paper of Becker's in the 82nd issue of the JPE (1974) develops a macroeconomic analogue of the same model, where the representative agent is a dynastic sequence of altruistically related generations. This construct has often been compared, in subsequent literature on the same topic, with Ramsey's Mathematical Theory of Savings (1928). It seems to me that, besides their undeniable practical virtues in terms of legibility and tractability, these models draw much of their obvious power of seduction from their metaphorical resonance with an archetype, nicely characterized by Karl Polanyi under the label of *redistribution* (and contrasted by him with the market on the one hand and with reciprocity on the other hand: *The Great Transformation*, 1944, Chapter 4; see also Max Weber 1921).

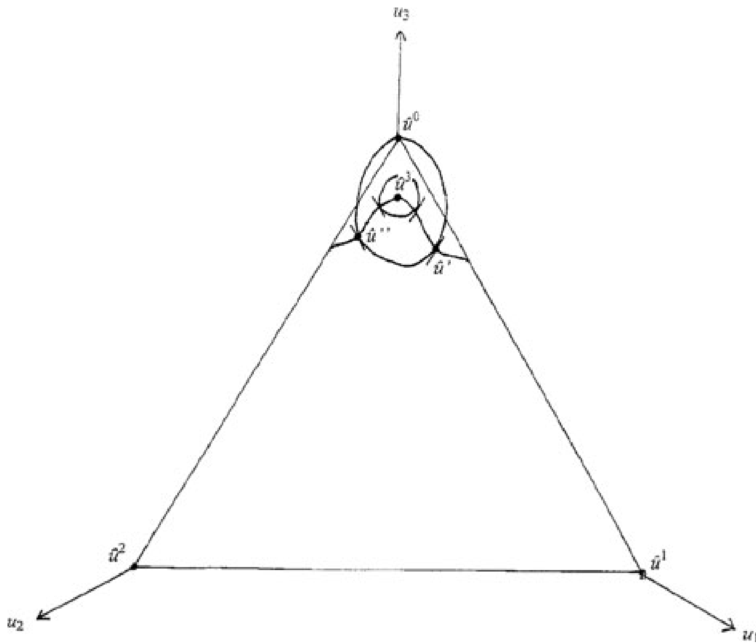

Figure 2: Becker' social system and the distributive liberal social contract.

social utility in the set of feasible allocations and suppose that it is $\gg 0$. If one assumes, for simplicity, that the initial distribution ω is a Walrasian equilibrium, the achievement of Pharaoh's optimum then supposes some redistribution of wealth and numeraire from himself to all others. Therefore, $w(u(x^*)) \gg w(u(\omega))$, and $x \notin P_w$. Because Pharaoh has a complete control over the resources in numeraire, the natural distributive outcome for this social system is allocation x^*. The latter is a distributive optimum unanimously preferred to the initial Walrasian equilibrium. It corresponds, consequently, to a distributive liberal social contract in the formal sense of Definition 4. This social contract is not inclusive, and actually there cannot be any more exclusive social contract, in a formal sense, than this one, as the "social-social" utility function that it maximizes coincides with the sole social utility function of Pharaoh. Figure 2 displays the variant of Figure 1 that corresponds to this configuration of the social system: $u(P_u)$ is represented by an isosceles triangle of base $\sqrt{2}$ obtained from S_3 by means of translation $(z_1, z_2, z_3) \rightarrow (z_1 + b_1(p^*), z_2 + b_2(p^*), z_3 + b_3(p^*))$, $\hat{u}^3 = u(x^*)$, $\hat{u}^0 = u(\omega)$; the curve connecting points $\hat{u}^0$, $\hat{u}'$, and $\hat{u}''$ is Pharaoh's indifference curve through $u(\omega)$; and the set of ophelimity distributions associated with the inclusive social contract solutions such that $w(u(x)) \gg w(u(\omega))$ is, consequently, the interior of surface $\hat{u}'\hat{u}^3\hat{u}''$.

Example 3: Arrow–Debreu social system

Let (w, u, ρ) verify Assumption 1, and suppose that individual distributive preferences are nonmalevolent, such that $w_i = pr_i$ for all i in $\{x \in \mathbb{R}^{nl}_+ : x_i \gg \varepsilon\rho \text{ for all } i\}$, where ε is a > 0 real number that can be taken arbitrarily close to 0. That is, all individuals are indifferent to the private wealth or welfare of others (universal distributive indifference) when all individual consumptions are above some $\gg 0$ threshold close to 0. We interpret this threshold as a survival or social minimum, and accordingly, we let w be such that $P^{**}_w \subset \{z \in P_u : z_i \gg \varepsilon\rho \text{ for all } i\}$. This social system verifies all the assumptions of Theorem 2, and, notably in particular, Assumption 2 (from nonmalevolence and Assumption 1); the first regularity condition, since $\partial w(\hat{u}) = 1_n$ for all $\hat{u} \in u(P^{**}_w)$; and the second regularity condition, for the differentiable strict concavity of all private utility functions implies the differentiable strict concavity of $x \to \sum_{i \in N} \mu_i u_i(x_i)$ for all $\mu \gg 0$ (see Proposition 3 later). The social system (w, u, ρ) then identifies, essentially, with the Walrasian exchange economy (u, ρ) whenever the associate Walrasian equilibria match the norm of the social minimum. In particular, all market optima above the social minimum are distributive optima; that is, $\{x \in P_u : x_i \gg \varepsilon\rho \text{ for all } i\} \subset P_w$, and, of course, the distributive liberal social contract implies status quo at all Walrasian equilibrium meeting the norm; that is, $\{z \in P_w : w(u(z)) \geq w(u(x))\} = \{x\}$ for all $x \in \{z \in P_u : z_i \gg \varepsilon\rho \text{ for all } i\}$ by the strict convexity of private preferences. As is well known, general Walrasian exchange economies, such as characterized by Assumption 1-(i), do not have representative agents in general (Balasko 1988, 7.Ann.3).

5.3. Regular Social Systems

This last Section makes a brief first exploration of the restrictions on admissible social systems required for a well-behaved liberal social contract solution to optimal redistribution. By *social contract solution*, we mean any distributive optimum unanimously weakly preferred to the initial market equilibrium (see the end of Section 2) or also, by extension, the set they constitute.

The social contract solutions are well behaved if, notably, they are inclusive; they are not, or not always, a status quo; and they make a simply connected subset of the set of market optima, of same dimension as the latter (that is, of dimension $n - 1$). We consider each of these characteristics in turn and some of their implications for the underlying social systems.

Inclusiveness is a basic normative requirement, designed to provide a universal foundation to the social contract by ensuring the effective inclusion of all individual preferences in the design of aggregate social utility functions. It notably implies the use of the weak Pareto Principle (the weak distributive preordering of Pareto) for comparing allocations and, consequently, of the strong Pareto optimum for the definition of distributive optimum, but actually demands still more than that (because the inclusion $P_w^{**} \subset P_w^*$ is proper, as noticed in 5.1 earlier).

The variant of Becker's social equilibrium analyzed in Example 2 of Section 5.2 suggests that the implementation of an inclusive social contract might require a sufficiently balanced initial distribution or, at least, may be greatly eased by it. It should not be the case, in other words, that a single agent or a group of agents (say, for example, "the Rich") are able and willing to take advantage of their dominant position at the initial allocation in order to implement their own optimum and so performing a literal interpretation of redistribution as unilateral Charity from benevolent benefactors to passive and silent beneficiaries (see Mercier Ythier 2006, notably 3.3.3 and 6.2, for a discussion of the theoretical literature on charitable donations). Note that such exclusive social contracts are always accessible from any initial market optimum $x \notin P_w$. (Formally, $\partial P_w \cap \{z \in \mathbb{R}_+^{nl} : w(u(z)) \geq w(u(x))\}$ generally is nonempty, as clearly appears in Figure 1.) The statement above, therefore, does not refer so much to the logical possibility or impossibility of exclusive solutions as to the plausibility of the selection of an inclusive outcome and the general characteristics of the social system, which condition the latter. A reasonably balanced initial distribution certainly is a favourable circumstance. A pervasive awareness of the robustness conferred to social contract by universal participation is another, still more important than the former. It seems reasonable to think that the real counterpart of the abstract notion of liberal social contract studied in this paper, if any, supposes both of them and their mutual reinforcement, in its state of maturity at least.

The second condition for a well-behaved social contract is that it explains effective redistribution, that is, that the social contract solution is not or not always the status quo. In a minimal interpretation of this requirement, this supposes that some market optima at least are not distributive optima; that is, formally, that inclusion $P_w \subset P_u$ is proper. The latter supposes in turn that preferences exhibit some taste for redistribution such as, for example, some degree of inequality aversion, at the individual level of course (see the social system of the Homo Economicus of Example 3) but also at the aggregate level (see the Marshallian social system of Example 2). The second regularity condition of Definition 10 essentially supposes the latter, that is, a taste for averaging exhibited by the positively weighted sums of individual social utility functions at associate inclusive distributive optima. We establish below that, for two complementary reasons, this regularity condition does not impose any serious restrictions on *nonmalevolent* individual distributive preferences.

First of all, the set of smooth (C^2), monotone preference preorderings on $\mathbb{R}_+^{nl} \backslash \{0\}$ that are differentiably strictly convex in A is open and dense in the set of smooth monotone distributive preference preorderings on $\mathbb{R}_+^{nl} \backslash \{0\}$, as a consequence of Mas-Colell 1985, 8.4.1, and its elements admit utility representations that are differentiably strictly concave in A, as a consequence of Mas-Colell 1985, 2.6.4. In other words, the strict concavity of utility representations in the set of admissible allocations is a generic property of smooth convex monotone social preferences at the individual level, hence also at the aggregate level.

Nevertheless, the genericity argument above is not completely satisfactory because, first, it is mute on nonmonotone (that is, malevolent) social preferences and, second, it derives the strict concavity of the "social–social" utility function from the strict concavity of individual social utility functions. The latter is not realistic, due to the large-scale character of the object of preferences (inter-individual wealth distribution in the whole society) and the distributive indifference that it seems normally to imply within widespread parts of their domain of definition. Fortunately enough, it can easily be established (see Proposition 3 later) that the *concavity* of individual *distributive* utility functions and *strict concavity* of *private* utility functions in A, which are much easier to defend, suffice for the strict concavity of *positively* weighted sums of individual social utilities in A, provided that individual distributive utility functions are nondecreasing (nonmalevolence) and increasing in their own ophelimities.

The violation of the second regularity condition in the first variant of Example 2, therefore, is not robust, for it appears as a consequence of the joint use of linear distributive utility functions and quasi-linear private utility functions. Robust difficulties with this regularity condition, if any, will stem from distributive malevolence.

PROPOSITION 3: *Suppose that for all I, u_i is strictly concave in $pr_i A$ and w_i is concave in $u(A)$, nondecreasing and increasing in its ith argument. Then, $\sum_{i \in N} \mu_i (w_i \circ u)$ is strictly concave in A for all $\mu \gg 0$.*

The third condition for a well-behaved social contract solution concerns the global structure of the solution set as a simply connected set of dimension $n-1$ (Theorem 2-(ii)). The latter obtains as a simple consequence of the same properties of the set P_w^{**} of inclusive distributive optima (see Step 3 of the proof of Theorem 2).

The simple connectedness of P_w^{**} means, essentially, that this set has no "holes." The set of market optima P_u also is simply connected (Balasko 1988, 3.2 and 3.3). This mathematical property is suggestive of the possibility of performing redistribution along a continuous path of minimal length in P_u or P_w^{**}, by means of continuous adjustments in the distribution of endowments (see Balasko 1988, 3.2, for further developments of this interpretation). It follows from the first and second regularity conditions of Definition 10 (see Step 1 of the proof of Theorem 2).

The dimensional property $\dim P_w^{**} = n-1$ states that the set of inclusive distributive optima has the maximum dimension consistent with inclusion $P_w^{**} \subset P_u$ (since $\dim \operatorname{Int} P_u = n-1$). This corresponds to a property of nondegeneracy in the strict (mathematical) sense. The first regularity condition is the minimal sufficient condition for the latter, as appears clearly from Step 2 of the proof of Theorem 2. This regularity condition supposes, essentially, that individuals have diverging views on desirable redistribution at any inclusive distributive optimum. More formally, the rows of matrix $\partial w(u(x))$ at $x \in P_w^{**}$ are the Jacobian vectors $\partial w_i(u(x))$, pointing in the direction of the best (local) redistributions from $u(x)$ from the perspective of individual i. The first regularity condition, therefore, states, equivalently, that the families of Jacobian vectors $\{\partial w_i(u(x)) : i \in I\}$ have maximal rank for any nonempty $I \subset N$ at any inclusive distributive optimum. Hence, the interpretation above.

The need for this regularity condition is a direct consequence of the public good character of private wealth and welfare distributions in this setup. The condition is automatically verified, for example, and can therefore remain implicit in the social system of the Homo Economicus of Example 3. ($x \to u(x)$ is a homeomorphism $P_u \to u(P_u)$ for monotone strictly convex private preferences, as is well-known. See footnote 14 above.) The very existence of a distributive liberal social contract, if any, supposes a balance between (i) on the one hand, some degree of conformity in individuals' tastes for redistribution, which must be sufficient to imply unanimous agreement relative to some acts of redistribution at least and (ii) on the other hand, divergences in individual views relative to distribution, which must be sufficient to make a contractual solution meaningful, as opposed to the more centralized modes of collective action that would proceed from the exact conformity of individual distributive preferences in large subsets of N (with the social system of Example 1 as a limit case). This balance of the social contract deduces quite naturally from actual characteristics of individual preferences, which commonly balance propensities to redistribute associated with altruistic feelings, empathy, or sense of distributive justice, on the one hand, against care for one's own wealth and welfare on the other hand.

A major, if not unique, source of divergence of individual views on redistribution is self-centredness, which consists of an individual placing a greater importance on his own wealth than on the individual wealth of others. The following Proposition derives, from this simple basic pattern, two assumptions on the system of individual social preferences that imply the first regularity condition, namely, the *distributive indifference to the wealthier*, which supposes that every individual puts, so to speak, a "null weight" on the wealth of any other individual at least as rich as himself at any inclusive distributive optimum, and the *positive diagonal dominance* of the Jacobian matrix of $r \to w(v(p, r))$ at any inclusive distributive optimum. These results should only be viewed as simple indications about a possible line of research for obtaining general characterizations of systems of preferences compatible with the first regularity condition. There appears to be room for substantial improvements on this topic, quite clearly.

PROPOSITION 4: *Let (w, u, ρ) verify Assumption 1, and suppose that, for any weak price-wealth distributive optimum $(p, r) \gg 0$ such that $f(p, r) \in P_w^{**}$, (i) either $\partial_j w_i(v(p, r_i)) = 0$ for all pair of distinct individuals (i, j) such that $r_i \leq r_j$*

or (ii) matrix $\partial w(v(p, r)).\partial_r v(p, r)$ *has a positive dominant diagonal. Then,* (w, u, ρ) *verifies the first regularity condition of Definition 10.*

6. Social Contract Equilibrium

To conclude the formal developments of this paper, we very briefly return to the notion of social contract equilibrium.

The set of social contract solutions addressed here leaves, when it is well behaved, a substantial amount of mathematical indeterminacy relative to distribution, as measured by the dimension ($= n - 1$) of the manifold of price-wealth social contract equilibria or, equivalently, by the dimension of the set of supporting vectors of weights of the associate "social-social" utility functions (Theorem 2-(ii)).[17] A natural solution for removing this remaining indeterminacy in our setup is Lindahl equilibrium, construed as a process of social communication that uses Lindahl pricing to elicit and coordinate individual preferences relative to distribution treated as a public good. Mercier Ythier 2004, defines the notion and analyzes its existence and some of its determinacy properties in the one-commodity case.[18] We extend that analysis to the present setup in Definition 11 below and establish, as a corollary of Theorem 1, that it actually yields an inclusive social contract solution. The associate wealth distribution, moreover, is unanimously strictly preferred to the wealth distribution induced by the initial market equilibrium allocation evaluated at social equilibrium market prices, when the initial market equilibrium allocation is not itself an inclusive distributive optimum. These properties of social equilibrium hold true provided that indirect individual social utility functions $r \to w_i(v(p, r))$ exhibit suitable properties of preference for averages at social equilibrium market prices.

[17]Note that indeterminacy in the sense above does not preclude a substantial explanation power of the notion, as measured by the ratio of the magnitude of hypersurface $u(L)$, computed from the relevant integral, relative to the magnitude of hypersurface $u(P_w)$ or $u(P_u)$ (see Figure 1 and the corresponding remarks, following the proof of Theorem 2). In other words, the set of social contract solutions could represent a very small fraction of the set of Pareto-efficient distributions in the distributive sense and, a fortiori, in the market sense. This might be the case, notably, if the initial market allocation is close to the set of distributive optima or, equivalently, if the value of the transfers of the social contract represents a small fraction of the total value of the equilibrium allocation. This could very well be the case in practice, as genuine redistribution seems to represent only a small fraction of aggregate market wealth in real economies.

[18]This notion of Lindahl equilibrium reduces to the standard notion in the general equilibrium model with public goods of footnote 4 above, when there is a single private commodity. This simple fact is established in Mercier Ythier 2006, Theorem 16-(i). The footnote 70 of the same reference also translates into this setup (general equilibrium with public goods and a single private commodity) Foley's 1970 proof that his notion of core with public goods contains the Lindahl equilibria. Note that the Foley-core necessarily is contained in the set of distributive liberal social contract solutions when the private commodity is unique (Mercier Ythier 2006, footnotes 62 and 69).

We let Π denote set $\{\pi = (\pi_1, \ldots, \pi_n) \in \prod_{i \in N} \mathbb{R}^n : \sum_{i \in N} \pi_{ij} = 1$ *for all* $j\}$.

DEFINITION 11: $(\pi, p^*, x^*) \in \Pi \times S_l \times A$ *is a social contract equilibrium of* (w, u, ω)*, relative to competitive market equilibrium with free disposal* (p^0, x^0) *of* (w, u, ω)*, if (i)*$w(u(x^*)) \geq w(u(x^0))$*; (ii)* x^* *is a market price equilibrium supported by* p^**; (iii) for all i,* $r^* = (p^*.x_1^*, \ldots, p^*.x_n^*)$ *maximizes* $r \to w_i(v(p^*, r))$ *in* $\{r \in \mathbb{R}_+^n : \pi_i.r \leq \pi_i.(p^*.x_1^0, \ldots, p^*.x_n^0)\}$.

The notion differs from the social contract price equilibrium of Definition 9 by maintaining the initial market equilibrium allocation x^0 in the specification of the right-hand side of individual "budget constraints." It shares with the former the fundamental feature of endorsing the separation of allocation and distribution as autonomous processes of coordination of (i) on the one hand, individual decisions relative to market demand, coordinated by market prices for given distribution and (ii) on the other hand, individual choices relative to distribution, coordinated by Lindahl shares for given market prices.

COROLLARY: *Let* (w, u, ω) *verify Assumptions* 1 and 2, *and suppose that, for all* $\mu \in S_n$ *and all* $\hat{u} \in u(A) \cap \mathbb{R}_{++}^n$, $\sum_{i \in N} \mu_i \partial w_i(\hat{u}) \neq 0$ *and, for all* $p \gg 0$ *and all* $i \in N$*, function* $r \to w_i(v(p, r))$ *is quasi-concave in* $\mathbb{R}_{++}^n$*. If* (π, p^*, x^*) *is a social contract equilibrium of* (w, u, ω)*, relative to competitive market equilibrium with free disposal* (p^0, x^0) *of* (w, u, ω) *such that* $x^* \gg 0$*, then endowment distribution* $\omega^* = x^*$ *is both (a) an inclusive distributive optimum of* (w, u, ρ) *and (b) an inclusive distributive liberal social contract of* (w, u, ω)*, relative to competitive market equilibrium with free disposal* (p^0, x^0) *of* (w, u, ω)*. If, moreover,* $x^0 \notin P_w^{**}$ *and* $r \to w_i(v(p^*, r))$ *is strictly quasi-concave for all i, then* $w(u(x^*)) \gg w(v(p^*, (p^*.x_1^0, \ldots, p^*.x_n^0)))$.

In closing, let us pose an interesting open question. We noted above that the liberal social contract ruled out transfer paradoxes in Arrow–Debreu social systems. The transfer problem is intimately associated with the role of income effects in the determination of aggregate demand (Guesnerie and Laffont 1978). It may be said, therefore, that the rational control that the liberal social contract exerts over the distribution of wealth imposes restrictions on income effects that are sufficient to eliminate transfer paradoxes in such social systems. Then the following question arises naturally: Does the rational control of wealth distribution within the liberal social contract of general (well-behaved) social systems, in the indeterminate variant of Definition 4 or in the determinate variant of Definition 11, imply systematic restrictions on the size or direction of income effects, and if this is the case, with what consequences for market functioning (the law of demand, the stability of equilibrium, transfer paradoxes, etc.)? A positive answer would open new perspectives for the study of the relations between allocation, distribution, and the dynamics and regulation of economic equilibrium in a setup richer, if not more tractable, than the models of representative agent that have been developed on this subject in the last 30 years or so (notably, by real business cycle theory).

7. Conclusion

This paper has examined the rational foundation of the distributive (and, by extension, productive) welfare state on the liberal social contract. The latter deduces from the unanimous agreement of individual members of society, as follows from their actual preferences and rights, including their common concerns relative to the distribution of wealth. We notably elicit general conditions over preferences and rights that make the liberal social contract an interesting, nontrivial solution to the public good problem of redistribution. The analysis relies, in the main, on the precise formulation of the integration of (rational) social contract redistribution with (competitive) market equilibrium. It introduces new questions concerning the implications of the rational control of wealth distribution in social contract redistribution for market functioning (especially, the combination of income effects in the determination of aggregate demand) and, consequently, for the interaction of the allocation, distribution, and regulation branches of public finance.

Appendix

A.1. First-Order Conditions for Distributive Efficiency

For the reader's convenience, we reproduce below, as Theorem 3, the characterization of weak distributive optima derived in Mercier Ythier 2009, Theorems 1 and 2 and proofs.

THEOREM 3: *Let (w, u, ρ) verify Assumptions 1 and 2. The following three propositions are then equivalent: (i) x is a weak distributive optimum (w, u, ρ); (ii) x is $\gg 0$, such that $\sum_{i\in N} x_i = \rho$, and there exists $(\mu, p) \in S_n \times \mathbb{R}^l_{++}$ such that, for all $j \in N$, $\sum_{i\in N} \mu_i \partial_j w_i(u(x)) = 1/\partial_{r_j} v_j(p, p.x_j) > 0$ and $(\sum_{i\in N} \mu_i \partial_j w_i(u(x)))\partial u_j(x_j) = p$; (iii) there exists $\mu \in S_n$, such that x maximizes $\sum_{i\in N} \mu_i(w_i \circ u)$ in A.*

A.2. Proofs

Proof of Theorem 1: The last part of Theorem 1 is a simple consequence of the first part and Definition 8. Let us prove the first part, that is, (i)$\Leftrightarrow$(ii).

(i) We first prove that (i)$\Rightarrow$(ii). Let x^* be a $\gg 0$ social contract price equilibrium relative to competitive market equilibrium with free disposal

(p^0, x^0) of (w, u, ω). Then x^* is a market price equilibrium by Definition 9. It is supported by a $\gg 0$ system of market prices p^*, such that $\sum_{i\in N} x_i = \rho$. Since x^* is $\gg 0$, we have $\partial u_i(x_i^*) = \partial_{r_i} v_i(p^*, p^*.x_i^*)p^*$ for all i. Moreover, for all i, there exists $\nu_i \in \mathbb{R}_{++}$ such that $\partial_j w_i(v(p^*, r^*))\partial_{r_j} v_j(p^*, r_j^*) = \nu_i \pi_{ij}$ for all $j \in N$, by the first-order conditions for a $\gg 0$ maximum of $r \to w_i(v(p^*, r))$ in $\{r \in \mathbb{R}^n_{++} : \pi_i.r \leq \pi_i.r^*\}$ (where $r^* = (p^*.x_1^*, \ldots, p^*.x_n^*)$). Dividing both sides of the f.o.c. by ν_i, adding over i for any given j, and using the fact that $\sum_{i\in N} \pi_{ij} = 1$ by Definition 9, one gets the set of Bowen–Lindahl–Samuelson conditions: $\sum_{i\in N} (1/\nu_i)\partial_j w_i(v(p^*, r^*))\partial_{r_j} v_j(p^*, r_j^*) = 1$ for all j. Letting $\mu = (1/\nu_1, \ldots, 1/\nu_n)$ and combining the findings above, we arrive at the following: x^* is $\gg 0$, such that $\sum_{i\in N} x_i^* = \rho$, and there exists $(\mu, p^*) \in \mathbb{R}^n_{++} \times \mathbb{R}^l_{++}$, such that, for all $j \in N$, $\sum_{i\in N} \mu_i \partial_j w_i(u(x^*)) > 0$ and $\sum_{i\in N} \mu_i \partial_j w_i(u(x^*))\partial u_j(x_j^*) = p^*$. The conclusion follows from Theorem 3 with a suitable normalization of μ.

(ii) We now prove the converse (ii)$\Rightarrow$(i). Let endowment distribution ω^* be an inclusive distributive optimum of (w, u, ρ) and an inclusive distributive liberal social contract of (w, u, ω) relative to competitive market equilibrium with free disposal (p^0, x^0) of (w, u, ω). From Theorem 3 and the definition of an inclusive distributive optimum, ω^* is $\gg 0$, such that $\sum_{i\in N} \omega_i^* = \rho$, and there exists a $\mu \in \mathbb{R}^n_{++}$ and a unique $p^* \in S_l$, such that, for all $j \in N$, $\sum_{i\in N} \mu_i \partial_j w_i(u(\omega^*)) > 0$ and $\sum_{i\in N} \mu_i \partial_j w_i(u(\omega^*))\partial u_j(\omega_j^*) = p^*$. Consequently, we know that ω^* is a market price equilibrium with free disposal of (w, u, ρ), supported by p^*, and that $\partial_{r_j} v_j(p^*, p^*.\omega_j^*) = 1/\sum_{i\in N} \mu_i \partial_j w_i(u(\omega^*))$ for all j. Let $r^* = (p^*.\omega_1^*, \ldots, p^*.\omega_n^*)$; $\pi_{ij} = \mu_i \partial_j w_i(v(p^*, r^*))\partial_{r_j} v_j(p^*, r_j^*)$ for all (i,j). Then $\sum_{i\in N} \pi_{ij} = 1$ for all j. And for all (i,j), $\partial_j w_i(v(p^*, r^*))\partial_{r_j} v_j(p^*, r_j^*) = (1/\mu_i)\pi_{ij}$, with $1/\mu_i > 0$.

At this stage, we have proved that: there exists a system of market prices $p^* \gg 0$, which supports ω^* as a market price equilibrium of (w, u, ρ), and a system of Lindahl prices π, such that $\pi_{ij} = \mu_i \partial_j w_i(v(p^*, r^*))\partial_{r_j} v_j(p^*, r_j^*)$ for all (i,j); $\sum_{i\in N} \pi_{ij} = 1$ for all j; and, for all i, r^* verifies the first-order necessary conditions for a local maximum of $r \to w_i(v(p^*, r))$ in $\{r \in \mathbb{R}^n_+ : \pi_i.r \leq \pi_i.r^*\}$. There remains to establish that $w_i(u(\omega^*)) \geq w_i(u(x^0))$ for all i and r^* is a global maximum of $r \to w_i(v(p^*, r))$ in $\{r \in \mathbb{R}^n_+ : \pi_i.r \leq \pi_i.r^*\}$ for all i.

Endowment distribution ω^* being a market price equilibrium of (w, u, ρ) necessarily is the unique Walrasian equilibrium allocation of (w, u, ω^*) under Assumption 1-(i) (Balasko 1988, 3.4.4).[19] The

[19]See the Appendix of Mercier Ythier 2007, for a discussion of the relations between our Assumption 1 and Balasko's setup and associate conditions for a valid transposition of Balasko's results into our setup.

definition of a liberal distributive social contract then readily implies that $w_i(u(\omega^*)) \geq w_i(u(x^0))$ for all i.

Finally, the functions $r \to w_i(v(p^*, r))$ being quasi-concave in $\mathbb{R}^n_{++}$ by assumption, the first-order necessary conditions for a local maximum of $r \to w_i(v(p^*, r))$ in $\{r \in \mathbb{R}^n_+ : \pi_i.r \leq \pi_i.r^*\}$ are also sufficient conditions for a global maximum of the same program as a consequence of the Theorem 1 of Arrow and Enthoven 1961. ■

Proof of Proposition 1: The bordered Hessian of $\hat{u} \to w_i(\hat{u})$, evaluated at $\hat{u} \gg 0$, is matrix $H_i(\hat{u}) = \left(\begin{smallmatrix} \partial^2 w_i(\hat{u}) & [\partial w_i(\hat{u})]^T \\ \partial w_i(\hat{u}) & 0 \end{smallmatrix}\right)$. The bordered Hessian of $r \to w_i(v(p, r))$, evaluated at $r \gg 0$, is matrix $H_i'(r) = \left(\begin{smallmatrix} \partial^2 (w_i \circ v)(p,r) & [\partial (w_i \circ v)(p,r)]^T \\ \partial (w_i \circ v)(p,r) & 0 \end{smallmatrix}\right)$. The generic entry of $\partial^2(w_i \circ v)(p, r)$, which is located on the jth row and kth column of $H_i'(r)$, is $\partial^2_{jk} w_i(v(p, r))\partial_{r_j} v_j(p, r_j)\partial_{r_k} v_k(p, r_k)$. The generic entry of $\partial(w_i \circ v)(p, r)$ (respectively, $[\partial(w_i \circ v)(p, r)]^T$), which is located on the kth column (respectively, jth row) of $H_i'(r)$, with $k \leq n$ (respectively, $j \leq n$), is $\partial_k w_i(v(p, r))\partial_{r_k} v_k(p, r_k)$ (respectively, $\partial_j w_i(v(p, r))\partial_{r_j} v_j(p, r_j)$). The multilinearity of the determinant then implies $D_{ij}'(r) = (\prod_{k \leq j} \partial_{r_k} v_k(p, r_k))^2 D_{ij}(v(p, r))$. The marginal ophelimities of wealth $\partial_{r_k} v_k(p, r_k)$ being > 0 for all k, $D_{ij}'(r)$ is equal to 0 if and only if $D_{ij}'(v(p, r)) = 0$ and, otherwise, has the same sign as $D_j(v(p, r))$. The second part of the proposition is a simple consequence of these facts and of Theorem 3 of Arrow and Enthoven 1961. ■

Proof of Proposition 2: The continuity of functions $\sum_{i \in N} \mu_i(w_i \circ u)$ for all $\mu \in S_n$ and compactness of A readily implies that φ is well defined; that is, that argmax$\{\sum_{i \in N} \mu_i w_i(u(x)) : x \in A\}$ is a nonempty subset of A for all $\mu \in S_n$. The convex-valuedness of φ is a straightforward consequence of the convexity of set A and quasi-concavity of functions $w_i \circ u$ for all i. $P_w = \cup_{\mu \in S_n} \varphi(\mu)$ by Theorem 3. It will suffice, therefore, to establish that Graphφ is closed (see Mas-Colell 1985, A.6). Let (μ^q, x^q) be a converging sequence of elements of Graphφ, and denote by (μ, x) its limit. We want to prove that $\mu = \varphi(x)$. From Theorem 3 and the continuity of functions ∂w_i, u_i, and ∂u_i for all i: x is ≥ 0, such that $\sum_{i \in N} x_i = \rho$, and there exists $p \in \mathbb{R}^l_+$, such that, for all $(i, j) \in N \times N$, $\sum_{i \in N} \mu_i \partial_j w_i(u(x))\partial u_j(x_j) = p$. μ belongs to S_n by closedness of the latter, so that $\mu > 0$. Therefore, x verifies the first-order necessary conditions for a weak maximum of w in A. The f.o.c. is also sufficient, by Assumption 1 and Theorem 1 of Arrow and Enthoven 1961. Therefore, $x \in P_w$, and the conclusion then comes as a simple consequence of Theorem 3. ■

Proof of Theorem 2: The proof proceeds in three steps.

(i) In Step 1, we prove that: The restriction of φ to $\operatorname{Int} S_n$ is a homeomorphism $\operatorname{Int} S_n \to P_w^{**}$ with a C^1 inverse; in particular, P_w^{**} is simply connected.

We first prove that the second regularity condition implies that $\varphi(\mu)$ is single valued for all $\mu \in \operatorname{Int} S_n$. Let $\mu \in \operatorname{Int} S_n$. We suppose that $\varphi(\mu)$ contains two distinct elements x and x', and derive a contradiction. The definition of φ and the quasi-concavity of functions $w_i \circ u$ together imply that $w(u(\alpha x + (1-\alpha)x')) \geq w(u(x)) = w(u(x'))$ for all real numbers $\alpha \in [0, 1]$. The second regularity condition readily implies that the C^2 functions $w_i \circ u$ are all strictly concave in some neighbourhood U of x in $\mathbb{R}^{nl}$. For $\alpha < 1$ sufficiently close to 1, we must therefore have $w(u(\alpha x + (1-\alpha)x')) \gg w(u(x))$. But $\alpha x + (1-\alpha)x' \in A$, due to the convexity of the latter set. Therefore, $x \notin \varphi(\mu)$, the contradiction.

We next prove that, for any $x \in P_w^{**}$, $\varphi^{-1}(x)$ is single-valued and C^1.

From Theorem 3, $x \in P_w^{**}$ is a $\gg 0$ market price equilibrium supported by a $\gg 0$ price system p, which is unique up to a positive multiplicative constant. Let p^* denote the unique supporting price system of x that belongs to S_l. Theorem 5 implies that for any $\mu \in \varphi^{-1}(x)$, there exists a unique price system αp^*, proportional to p^* with $\alpha \in \mathbb{R}_{++}$, such that, for all $j \in N$, $\sum_{i\in N} \mu_i \partial_j w_i(u(x)) = 1/\partial_{r_j} v_j(\alpha p^*, \alpha p^*.x_j)$.

The homogeneity of degree 0 of indirect ophelimity functions implies that $\partial_{r_j} v_j(\beta\alpha p^*, \beta\alpha p^*.x_j) = (1/\beta)\partial_{r_j} v_j(\alpha p^*, \alpha p^*.x_j)$ for all $\beta > 0$ (positive homogeneity of degree -1 of the derivative). Letting $\beta = \partial_{r_1} v_1(\alpha p^*, \alpha p^*.x_1)$ and applying f.o.c. $\partial u_1(x_1) = \partial_{r_1} v_1(\alpha p^*, \alpha p^*.x_1)\alpha p^*$, one gets $\partial_{r_j} v_j(\alpha p^*, \alpha p^*.x_j)/\partial_{r_1} v_1(\alpha p^*, \alpha p^*.x_1) = \partial_{r_j} v_j(\partial u_1(x_1), \partial u_1(x_1).x_j)$ for all $j > 1$.

Dividing f.o.c. $\sum_{i\in N} \mu_i \partial_j w_i(u(x)) = 1/\partial_{r_j} v_j(\alpha p^*, \alpha p^*.x_j)$ by f.o.c. $\sum_{i\in N} \mu_i \partial_1 w_i(u(x)) = 1/\partial_{r_1} v_1(\alpha p^*, \alpha p^*.x_1)$ for all $j > 1$, and using the result of the former paragraph, one gets the following equivalent system of $n-1$ equations: $(1/\sum_{i\in N} \mu_i \partial_1 w_i(u(x)))\sum_{i\in N} \mu_i \partial_j w_i(u(x)) = 1/\partial_{r_j} v_j(\partial u_1(x_1), \partial u_1(x_1).x_j)$. Multiplying both sides by $\sum_{i\in N} \mu_i \partial_1 w_i(u(x))$ and rearranging, one finally gets $\sum_{i\in N} \mu_i(\partial_j w_i(u(x))$-$(1/\partial_{r_j} v_j(\partial u_1(x_1), \partial u_1(x_1).x_j))\partial_1 w_i(u(x))) = 0, j > 1$.

Denote by $B(x)$ the $n \times n$ matrix obtained from Jacobian matrix $\partial w(u(x))$ by substracting column-vector $(1/\partial_{r_j} v_j(\partial u_1(x_1), \partial u_1(x_1).x_j)).\partial_1 w(u(x))$ from the first and jth columns of $\partial w(u(x))$ for all $j > 1$ and by $C(x)$ the $n \times (n-1)$ matrix obtained from $B(x)$ by deleting its first column. The system of f.o.c. obtained at the end of the former paragraph writes in matrix form $\mu.C(x) = 0$ or, equivalently, $[C(x)]^T.\mu^T = 0$, which, for any given x, characterizes the kernel of the transpose of $C(x)$. The first regularity condition of Definition 10 and the multilinearity of the determinant imply $|\partial w(u(x))| = |B(x)| \neq 0$, hence rank $C(x) = \operatorname{rank}[C(x)]^T = n-1$. Therefore, $\dim \operatorname{Kernel}[C(x)]^T =$

$n-(n-1)=1$; that is, the kernel of $[C(x)]^T$ is a homogeneous line of $\mathbb{R}^n$, which moreover admits a >0 directing vector because $\varphi^{-1}(x) \subset \mathrm{Kernel}[C(x)]^T$. Its intersection with hyperplane $\{z \in \mathbb{R}^n : \sum_{i\in N} z_i = 0\}$ reduces, consequently, to $\{0\}$. This implies, in turn, that the $n \times n$ matrix $D(x)$ obtained from $B(x)$ by substituting the transpose of the unit diagonal row-vector $(1, \ldots, 1)$ of $\mathbb{R}^n$ for its first column is nonsingular, for $\mathrm{rank}\, D(x) = \mathrm{rank}[D(x)]^T = n - \mathrm{dimKernel}[D(x)]^T = n - \dim\{z \in \mathbb{R}^n : \sum_{i\in N} z_i = 0\} \cap \mathrm{Kernel}[C(x)]^T = n$. Therefore, equation $\mu.D(x) - (1, 0, \ldots, 0) = 0$, viewed as a linear equation in μ for any fixed $x \in P_w^{**}$, admits a unique solution, $= (1, 0, \ldots, 0).[D(x)]^{-1}$. We can let $\varphi^{-1}(x) = (1, 0, \ldots, 0).[D(x)]^{-1}$. Moreover, φ^{-1} is C^1 by Assumptions 1-(i)-(b) and 1-(ii)-(b) (C^2 utility functions) and the implicit function theorem applied to function $\mathbb{R}^n \times \mathbb{R}^{nl}_{++} \to \mathbb{R}^n : (\mu, x) \to \mu.D(x) - (1, 0, \ldots, 0)$ at any point $(\mu, x) \in S_n \times P_w^{**}$, such that $\mu \in \varphi^{-1}(x)$.

From there on, the restriction of φ' to $\mathrm{Int}\, S_n$ is denoted by φ.

Theorem 3 and the definition of inclusive distributive optimum readily imply that $\varphi(\mathrm{Int}\, S_n) = P_w^{**}$. Function φ', therefore, is a one-to-one mapping $\mathrm{Int}\, S_n \to P_w^{**}$ with a C^1 inverse. We now prove that φ' is continuous. Let sequence μ^q converge to μ in $\mathrm{Int}\, S_n$. The compactness of A implies that sequence $\varphi'(\mu^q)$ admits a converging subsequence in A. Let x be the latter's limit. The continuity of $(\mu, x) \to \sum_{i\in N} \mu_i w_i(u(x))$ implies that inequalities $\sum_{i\in N} \mu_i^q w_i(u(\varphi'(\mu^q))) \geq \sum_{i\in N} \mu_i^q w_i(u(z))$, which hold true for all pairs $(\mu^q, \varphi(\mu^q))$ and all $z \in A$ by definition of φ', extend to the limit pair (μ, x). That is, $x = \varphi'(\mu)$.

Finally, $\mathrm{Int}\, S_n$ is simply connected, as a convex set. Therefore, $P_w^{**} = \varphi(\mathrm{Int}\, S_n)$ is simply connected as homeomorphic to the former. This completes the proof of the first step.

(ii) In Step 2, we prove that: P_w^{**} *is a C^1 manifold of dimension* $n-1$.

Let g denote the C^1 function $\mathbb{R}^n_{++} \times \mathrm{Int}\, P_u \to \mathbb{R}^n$ defined by $g(\mu, x) = \mu.D(x) - (1, 0, \ldots, 0)$ (see Step 1 earlier). Under Assumption 1-(i), $\mathrm{Int}\, P_u$ is a C^1 manifold of dimension $n-1$ (Mas-Colell 1985, 4.6.9). Function g, therefore, is a C^1 function on a C^1 manifold of dimension $2n-1$, mapping into a C^∞ manifold of dimension n. From Theorem 3, $\mathrm{Graph}\, \varphi = g^{-1}(0)$. $\partial_\mu g(\mu, x) = D(x)$, which is a nonsingular $n \times n$ matrix at any $x \in P_w^{**}$ by the first regularity condition (see Step 1). Therefore, $\mathrm{rank}\, \partial g(\mu, x) = n$ everywhere in $\mathrm{Graph}\, \varphi'$; that is, 0 is a regular value of g. The Regular Value Theorem (see Mas-Colell 1985, H.2.2) then implies that $\mathrm{Graph}\, \varphi'$ is a C^1 manifold, whose dimension is equal to $\dim\, (\mathbb{R}^n_{++} \times \mathrm{Int}\, P_u) - \dim \mathbb{R}^n = n-1$. Finally, denote by $h_{(\mu,x)}$ a local C^1 diffeomorphism $\mathbb{R}^{n-1} \to \mathrm{Graph}\, \varphi'$ at some point (μ, x) of $\mathrm{Graph}\, \varphi'$; pr_2 the projection $\mathrm{Graph}\, \varphi' \to P_w^{**}$ defined by $\mathrm{pr}_2(\mu, x) = x$; and Φ function $P_w^{**} \to \mathrm{Graph}\, \varphi'$ defined by $\Phi(x) = (\varphi^{-1}(x), x)$. Note that pr_2 is C^∞, while Φ is C^1 by Step 1 of this proof. Therefore, $\mathrm{pr}_2 \circ h_{(\mu,x)}$ is a local C^1

diffeomorphism $\mathbb{R}^{n-1} \to P_w^{**}$ at (μ, x), whose C^1 inverse is $(h_{(\mu,x)})^{-1} \circ \Phi$. This completes the proof of Step 2.

(iii) In Step 3, finally, we prove the second part of Theorem 3.

Let L denote the set of social contract price equilibria of (w, u, ω) relative to the Walrasian equilibrium (p, x) of the latter, and suppose that $x \notin P_w$. From Theorem 1, $L \cap \mathbb{R}^{nl}_{++} = P_w^{**} \cap \{z \in \mathbb{R}^{nl} : w(u(z)) \geq w(u(x))\}$. The continuity of w and u and the openness of P_w^{**} then imply that $\operatorname{Int} L$ is equal to $P_w^{**} \cap \{z \in \mathbb{R}^{nl} : w(u(z)) \gg w(u(x))\}$. Since $x \notin P_w$, open set $\{z \in \mathbb{R}^{nl} : w(u(z)) \gg w(u(x))\}$ is nonempty. $P_w^{**} \cap \{z \in \mathbb{R}^{nl} : w(u(z)) \geq w(u(x))\}$ is nonempty by Mercier Ythier (2009) Theorem 1-(ii). Therefore, so is $\operatorname{Int} L$ (since P_w^{**} is open). Functions $w_i \circ u$ being quasi-concave, set $\{z \in \mathbb{R}^{nl} : w(u(z)) \gg w(u(x))\}$ is convex and is, therefore, an open convex subset of $\mathbb{R}^{nl}$, hence is a simply connected C^∞ manifold of dimension ln. P_w^{**} is a simply connected C^1 manifold of dimension $n-1 < nl$ by Steps 1 and 2 above, and so is its intersection with $\{z \in \mathbb{R}^{nl} : w(u(z)) \gg w(u(x))\}$. That is, $\operatorname{Int}L$ is a simply connected C^1 submanifold of P_w^{**}, of the same dimension as the latter. Consequently, $\varphi^{-1}(\operatorname{Int}L)$ is a simply connected, open subset of $\operatorname{Int} S_n$. ■

Proof of Proposition 3: For any pair of distinct attainable allocations (x, x') and any $0 < \alpha < 1$, we have $u(\alpha x + (1-\alpha)x') > \alpha u(x) + (1-\alpha)u(x')$ since u_i is strictly concave in $pr_i A$ for all i and x_i is different from x_i' for at least one i. Therefore, $w_i(u(\alpha x + (1-\alpha)x')) \geq w_i(\alpha u(x) + (1-\alpha)u(x'))$ for all i, with a strict inequality for any I, such that $u_i(\alpha x + (1-\alpha)x') > \alpha u_i(x) + (1-\alpha)u_i(x')$ by the monotonicity assumptions, and $w_i(\alpha u(x) + (1-\alpha)u(x')) \geq \alpha w_i(u(x)) + (1-\alpha)w_i(u(x'))$ by concavity for all i. Hence: $w(u(\alpha x + (1-\alpha)x')) > \alpha w(u(x)) + (1-\alpha)w(u(x'))$. Therefore, for any $\mu \gg 0$: $\mu.w(u(\alpha x + (1-\alpha)x')) > \alpha\mu.w(u(x)) + (1-\alpha)\mu.w(u(x'))$. ■

Proof of Proposition 4:

(i) Let (w, u, ρ) verify the first assumption, and suppose, without loss of generality, that $r_1 \geq r_2 \geq \ldots \geq r_n$. Then $\partial w(v(p, r))$ is a triangular matrix, whose sub-diagonal entries are all $= 0$. Therefore, $|\partial w(v(p, r))| = \prod_{i \in N} \partial_i w_i(v(p, r))$, which is >0 by Assumption 1-(ii)-(a). The conclusion follows from the equivalence of weak price-wealth distributive and weak distributive optimum (Theorem 3).

(ii) Let (w, u, ρ) verify the second assumption. Note that the generic entry located on the ith row and j-th column of matrix $\partial w(v(p, r)).\partial_r v(p, r)$ is $\partial_j w_i(v(p, r)).\partial_{r_j} v_j(p, r_j)$. The multilinearity of the determinant, therefore, implies that $|\partial w(v(p, r)).\partial_r v(p, r)| = (\prod_{i \in N} \partial_{r_i} v_i(p, r_i))|\partial w(v(p, r))|$, where $\prod_{i \in N} \partial_{r_i} v_i(p, r_i)$ is >0. The diagonal dominance assumption implies that $|\partial w(v(p, r)).\partial_r v(p, r)|$ is >0. Therefore, $|\partial w(v(p, r))|$ is >0, and the conclusion follows from the equivalence of weak price-wealth distributive and weak distributive optimum as above. ■

Proof of Corollary: Let (π, p^*, x^*) be a social contract equilibrium of (w, u, ω), relative to competitive market equilibrium with free disposal (p^0, x^0) of (w, u, ω), such that $x^* \gg 0$, and denote $r^* = (p^* . x_1^*, \ldots, p^* . x_n^*)$. Function $r \to w_i(v(p^*, r))$ being strictly increasing in r_i, the budget constraint must be satiated at any of its maxima in $\{r \in \mathbb{R}_+^n : \pi_i.r \leq \pi_i.(p^*.x_1^0, \ldots, p^*.x_n^0)\}$. Therefore, $\pi_i.r^* = \pi_i.(p^*.x_1^0, \ldots, p^*.x_n^0)$, and r^* also is a maximum of $r \to w_i(v(p^*, r))$ in $\{r \in \mathbb{R}_+^n : \pi_i.r \leq \pi_i.r^*\}$. Hence, x^* is a $\gg 0$ social contract price equilibrium of (w, u, ω), relative to competitive market equilibrium with free disposal (p^0, x^0) of (w, u, ω), and the first part of the Corollary follows from the application of Theorem 1.

Suppose that, moreover, $x^0 \notin P_w^{**}$, and $r \to w_i(v(p^*, r))$ is strictly quasi-concave for all i. We have $w(u(x^*)) \geq w(v(p^*, (p^*.x_1^0, \ldots, p^*.x_n^0)))$, r^* being a maximum of $r \to w_i(v(p^*, r))$ in $\{r \in \mathbb{R}_+^n : \pi_i.r \leq \pi_i.(p^*.x_1^0, \ldots, p^*.x_n^0)\}$ for all i by definition of a social contract equilibrium. Suppose that $w_i(u(x^*)) = w_i(v(p^*, (p^*.x_1^0, \ldots, p^*.x_n^0)))$ for some i, and let us derive a contradiction. The strict quasi-concavity of $r \to w_i(v(p^*, r))$ implies that any strict convex combination $\alpha r^* + (1-\alpha)(p^*.x_1^0, \ldots, p^*.x_n^0)$, $0 < \alpha < 1$, is strictly preferred by i to both r^* and $(p^*.x_1^0, \ldots, p^*.x_n^0)$ (since $w_i(v(p^*, r^*)) = w_i(u(x^*)) = w_i(v(p^*, (p^*.x_1^0, \ldots, p^*.x_n^0)))$. Since, moreover, $\alpha r^* + (1-\alpha)(p^*.x_1^0, \ldots, p^*.x_n^0) \in \{r \in \mathbb{R}_+^n : \pi_i.r \leq \pi_i.(p^*.x_1^0, \ldots, p^*.x_n^0)\}$, r^* cannot be a maximum of $r \to w_i(v(p^*, r))$ in $\{r \in \mathbb{R}_+^n : \pi_i.r \leq \pi_i.(p^*.x_1^0, \ldots, p^*.x_n^0)\}$, which yields the sought for contradiction. ■

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