The Effectiveness of Distributive Policy in a Competitive Economy

JEAN MERCIER YTHIER
Université de Paris and GREQAM

Abstract

I consider an abstract social system made of individual owners endowed with nonpaternalistic interdependent preferences, who interact by means of individual gifts and by exchanges on competitive markets. The existence of equilibrium is established. I identify the set of allocations that are decentralizable in the sense that they are general equilibria for some vectors of market prices and initial endowments. This set is characterized in a simple way from the social endowment and individual market and distributive preferences. Decentralizable allocations are all accessible to distributive policy, unless public transfers are confined to some neighborhood of 0. In the latter case, distributive policy remains free to perform local redistributions of wealth across the components of the graph of equilibrium gifts.

1. Introduction

This paper studies the effectiveness of distributive policy in the context of an "abstract social system" (Debreu 1952), involving the interaction of competitive exchange and individual gift. The agents are individual owners, whose preferences verify the assumption of nonpaternalistic utility interdependence. Borrowing the vocabulary of Pareto's *Traité de Sociologie Générale* (1916, Chap. XII), an early promoter of this assumption as well as of the application of the methods and concepts of economic equilibrium analysis to the study of wider social equilibria, we suppose more precisely that each individual is endowed with both an ophelimity function, describing the (ordinal) satisfaction he derives from his own consumptions of market commodities, and a utility function, representing his preferences on the profiles of individual ophelimities. These agents act in the follow-

12, Place du Panthéon, 75005 Paris, France, and GREQAM (CNRS-EHESS), 2, rue de la Charité, 13002 Marseille (jean.mercier-ythier@wanadoo.fr).

Received 5 February 1999; Accepted 9 June 1999.

Journal of Public Economic Theory, 2 (1), 2000, pp. 43–69.

ing general equilibrium context: They choose their individual market excess demands and individual gifts in order to maximize their utility functions subject to their individual budget constraints, given market prices and the excess demands and individual gifts of the other agents. A social equilibrium is then a system of market prices and individual actions which clears all markets and solves simultaneously all individual maximization problems.

The effectiveness of distributive policy is a positive matter. It consists of the ability of the government to achieve independent objectives concerning the distribution of wealth. Distributive implications are involved, as means, ends, or mere consequences, in virtually all actions of economic policy, whether they aim at allocative efficiency, distributive justice, or macroeconomic regulation (Musgrave 1959). So the evaluation of the effectiveness of distributive policy is essentially equivalent to (both implies and is implied by) an evaluation of the effectiveness of general economic policy.

The debate on the effectiveness of distributive policy has largely concentrated on the existence and extension of a crowding out of private transfers by public ones. Barro (1974) and Becker (1974) analyze models where distributive policy is neutral in the sense that variations in public transfers are exactly offset by opposite variations in private ones. These neutrality theorems have been challenged on essentially two grounds. One line of argument considers situations where some private transfers are corner solutions (Roberts 1984; Bergstrom, Blume, and Varian 1986). And an abundant literature exhibits nonneutralities in situations where private and public transfers are not perfect substitutes, noticeably because gift matters per se (for instance because there is a "warm glow of giving;" Andreoni 1989); because gift stems from an exchange motive rather than from an altruistic one (e.g., Cox and Jakubson 1995); and because of capital market imperfections (e.g., Altig and Davis 1993).

This article builds on the first line of argument. Public and private transfers are perfect substitutes, in the sense that they both consist of lump-sum transfers aiming at correcting significant wealth differences between the donor (voluntary or not) and the beneficiary. Markets are complete and perfect. I establish a rigorous relation between the effectiveness of distributive policy and the structure of the graph of equilibrium transfers in this context. I show that the dimension of the set of wealth distributions accessible to a public distributive policy operating by lump-sum transfers chosen in a neighborhood of 0 is the number of components of the graph of equilibrium transfers (minus 1); that is, small public transfers can redistribute wealth "across" the components of this graph but not inside each of them. Thus, the neutrality of distributive policy holds true if and only if the graph of equilibrium gifts is connected.

The paper is organized as follows. Section 2 defines social equilibrium, Section 3 characterizes the set of decentralizable allocations, and

Section 4 establishes existence. Section 5 presents the main theorem and discusses the relevance of neutrality properties in the light of this result. Section 6 concludes and the Appendix gathers the proofs.

2. Social Equilibrium

There are l *commodities,* identified by an index h running in $L = \{1, \ldots, l\}$, and n *agents,* identified by an index i, running in $N = \{1, \ldots, n\}$.

The commodities are divisible consumption goods. The total quantity of each of them available in the social system is given once and for all—in other words, we have an exchange economy.[1] Physical units are chosen so that the *endowment of the social system* in any consumption good h is 1. We denote by e the element of the space of goods $\mathbb{R}^l$ whose components are all equal to 1.

The agents are individuals. They privately own the social endowment in consumption goods (social system of private property). Denoting agent i's *initial endowment* as ω_i we have, formally,

$$\sum_{i \in N} \omega_i = e.$$

Then ω_{ih} denotes the nonnegative quantity of commodity h initially owned by individual i. The vector $(\omega_1, \ldots, \omega_n)$ of individual endowments is denoted by ω.

The agents can use commodities in three different ways: private consumption, individual gift, and exchange on competitive markets. We will ignore all other conceivable individual uses, such as disposal or production.

A *consumption* of generic agent i is represented by an element x_i of the commodity space $\mathbb{R}^l$. Its hth component x_{ih} is a quantity of commodity h consumed by individual i. Agent i's *consumption set* is the positive orthant $\mathbb{R}^l_+$ of the space of commodities. An *allocation* is then an element $x = (x_1, \ldots, x_i, \ldots, x_n)$ of $\mathbb{R}^{ln}$. We denote by X_i the subset of $\mathbb{R}^{ln}$ whose ith projection on $\mathbb{R}^l$ is agent i's consumption set $\mathbb{R}^l_+$, and jth projection on $\mathbb{R}^l$ is $\mathbb{R}^l$ for all $j \neq i$. An allocation x is *feasible* if it belongs to $\mathbb{R}^{ln}_+$ and verifies the global resource constraint

$$\sum_{i \in N} x_i \leq e.$$

The set $\{x \in \mathbb{R}^{ln}_+ | \sum_{i \in N} x_i = e\}$ of feasible allocations that exhaust the social endowment is denoted by F.

A *gift* of agent i to individual j is represented by a nonnegative element t_{ij} of the commodity space $\mathbb{R}^l$. Its hth component t_{ijh} is a nonnega-

[1]The introduction of production and disposal leaves our analysis essentially unchanged under the following conditions: firms are price-takers (perfect competition) and maximize profits in convex production sets; and disposal is free.

tive quantity of consumption good h, transferred by agent i to individual j. We denote by t_i, and name *gift of i*, a vector whose ith projection t_{ii} on $\mathbb{R}^l$ is equal to 0, and jth projection t_{ij} on $\mathbb{R}^l$ is a gift from agent i to individual j for all $j \neq i$. Agent i's *gift set* T_i is the subset of $\mathbb{R}^{ln}_+$ whose ith projection on $\mathbb{R}^l$ is $\{0\}$, and jth projection on $\mathbb{R}^l$ is $\mathbb{R}^l_+$ for all $j \neq i$. A vector of individual gifts $(t_1, \ldots, t_i, \ldots, t_n)$ is then named a *gift vector* and denoted by t. For all gift vectors t and all individual gifts t_i^*, we will use the following standard notations: $t_{n/i}$ will be the vector of individual gifts obtained from t by deleting its ith component t_i; $(t_{n/i}, t_i^*)$ will be the gift vector obtained from t by replacing its ith component t_i by t_i^*; and $\Delta_i t$ is the net gift

$$\sum_{j \in N} (t_{ji} - t_{ij})$$

accruing to individual i when the gift vector is t.

A *net trade* of agent i is represented by a vector z_i of the space of commodities. Its hth component z_{ih} is the net trade of agent i in good h—that is, the difference between his physical purchases and sales of commodity h. We denote by z a vector $(z_1, \ldots, z_i, \ldots, z_n)$ of individual net trades.

A *social state* is then a vector (x, t, z). Since the individual uses of commodities are here restricted to private consumption, individual gift, and exchange, a state (x, t, z) must verify the following physical accounting identities for all i:

$$x_i = z_i + \omega_i + \Delta_i t,$$

equating consumptions to net physical inflows from trade, gift-giving, and initial endowment, for all individuals and commodities.

An *action* of individual i, denoted by a_i, is a pair (z_i, t_i). An *action vector* is then a vector $a = (a_1, \ldots, a_i, \ldots, a_n)$ of individual actions. For all action vector a and all individual action a_i^*, we denote, as above, $a_{n/i}$ the vector of individual actions obtained from a by deleting its ith component a_i; $(a_{n/i}, a_i^*)$ the action vector obtained from a by replacing its ith component a_i by a_i^*. We suppose that every agent considers the others' actions as independent of his own actions (takes them as given). It follows from this and the accounting identities above that, given some $a_{n/i}$, the choice by agent i of some action $a_i^* = (z_i^*, t_i^*)$ determines the realization of one and only one allocation, namely allocation $x((a_{n/i}, a_i^*))$ whose jth component is $z_j + \omega_j + \Delta_j(t_{n/i}, t_i^*)$ for all j. We suppose, too, that every agent considers market prices as independent from his individual actions (competitive markets). The vector of market prices is denoted by p. The unique social state determined by the action vector a is denoted by $(x(a), t(a), z(a))$.

Individual preference preorderings are defined directly, for the sake of brevity, by means of their utility representations.[2] Using Pareto's words, we define an *ophelimity function* of agent i, $u_i : \mathbb{R}^l \to \mathbb{R}$, which describes i's preferences on his own consumptions. Function $(x_1, \ldots, x_n) \to (u_1(x_1), \ldots, u_n(x_n))$, mapping $\mathbb{R}^{ln}$ into $\mathbb{R}^n$, is denoted by u. The *utility function* of agent i, denoted by w_i, maps the set $u(\mathbb{R}^{ln})$ of ophelimity profiles into the real line. Function $w_i \circ u$ then describes i's preferences on allocations. This particular shape of individual utility functions corresponds to the definition of nonpaternalistic utility interdependence as stated by Archibald and Donaldson (1976). It allows for the representation of moral sentiments such as benevolence, malevolence, or indifference to others, provided that they do not involve merit wants (individuals are not sensitive to the others' consumptions per se but only through their consequences on ophelimities). Without loss of generality, we will let $u_i(0) = 0$ for all i.

The picture concerning individual behavior is, at this point, the following: Each agent makes the choice of his gifts and net trades in order to achieve some allocation of resources according to his nonpaternalistic preferences. We can now complete this description of individual behavior by a specification of the constraints binding individual choices. Consider some price-action vector (p^*, a^*), defining an environment for individual decisions. Individual i will choose his action in the *budget set* $B_i(p^*, a^*) = \{a_i = (z_i, t_i) \in \mathbb{R}^l \times T_i \mid x_i((a^*_{n/i}, a_i)) \in \mathbb{R}^l_+ \text{ and } p^* z_i \leq 0\}$, in order to maximize his utility function according to the program: $\text{Max}\{w_i(u(x((a^*_{n/i}, a_i)))) \mid a_i \in B_i(p^*, a^*)\}$.

A *social system* is a list $(w_1 \circ u, \ldots, w_n \circ u)$ and is denoted by w. A *market optimum* of social system w is a (strong) Pareto optimum with respect to the

---

[2] Axiomatic definitions of nonpaternalistic utility interdependence, building on preference preorderings, can be found, for instance, in Winter (1969), Bergstrom (1970), or Lemche (1986). Winter considers private complete preorderings, defined on individual consumptions, and social ones, defined on allocations, and assumes that if x_j (resp. x_i) is preferred (resp. strictly preferred) to x'_j (resp. x'_i) for j's (resp. i's) private preordering, then it must be preferred (resp. strictly preferred) for the preordering induced by i's social preferences on j's consumptions. Bergstrom starts with individual social complete preorderings, supposes them weakly separable in individual consumptions (i.e., the preordering induced by i's social preferences on j's consumptions does not depend on the others' consumptions whatever i and j) and assumes that if x_j is preferred to x'_j for j's induced preordering, then the same is true for i's. Lemche considers utility representations of individual social preorderings and defines interdependence by the following condition: If x_j is indifferent to x'_j for j's induced preordering, then the same is true for i's.

Lemche's definition implies Winter's. It implies the definition of Archibald and Donaldson (1976), too, provided that the latter is strengthened by the requirement that the utility of an individual must be increasing in his own ophelimity. The definition of Archibald and Donaldson implies Bergstrom's, which implies Winter's. All these implications are strict. Winter's definition is essentially identical to Pareto's original formulation ophelimity functions of its members, that is, a feasible allocation x such that there exists no feasible allocation x' verifying both $u(x') \geq u(x)$ and $u(x') \neq u(x)$. The set of market optima of w is denoted by O. A *social system of private property* is a pair (w, ω). A *social equilibrium* of (w, ω) is a price-action vector (p^*, a^*) such that (i) $\sum_{i \in N} z_i^* = 0$ (all markets clear); and (ii) a_i^* solves $\text{Max}\{w_i(u(x((a_{n/i}^*, a_i)))) \mid a_i \in B_i(p^*, a^*)\}$ for all i (everyone is satisfied with his own choice, given prices and the others' actions).

The following assumptions on preferences and endowments will be maintained throughout the sequel.

Assumption 1:

(i) *For all i, u_i is: (a) continuous in $\mathbb{R}_+^l$ and differentiable in $\mathbb{R}_{++}^l$ (the interior of $\mathbb{R}_+^l$); (b) increasing in $\mathbb{R}_{++}^l$ (i.e., $u_i(x_i) > u_i(x_i')$ for all $(x_i, x_i') \in \mathbb{R}_{++}^l \times \mathbb{R}_{++}^l$ such that $x_i > x_i'$)*[3]*; (c) such that $x_i \gg 0$ whenever $u_i(x_i) > 0$ $(= u_i(0))$.*

(ii) *For all i, w_i is (a) continuous in $\mathbb{R}_+^n$ and differentiable with respect to its jth argument in $\{\hat{u} \in \mathbb{R}_+^n \mid \hat{u}_j > 0\}$ for all j, and (b) increasing in its ith argument.*

(iii) *For all i, $w_i \circ u$ verifies that (a) $w_i(u(\lambda x + (1 - \lambda)x')) > w_i(u(x'))$ for all real number $\lambda \in [0,1]$ and all $(x, x') \in \mathbb{R}_+^{ln} \times \mathbb{R}_+^{ln}$ such that $w_i(u(x)) > w_i(u(x'))$, and (b) $w_i(u(x)) = 0$ whenever $u_i(x_i) = 0$.*

(iv) *For all i, $\omega_i > 0$.*

We will consider, therefore, differentiable social systems[4].

Assumptions (i)(b) and (i)(c) are commonly used in the study of differentiable economies. Together with (ii)(b), (i)(b) implies that prices are positive at equilibrium, and (i)(c) implies that an agent whose after-transfer wealth is positive will consume a positive amount of all goods (thereby eliminating inessential technicalities associated with nonnegativity constraints on consumption).

Assumption 1(ii)(b), stating that utility is increasing in its own ophelimity, appears natural enough in the context of this study. It can be viewed as a component of a sensible definition of nonpaternalistic utility interdependence (Lemche 1986, Rem. 1).

Parts (iii) and (iv) of Assumption 1 ensure that individual behavioral correspondences will have the relevant continuity property required for

[3]We write $x_i \geq x_i'$ whenever $x_{ih} \geq x_{ih}'$ for all h, $x_i > x_i'$ whenever $x_i \geq x_i'$ and $x_i \neq x_i'$, and $x_i \gg x_i'$ whenever $x_{ih} > x_{ih}'$ for all h.

[4]A natural strategy for the study of continuous social systems will consist of "smoothing" them by means of appropriate approximation techniques, and then examining whether, as is often the case, the properties of smooth social systems extend by continuity to continuous ones. This is done, for instance, in Mercier Ythier (1992) for the existence of a social equilibrium in the case of a pure distributive social system ($l = 1$).

the existence of a social (hence competitive market) equilibrium. Parts (iii)(b) and (iv), together with (i)(c) and (ii)(b), are designed to imply, noticeably, the seemingly reasonable consequence that every agent will wish and be able to keep a positive after-transfer wealth for all positive price vector, which ensures in turn the continuity of budget correspondences on relevant domains. The convexity of preferences (iii)(a) implies then the upper hemicontinuity of behavioral correspondences.

3. Decentralizable Allocations

This section characterizes the subset of allocations that can be reached as equilibrium allocations of the social system. The following definitions will prove useful. An allocation x^* is (i,j)-*maximal* if there exists an ophelimity profile $(\hat{u}_1,\ldots,\hat{u}_n) \in u(\mathbb{R}_+^{ln})$ such that x^* solves $\mathrm{Max}\{w_i(u(x)) \mid x$ is feasible and $u_k(x_k) \geq \hat{u}_k$ for all $k \neq j\}$.[5] The set of (i,j)-maximal allocations is denoted by M_{ij}. The set $\bigcap_{i\in N} M_{ii}$ is denoted by M.

The following characterization of social equilibrium follows then easily from Kuhn and Tucker first-order conditions (proofs of the theorems are in the Appendix).

THEOREM 1: *Suppose that (w,ω) verifies Assumption 1. The price-action vector (p,a) is then a social equilibrium of this system if and only if it verifies the following three conditions: (i) $px_i(a) = p(\omega_i + \Delta_i t(a))$ for all i; (ii) $x(a) \gg 0$ and $x(a)$ is (i,i)-maximal for all i; (iii) $x(a)$ is (i,j)-maximal whenever $t_{ij}(a)>0$.*

It follows readily from condition (ii) of Theorem 1 that social equilibrium allocations must lie in set $M \cap \mathbb{R}_{++}^{ln}$.

Conditions (ii) and (iii) can receive the following interpretation, building on the fact that M_{ij} (and therefore M) is a subset of the set O of market optima for all (i,j) (as a consequence of Lemma 4(iv) of the Appendix). The (i,j)-maximality of an interior allocation means then that agent i's utility is nonincreasing in wealth transfers, evaluated at *supporting* market prices, from j to any other agent. Condition (ii) of Theorem 1 says therefore that the equilibrium allocation is a market optimum, and that nobody wants to increase his own transfers of wealth to the other agents, evaluated at equilibrium (hence supporting) prices. Condition (iii) says that, if there is a gift from i to j, the former does not want to diminish this transfer of wealth, evaluated at supporting market prices.

[5]The relevant definition of (i,j)-maximality becomes the following in a production economy. Let Y_j be the production set of firm j, and let y_j denote an input-output vector of firm j, $j = 1,\ldots,m$: (x^*,y^*) is (i,j)-*maximal* if there exists an ophelimity profile $(\hat{u}_1,\ldots,\hat{u}_n) \in u(\mathbb{R}_+^{ln})$ such that (x^*,y^*) solves $\mathrm{Max}\{w_i(u(x))|(x,y)$ is feasible and $u_k(x_k) \geq \hat{u}_k$ for all $k \neq j\}$, where feasibility is defined in the usual way by individual constraints on consumption and production embodied in consumption and production sets and by the global resource constraint $\sum_i x_i \leq e + \sum_j y_j$.

A partial converse is established in Theorem 2.

THEOREM 2: *Suppose that* (w, ω) *verifies Assumption 1. For any allocation* $x \gg 0$ *in M, there exist a price vector* $p \neq 0$ *and a vector* ω^* *of individual endowments, with* $\omega^* = x$*, such that the price-action vector* $(p, 0)$ *is a social equilibrium of* (w, ω^*).

Theorem 2 states that when the vector of individual endowments is in set $M \cap \mathbb{R}^{ln}_{++}$ there is a vector of market prices such that the status quo is an equilibrium (status quo meaning zero gifts and consumption of own endowment for all agents).

Set $M \cap \mathbb{R}^{ln}_{++}$, defined in a simple way from the fundamentals of the social system (namely, social endowment, individual market preferences described by ophelimity functions, and individual distributive preferences described by utility functions), is therefore the set of decentralizable allocations, which means that any equilibrium allocation lies in this set and any allocation in this set is an equilibrium allocation for properly chosen vectors of market prices and individual endowments.

Example—The Cobb–Douglas social system: We take $n = 3$. The number of goods l is left unspecified. The three agents have identical market preferences, represented by Cobb–Douglas ophelimity functions $u_i\colon x_i \to \prod_{h=1}^{l} x_{ih}^{1/l}$, $i = 1, 2, 3$. Their utility functions are the Cobb–Douglas $w_i\colon u \to \prod_{j=1}^{3} u_j^{\alpha_{ij}}$, where the vector $\alpha^{\mathrm{i}} = (\alpha_{i1}, \alpha_{i2}, \alpha_{i3})$ is $\gg 0$ and belongs to the unit simplex $S_3 = \{s = (s_1, s_2, s_3) \in \mathbb{R}^3_+ | \sum_{i=1}^{3} s_i = 1\}$. These preferences verify Assumption 1.

The market efficiency frontier is then $O = \{x \in F | \exists s \in S_3$ such that $x_i = s_i e \ \forall \ i\}$, that is, the set of feasible allocations where each agent i consumes some fraction s_i of the social endowment e. The ophelimity frontier $u(O)$ is therefore the simplex S_3, represented in Figure 1 by triangle $u^1 u^2 u^3$, where u^i denotes the ophelimity distribution such that agent i's ophelimity is 1 $(= u_i(e))$ and agent j's ophelimity is 0 $(= u_j(0))$ for all $j \neq i$. All the elements of O have the same unique supporting price vector in the unit simplex $S_l = \{s \in \mathbb{R}^l_+ | \sum_{i=1}^{l} s_i = 1\}$, namely, price vector $p^* = \left(\frac{1}{l}, \ldots, \frac{1}{l}\right)$. And we have therefore, for any x in O, $u(x) = (p^* x_1, p^* x_2, p^* x_3)$, so that ophelimity distribution and wealth distribution can be identified everywhere on the market efficiency frontier.[6]

The maximum of w_i in $u(O)$ is the vector of weights α^i. The set $u(M_{ij})$ of (i, j)-maximal ophelimity or wealth profiles is triangular surface $u^k \alpha^i u^m$,

[6] The possibility of identifying the ophelimity distribution with the wealth distribution on the frontier of market efficiency is of course peculiar to this example. Nevertheless, it expresses in a strong way a general feature of the type of social system examined in this paper, namely, the fundamental equivalence of ophelimity and wealth from the viewpoint of nonpaternalistic individuals pursuing distributive aims on the background of a competitive economy.

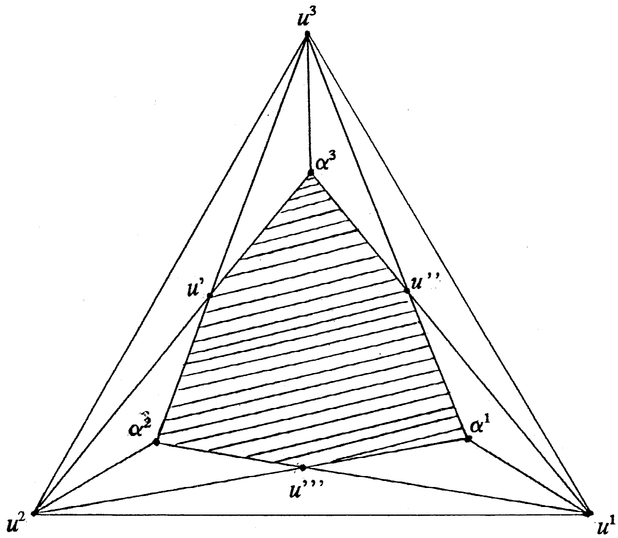

Figure 1: Decentralizable ophelimity profiles in a Cobb–Douglas social system.

with k and $m \neq j$ (e.g., $u(M_{33}) = u^1\alpha^3u^2$; cf. Figure 1).[7] The set $u(M)$ of decentralizable ophelimity or wealth profiles is therefore the surface $(u^2\alpha^1u^3) \cap (u^1\alpha^2u^3) \cap (u^1\alpha^3u^2)$ (the hatched area of Figure 1). The (unique) equilibrium distribution of wealth runs in this surface when the vector of initial individual endowments runs in $F \cap \mathbb{R}^{ln}_{++}$.

4. Existence of Social Equilibrium

One can easily build Cobb–Douglas social systems with an empty set M. Mercier Ythier (1989) or Stark (1993), for instance, provide simple examples of pure distributive social systems ($l = 1$) involving a "war of gifts" between two agents. One verifies, likewise, that M is empty in the Cobb–Douglas social system above, when distributive parameters α^i are chosen so that $\alpha_{ij} = \alpha_{ji} > \alpha_{ii}$ for all i and all $j \neq i$. There is then, at any market optimum and associate supporting price vector, at least one individual who wishes and is able to transfer some of his own wealth to another agent.

The existence property is analyzed in detail in Mercier Ythier (1992; 1993) for the one commodity case. I establish that nonexistence stems from the fact that gifts are virtually unbounded in the presence of a closed chain (a "directed circuit") of individual redistributive desires (gifts

[7] Mercier Ythier (1998b, Thm. 5).

received by an agent then being turned, like "hot potatoes," to the subsequent one in the circuit).[8] The compactness assumption embodied in Debreu's (1952) social equilibrium existence theorem, in particular, is violated in such cases. I prove too that a social equilibrium exists for all initial distributions of endowments when the agents never wish to make any transfer to individuals wealthier than themselves (Mercier Ythier 1992, Cor. 3; or 1993, Thm. 2).

Theorem 3 extends this last result to the case of multiple commodities. It makes use of the following definitions and assumption. Let v_i denote i's *indirect ophelimity function,* defined on the set of price-wealth vectors $(p, R_i) \in \mathbb{R}^l_{++} \times \mathbb{R}_+$, by $v_i(p, R_i) = \mathrm{Max}\{u_i(x_i) \mid x_i \in \mathbb{R}^l_+ \text{ and } px_i \leq R_i\}$. These are well-defined, continuous, $\mathbb{R}^l_{++} \times \mathbb{R}_+ \to \mathbb{R}$ functions under Assumption 1 (as a well-known consequence of the continuity of ophelimity functions and compactness and nonemptiness of $\{x_i \in \mathbb{R}^l_+ \mid px_i \leq R_i\}$ for all $(p, R_i) \in \mathbb{R}^l_{++} \times \mathbb{R}_+$). We let $R = (R_1, \ldots, R_n)$, $v : (p, R) \to (v_1(p, R_1), \ldots, v_n(p, R_n))$, e_{ij} be the vector of $\mathbb{R}^n$, the components of which are all equal to 0 except the ith one, equal to -1, and the jth one, equal to $+1$; moreover, we make the following assumption.

Assumption 2:

(i) *v_i is differentiable in $\mathbb{R}^l_{++} \times \mathbb{R}_{++}$ for all i.*

(ii) *$\tau \to w_i(v(p, R + \tau e_{ij}))$, defined on $[0, R_i]$, is nonincreasing whenever $R_j \geq R_i$.*

Assumption 2(ii) embodies a notion of *self-centeredness* in distributive preferences. It fits naturally in this class of models, where gift-giving is a mere instrument of redistribution of wealth between individuals, designed in practice to correct significant wealth differences between the donor and the beneficiary. A much tighter, in some sense the tightest (Mercier Ythier 1992), condition for existence is the hypothesis of Lemma 6 of the Appendix. The Cobb–Douglas social system defined above verifies Assumption 2 if and only if $\alpha_{ii} \geq \alpha_{ij}$ for all (i, j).

THEOREM 3: *Suppose that (w, ω) verifies Assumptions 1 and 2. Then there exists a social equilibrium.*

[8]This phenomenon is implicitly assumed away by Nakayama (1980) and Kranich (1988). Nakayama restricts transferable wealth to initial endowment, the latter being therefore the upper bound to individual gifts in his construct. Kranich embodies a priori upper bounds on individual gifts in individual gift sets T_i (assumed compact). These choices are not satisfactory in the context of these nonpaternalistic models, where gifts are essentially identical with transfers of wealth (money units). In such a context, Nakayama's choice appears counterfactual (wealth received through gifts is clearly transferable), and Kranich's seems artificial (what is the practical significance of an a priori upper bound on money transfers?).

5. Distributive Policy

The type of distributive policy that I consider now is the simplest conceivable one, namely, a discretionary redistribution of individual endowments by means of lump-sum transfers. I briefly examine below what can be said on the feasibility of these policies in this abstract social system.

It follows readily from Theorem 2 that a distributive policy operating by lump-sum transfers can reach, in principle, any allocation of M. The effectiveness of distributive policies has nevertheless been challenged by the neutrality theorems recalled in Section 1 of this article. Theorem 4, below, translates these neutrality properties into our framework in a way that allows us to appreciate their relevance and scope.

A preliminary statement, which is a simple corollary of Theorems 1 and 2, will help interpret the neutrality property. From now on, we let $\theta_{ij} \in \mathbb{R}^{ln}_{+}$ be a vector of public lump-sum transfer from i to j when $j \neq i$, $\theta_{ii} = 0$ for all i, $\theta = (\theta_{11}, \ldots, \theta_{1n}, \ldots, \theta_{n1}, \ldots, \theta_{nn})$. We use the simple notions of graph theory which are collected in Appendix A.II, and the following two formal definitions, built from Theorems 1 and 2. Consider a social system verifying Assumption 1, an allocation x in $M \cap \mathbb{R}^{ln}_{++}$ and its (unique) supporting price vector p in the unit simplex S_l. The *graph of gift desires* at x is the set $\gamma(x) = \{(i, j) \in N \times N \mid x \in M_{ij}\}$. It can be interpreted as the graph of potential equilibrium gifts at x, in the sense that (i, j) must belong to $\gamma(x)$ whenever (x, t, z) is an equilibrium state such that $t_{ij} > 0$ (Theorem 1(iii)). We define, second, the set $\Omega(x) = \{\omega \in \mathbb{R}^{ln} \mid \sum_{i \in N} \omega_i = e$ and $\exists t$ such that $t_i \in T_i$ for all i; $t_{ij} = 0$ whenever $x \notin M_{ij}$; $px_i = p(\omega_i + \Delta_i t)$ for all $i\}$. Given Theorems 1 and 2, $\Omega(x) \cap \mathbb{R}^{ln}_{++}$ is the (nonempty) set of vectors of interior individual endowments supporting x as a social equilibrium allocation.

COROLLARY 1: *Suppose that (w, ω^0) verifies Assumption 1, and let (p, a) be an equilibrium. (i) $(p, x(a))$ is an equilibrium price-allocation vector of $(w, (\omega_1^0 + \Delta_1\theta, \ldots, \omega_n^0 + \Delta_n\theta))$ if and only if there exists t such that $t_i \in T_i$ for all i, $t_{ij} = 0$ whenever $x(a) \notin M_{ij}$, and $p(t_{ij} - t_{ij}(a) - (t_{ji} - t_{ji}(a)) + p(\theta_{ij} - \theta_{ji}) = 0$ for all (i, j). (ii) In particular, $(p, x(a))$ is* not *an equilibrium price-allocation vector of $(w, (\omega_1^0 + \Delta_1\theta, \ldots, \omega_n^0 + \Delta_n\theta))$ whenever θ implies wealth transfers across components of $\gamma(x(a))$, that is, whenever there are two agents i and j belonging to two distinct components of $\gamma(x(a))$ such that $p(\theta_{ij} - \theta_{ji}) \neq 0$.*

The first part of the corollary states that an equilibrium survives public redistributions of endowments if and only if the corresponding wealth transfers can be *offset* by individual countertransfers that maintain the structure of the graph of transfers which is associated with the equilibrium allocation. The second part states that equilibrium does not survive public redistributions involving transfers of wealth *across* the components of the graph.

This result is intuitively appealing. It draws its logical strength from the fact that there is only one structure of potential equilibrium gifts associated with any potential equilibrium allocation (any element of M). It points both to a sufficient condition for the nonneutrality of distributive policy, namely that it performs redistributions of wealth across the components of the graph of gift desires, and to its interpretation, that is, that offsetting individual countertransfers will then be incompatible with the structure of this graph.

We proceed now to the definition and complete characterization of the neutrality of distributive policy. A distributive policy is *locally neutral* at some vector $\omega^0 \in F \cap \mathbb{R}^{ln}_{++}$ of individual endowments if there exist a neighborhood $V(\omega^0)$ of ω^0 in $F \cap \mathbb{R}^{ln}_{++}$ and an ophelimity profile u^0 such that u^0 is the unique social equilibrium ophelimity profile for all ω in $V(\omega^0)$. A distributive policy is, second, *globally neutral* if there exists an ophelimity profile u^0 such that u^0 is the unique social equilibrium ophelimity profile for all ω in $F \cap \mathbb{R}^{ln}_{++}$. Local neutrality means that lump-sum transfers will not modify the distributive outcome if they remain confined to some neighborhood of the initial distribution of individual endowments. Global neutrality means that lump-sum transfers cannot modify the distributive outcome at all.

Theorem 4 provides necessary and sufficient conditions for the local and global neutrality of distributive policy.

THEOREM 4: *Suppose that (w, ω) verifies Assumption 1. (i) Distributive policy is globally neutral if and only if set $u(M \cap \mathbb{R}^{ln}_{++})$ is a singleton. Consider, moreover, an $x \in M \cap \mathbb{R}^{ln}_{++}$. (ii) Set $\Omega(x) - \{x\} = \{\omega - x \mid \omega \in \Omega(x)\}$ is a nonempty, closed, convex cone of dimension $l(n - c(\gamma(x)))$, where $c(\gamma(x))$ denotes the number of components of graph $\gamma(x)$. (iii) If, in particular, $u(x)$ is the unique social equilibrium ophelimity profile for all $\omega \in \Omega(x) \cap \mathbb{R}^{ln}_{++}$, then distributive policy is locally neutral at any ω of the (nonempty) interior of $\Omega(x) \cap \mathbb{R}^{ln}_{++}$ in F if and only if $\gamma(x)$ is connected.*

The first part of Theorem 4 is an immediate consequence of Theorems 1 and 2: Since the elements of $M \cap \mathbb{R}^{ln}_{++}$, and only them, can be reached as interior equilibrium allocations, global neutrality is clearly equivalent to the single-valuedness of $u(M \cap \mathbb{R}^{ln}_{++})$.[9]

The last part of Theorem 4 is a less straightforward consequence of Theorems 1 and 2. It translates Barro's neutrality theorem into our frame-

---

[9]An interesting special case of single-valuedness of $u(M)$ is Ramsey's dynastic framework (1928), where the agents are generations and where, using Pareto's vocabulary, their ophelimities are integrated in a single utility function, common to all generations, consisting of the (nondiscounted) sum of generations' ophelimities. Such a framework leaves little room, of course, for a conflict or even a difference between public and private views on distribution, which lies at the heart of Barro's neutrality theorem.

work. The connectedness condition that characterizes local neutrality generalizes, in particular, Barro's sufficient condition for neutrality, namely, that "current generations are connected to all future generations by a chain of operative transfers" (1974, p. 1106).

Theorem 4(iii) extends straightforwardly to the connected components of $\gamma(x)$: Small public lump-sum transfers between the vertices of a component of the graph of equilibrium transfers leave the equilibrium ophelimity profile unchanged. Combining this result with Corollary 1 results in the following statement: Small public transfers can influence the distribution of wealth across the components of the graph of gift desires; but they cannot influence the distribution of wealth inside the components of the graph of equilibrium transfers, for they are then offset by private countertransfers (this part is conditional on the uniqueness of the equilibrium ophelimity profile).

Considered from a purely theoretical point of view, the local neutrality of distributive policy, and a fortiori its global neutrality, cannot be viewed as general properties, that is, as properties holding true under general assumptions on preferences and endowments (like the assumptions of Section 2). In the social system of Figure 1, for instance, there are only six decentralizable ophelimity profiles that verify connectedness, namely: α^1, α^2, α^3, u', u'', and u''' (where the first three correspond in fact to the situation analyzed by Becker 1974). The distributions of the interior of $u(M)$ are all accessible objectives for distributive policy, the achievement of which implies the crowding out of all individual gifts (the situation considered by Roberts 1984). Moreover, the Beckerian situations α^1, α^2, and α^3 of this example are the only accessible targets of distributive policy which do not imply a complete crowding out of private transfers and which are, at the same time, Pareto efficient with respect to utility functions. This is suggestive of a general property of this type of model: The achievement of social Pareto efficiency by distributive policy usually implies the crowding out of all private transfers when, as a familiar consequence of the public good problem (Kolm 1966; Hochman and Rodgers 1969), social equilibrium is Pareto inefficient with respect to utilities.[10]

[10]The set of distributive optima of Figure 1 (i.e., the ophelimity profiles that are Pareto efficient with respect to *utility* functions) is surface $\alpha^1\alpha^2\alpha^3$. Denoting $u(P)$ this set and $\partial u(M)$ the boundary of $u(M)$, we have $u(P) \subset u(M)$ and $u(P) \cap \partial u(M) = \{\alpha^1, \alpha^2, \alpha^3\}$ in the social system of Figure 1. Moreover, if there are gifts at equilibrium, then the corresponding ophelimity profile must be in $\partial u(M)$. These facts are general: An efficient distributive policy will normally crowd out all private transfers (except for targets taken a discrete set that contains the maxima of individual utility functions) *whenever* $u(P) \subset u(M)$. General social systems such that $u(P) \subset u(M)$ can be found in Roberts (1984, Sec. III), and in Mercier Ythier (1998a, Thm. 1). The interested reader can build easily Cobb–Douglas social systems such that $u(P)$ is not contained in $u(M)$ by moving properly the α^i's in Figure 1; $u(P) \cap \partial u(M)$ then contains a subset (segment) of dimension $n - 2$.

This theoretical skepticism[11] is nevertheless qualified, as far as the *positive* question of the effectiveness of distributive policy is concerned (as opposed to the normative question of its social efficiency), when one confronts the general properties stated in Theorem 4 with empirical evidence on the structure of the graph of actual private transfers. The interesting question, from a positive point of view, is not whether local neutrality holds true or not in any real social system (it certainly does not), but rather how many degrees of freedom distributive policy enjoys in such systems. Part (ii) of Theorem 4 states a general property that gives, at least in principle, a precise answer to this question: distributive policy can influence the distribution of wealth in a number of dimensions equal to the number of components of the graph of equilibrium gifts (minus 1). Combined with empirical evidence that intergenerational transfers within the family are very important in frequency and magnitude and are by far the most widespread type of private transfers,[12] this property suggests that it should be much easier for a distributive policy, operating by lump-sum transfers confined to some neighborhood of 0, to redistribute wealth from rich to poor than to perform intergenerational transfers, thereby reconciling, to some extent, the opposed statements of Barro (1974), Becker (1974), and Roberts (1984).[13]

This partial practical relevance of local neutrality does not extend, of course, to global neutrality. Clearly, there is generally a practical ability of distributive policy to influence the distribution of wealth, as soon as lump-sum transfers are no longer confined to a neighborhood of 0.

One should mention, for the sake of completeness, two limits to the effectiveness of distributive policy which can appear in our framework, besides offsetting private transfers. Both raise the question of the ability of the government to control the consequences of nonneutral transfers in order to achieve a determinate, a priori accessible, distributive objective (that is, a given element of $u(M)$).

---

[11] Bernheim and Bagwell (1988), expresses the theoretical skepticism of the authors with respect to Barro's theorem in a different way; they show that if this theorem is taken seriously, it implies that not only lump-sum public transfers but also "distorsionary" ones might be neutral, an implication that can hardly be viewed as reasonable.

[12] The stimulating paper of Kotlikoff and Summers (1981) revived empirical research concerning intergenerational transfers. Recent results and a comprehensive review can be found in Gale and Scholz (1994). Auten and Joulfaian (1996) confronts intergenerational transfers with charitable ones.

[13] One can imagine, for instance, that the components of the graph of transfers split the set of agents into connected intertemporal blocks (dynasties, so to speak, or extended families, but conceived as sets of distinct individual agents, and not, in the manner of Ramsey (1928), as single collective ones). It would then be possible to influence the distribution of wealth between two agents, living at different periods of time or not, if and only if they belonged to different "dynasties." This condition clearly imposes more restrictions on intertemporal resdistribution (one can not redistribute inside a "dynasty") than on simultaneous redistribution.

The first difficulty stems from the theoretical possibility of multiple social equilibria.[14] Social equilibrium might "jump" discontinuously, following small lump-sum transfers, either because of equilibrium multiplicity per se (indeterminacy of equilibrium) or because of an associate discontinuity of the equilibrium correspondence (the latter being in general upper hemicontinuous but not lower hemicontinuous). Such discontinuous jumps violate the neutrality property, without implying the effectiveness of distributive policy, since the distributive outcome is uncontrolled then by construction. Equilibrium multiplicity is, nevertheless, difficult to interpret. I tend to understand it as an intrinsic property of mathematical models of social equilibrium—an unfortunate consequence of abstractness so to speak. In such a view, it is but one manifestation of the natural uncompleteness of theoretical representations of reality, and should not, therefore, be taken as too serious a problem.

The second difficulty is an aspect of the well-known "transfer problem," that is, the fact that a lump-sum transfer on endowments can impoverish the beneficiary and/or enrich the donor, due to induced effects on their respective terms of trade (e.g., Postlewaite 1979). This phenomenon can be safely ignored for private transfers, which consist in our framework of *individual* gifts, whose consequences on market prices can be realistically viewed as negligible by the donor. But the same does not hold for public transfers. The government could face, in principle, paradoxical effects of his decisions on the distribution of wealth when his transfers become large. The learning process by which the government actually experiences his ability to influence the distribution of wealth would be made more complicated and costly in such circumstances.

6. Conclusions

I have identified the set of allocations that are decentralizable in the sense that they are general equilibria for some vectors of market prices and initial endowments. This set is characterized in a simple way from the social endowment and individual market and distributive preferences, and is generally nonempty. All decentralizable allocations are accessible to distributive policy unless public transfers are confined to some neighborhood of 0. In the latter case, distributive policy remains free to perform local redistributions of wealth across the components of the graph of equilibrium gifts.

[14]The multiplicity of equilibrium can stem from market exchange (multiple economic equilibria) as well as from gift-giving (multiple distributive equilibria). Status quo equilibrium is, nevertheless, unique in pure distributive social systems endowed with suitable regularity properties (Mercier Ythier 1998c); in such cases, equilibrium multiplicity remains an issue only for local distributive policy.

These properties are established in the least favorable context for the effectiveness of distributive policy, namely, a competitive economy where public and private transfers are perfect substitutes.

Relaxing perfect substitutability will increase the descriptive accuracy of the model but is unlikely to modify its properties substantially. In a realistic setting, small public transfers inside a component of the graph of equilibrium gifts should be at least partly offset by private countertransfers, thereby inducing only second-order effects on market prices and the allocation of resources. Practically, effective distributive policies will then either redistribute wealth across the components of the graph of equilibrium gifts or push public transfers up to the point where they become, in the main, complements with the corresponding private transfers.

Considering market imperfections does alter the analysis quite significantly, for then distributive effects interact with efficiency effects. Consequently, the positive question of the effectiveness of distributive policy cannot be discussed independently from the normative questions of its market and social efficiencies. The means and ends of economic policy, more generally, must be defined simultaneously, and its distributive and allocative objectives coordinated. These questions match some of the main issues confronting actual distributive policy. They should become the principal focus of future research on its effectiveness.

Appendix

Assumption 1 of Section 2 is maintained throughout.

AI. Proofs of Theorems 1 and 2

LEMMA 1: *Consider a price vector* $p \gg 0$, *an action vector* $\hat{a}$, *and an individual action* a_i^* *such that* $x((\hat{a}_{n/i}, a_i^*)) \gg 0$. *Then* $a_i^* \in \operatorname{argMax}\{w_i(u(x((\hat{a}_{n/i}, a_i))))| a_i \in B_i(p, \hat{a})\}$ *if and only if the following four conditions are verified:* (i) $px_i((\hat{a}_{n/i}, a_i^*)) = p(\omega_i + \Delta_i t((\hat{a}_{n/i}, a_i^*)))$; (ii) $\partial_{u_j} w_i(u(x((\hat{a}_{n/i}, a_i^*)))).\partial_{x_j} u_j(x_j((\hat{a}_{n/i}, a_i^*))) \leq \partial_{u_i} w_i(u(x((\hat{a}_{n/i}, a_i^*)))).\partial_{x_i} u_i(x_i((\hat{a}_{n/i}, a_i^*)))$ *for all* j; (iii) $[\partial_{u_i} w_i(u(x((\hat{a}_{n/i}, a_i^*)))).\partial_{x_i} u_i(x_i((\hat{a}_{n/i}, a_i^*))) - \partial_{u_j} w_i(u(x((\hat{a}_{n/i}, a_i^*)))).\partial_{x_j} u_j(x_j((\hat{a}_{n/i}, a_i^*)))].t_{ij}((\hat{a}_{n/i}, a_i^*)) = 0$ *for all* j; *and* (iv) *there exists a real number* $\mu_i > 0$ *such that* $\partial_{u_i} w_i(u(x((\hat{a}_{n/i}, a_i^*)))).\ \partial_{x_i} u_i(x_i((\hat{a}_{n/i}, a_i^*))) = \mu_i p$.

Proof: Conditions (i) to (iv) of Lemma 1 are the Kuhn and Tucker conditions for the program $\operatorname{Max}\{w_i(u(x((\hat{a}_{n/i}, a_i))))| a_i \in B_i(p, \hat{a})\}$ at a maximum a_i^* such that $x((\hat{a}_{n/i}, a_i^*)) \gg 0$ (with $\mu_i > 0$ because of our monotonicity assumptions on utility and ophelimity functions). These conditions are necessary and sufficient by our assumptions and by Arrow and Enthoven (1961, Thm. 1(b) and Thm. 2(a) or 2(b)). ■

LEMMA 2: *Consider an allocation* $x \gg 0$. *It is the case that* x *is* (i, j)*-maximal if and only if* (i) $x \in F$, *and* (ii) *there exist* $p \gg 0$ *in* $\mathbb{R}^l$ *and* $\lambda =$

$(\lambda_1, \ldots, \lambda_n) \gg 0$ *in* $\mathbb{R}^n$ *such that, for all* k *in* N, $\partial_{x_k} u_k(x_k) = \lambda_k p$ *and* $\partial_{u_j} w_i(u(x))\lambda_j \geq \partial_{u_k} w_i(u(x))\lambda_k$.

Proof: As an immediate consequence of definitions, we have $x^* \in M_{ij}$ if and only if there exists $u^* \in \mathbb{R}^n$ such that $u^* = u(x^*)$; and x^* solves the program: $\text{Max}\{w_i(u(x)) | x \text{ is feasible and } u_k(x_k) \geq u_k^* \ \forall \ k \neq j\}$. Using our monotonicity assumptions on utility and ophelimity functions, conditions (i) and (ii) of Lemma 2 are then the Kuhn and Tucker conditions for an interior solution to this program. These conditions are necessary and sufficient by Assumption 1 and by Arrow and Enthoven (1961, Thm. 1(b) and Thm. 2(b)). ■

Proof of Theorem 1: Consider a vector of individual endowments $\omega \in F$ such that $\omega_i > 0$ for all i, and some associate social equilibrium (p, a). Denote (x, t, z) the corresponding equilibrium state. It follows readily from our monotonicity assumptions on utility and ophelimity functions that $p \gg 0$ and $x \in F$. Moreover, we must have $x \gg 0$ since, for all i and all $x \in \mathbb{R}_+^{ln}$, $p\omega_i > 0$, $u_i(x_i) > 0$ whenever $w_i(u(x)) > 0$, and $x_i \gg 0$ whenever $u_i(x_i) > 0$. Conditions (i)–(iv) of Lemma 1 are then verified for all i at social equilibrium (p, a). Recalling that $\partial_{u_i} w_i(u(x)) > 0$ by Assumption 1(ii), and letting $\lambda_i = \mu_i / \partial_{u_i} w_i(u(x))$, one concludes then by Lemma 2. ■

Proof of Theorem 2: Let $x \in M \cap \mathbb{R}_{++}^{ln}$. By Lemma 2, $x \in F$; and there exist $p \gg 0$ in $\mathbb{R}^l$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \gg 0$ in $\mathbb{R}^n$ such that, for all $(i, k) \in N \times N$, $\partial_{x_i} u_i(x_i) = \lambda_i p$ and $\partial_{u_i} w_i(u(x))\lambda_i \geq \partial_{u_k} w_i(u(x))\lambda_k$. Consider the social system (w, x), and the price-action vector $(p, 0)$. All markets clear by construction. It will suffice to prove, therefore, that $(z_i, t_i) = 0$ solves $\text{Max}\{w_i(u(x((0, a_i)))) | a_i \in B_i(p, 0)\}$ for all i. Conditions (i) and (iii) of Lemma 1 are obviously verified, and conditions (ii) and (iv) are verified by the consequence of Lemma 2 written above (just let $\mu_i = \partial_{u_i} w_i(u(x)).\lambda_i$, and recall that $\partial_{u_i} w_i(u(x)) > 0$ by Assumption 1(ii) and that $p \gg 0$). ■

A2. Proof of Theorem 3

This section and the next make use of the following few concepts of graph theory.[15] For any $x \in M$, define the *graph of gift desires at x,* $\gamma(x) = \{(i, j) \in N \times N | x \in M_{ij}\}$; for any $x \in O$, define the *graph of redistributive desires at x,* $\gamma'(x) = \{(i, j) \in N \times N | x \in M_{ij} \text{ and } x \notin M_{ii}\}$; and for any gift vector t, define the *graph of individual gifts at t,* $g(t) = \{(i, j) \in N \times N | t_{ij} > 0\}$. The set $\gamma(x)$ (respectively, $\gamma'(x)$; $g(t)$) can be viewed as (in one-to-one correspondence with) a formal *directed graph* (in short, *digraph* or *graph*). Its *vertices* are the agents i such that either (i, j) or (j, i) belongs to $\gamma(x)$

[15] For a detailed presentation of the concepts and ideas of graph theory used in this Appendix Section AII, cf. Tutte (1984).

(respectively, $\gamma'(x)$; $g(t)$). An element (i,j) of $\gamma(x)$ (respectively, $\gamma'(x)$; $g(t)$) is a *dart,* whose associate vertices i and j are named, respectively, its *tail* and *head.* Two darts (i,j) and (i',j') are *adjacent* if they have at least one vertex in common (i.e., if $\{i,j\} \cap \{i',j'\} \neq \emptyset$). An *m-circuit* of $\gamma(x)$ (respectively, $\gamma'(x)$; $g(t)$) is a sequence $((i_k,j_k))_{1\leq k\leq m}$ of m distinct elements of $\gamma(x)$ (respectively, $\gamma'(x)$; $g(t)$) such that, for all $k \in \{1,\ldots,m\}$, dart (i_{k+1},j_{k+1}) is adjacent to dart (i_k,j_k) (setting, conventionally, $(i_{m+1},j_{m+1}) = (i_1,j_1)$). A *directed circuit* of $\gamma(x)$ (respectively, $\gamma'(x)$; g(t)) is a circuit $((i_k,j_k))_{1\leq \mathrm{k} \leq \mathrm{m}}$ such that, for all $k \in \{1,\ldots,m\}$, $j_k = i_{k+1}$. A *directed path* of $\gamma(x)$ (respectively, $\gamma'(x)$; $g(t)$) is a sequence $((i_k,j_k))_{1\leq k\leq m}$ of adjacent darts of $\gamma(x)$ (respectively, $\gamma'(x)$; $g(t)$) such that, for all $k \in \{1,\ldots,m-1\}$, $j_k = i_{k+1}$. A *subgraph* of digraph Γ is a digraph whose sets of vertices and darts are contained, respectively, in the set of vertices and in the set of darts of Γ. We say then that this digraph is "contained" in Γ. A subgraph of Γ *spans* the latter if it has the same set of vertices. A digraph is *connected* if, for any pair of distinct vertices i and j, there exists a path in the digraph which has i and j as vertices. A *component* of digraph Γ is a connected subgraph of Γ which is strictly contained in no connected subgraph of Γ. A digraph is a *forest* if it has no circuit.

LEMMA 3: (i) *O is closed.* (ii) *If $x^* \in O$, then, for all i, either $x_i^* = 0$ or $x_i^* \gg 0$.* (iii) *$x^* \in O$ if and only if: $x^* \in F$; and there exist $p \gg 0$ in $\mathbb{R}^l$ and, for all $i \in \{j \in N | x_j^* \gg 0\}$, a real number $\lambda_i > 0$ such that $\partial_{x_i} u_i(x_i^*) = \lambda_i p$.*

Proof:

(i) Consider some converging sequence $(x^q)_{q\in\mathbb{N}}$ of elements of O, denote x^* its limit, and suppose that $x^* \notin O$. Since x^* is feasible by closedness of the set of feasible allocations, there exists a feasible x^{**} such that $u(x^{**}) > u(x^*)$.

Let us prove first that ophelimity is nonnegative in $\mathbb{R}_+^l$ and positive in $\mathbb{R}_{++}^l$. Function $\lambda \to u_i(\lambda x_i)$, defined on $\mathbb{R}_+$, is continuous and decreasing whatever $x_i \gg 0$ as a consequence of Assumption 1(i)(a) and (i)(b). Therefore $u_i(x_i) > 0$ for all $x_i \in \mathbb{R}_{++}^l$. The conclusion follows from continuity and Assumption 1(i)(c).

From the paragraphs above, we have: $u(x^*) > 0$; and $x_i^{**} \gg 0$ whenever $u_i(x^{**}) > u_i(x^*)$. Using continuity and monotonicity, we can then assume without loss of generality that $u(x^{**}) \gg u(x^*)$. But then we must have, by continuity of ophelimity functions, $u(x^q) \ll u(x^{**})$ when q is large enough, a contradiction. Therefore O is closed.

(ii) Consider some feasible allocation x^*. Suppose first that for all agent i there is some commodity h such that $x_{ih}^* = 0$; we then have $u(x^*) = 0$, while $u(x) \gg 0$ for all x in the (nonempty) intersection of F with $\mathbb{R}_{++}^{ln}$; therefore $x^* \in O$. Suppose now that

$x_i^* \gg 0$ for some agent i and that $x_{jh}^* = 0$ and $x_{jh'}^* > 0$ for some agent j and some pair of commodities h and h'; we then have $u_j(x_j^*) = 0 = u_j(0)$ and $u(x^{**}) > u(x^*)$ for the feasible allocation x^{**} such that $x_i^{**} = x_i^* + x_j^*$, $x_j^{**} = 0$, and $x_k^{**} = x_k^*$ for all k distinct from i and j; therefore $x^* \notin O$ and the first point is established.

(iii) Consider now some feasible allocation $x^* > 0$ such that, for all i, either $x_i^* = 0$ or $x_i^* \gg 0$, and suppose without loss of generality that $x_i^* = 0$ if and only if $i > m$, with $m \geq 1$. Then, $x^* \in O$ if and only if $(x_1^*, \ldots, x_m^*)$ solves the program $\text{Max}\{u_1(x_1) | \sum_{i \leq m} x_i \leq e$ and $u_i(x_i) \geq u_i(x_i^*)$ for all $i = 2, \ldots, m\}$ (necessity follows from definitions, sufficiency from the increasingness of ophelimity functions in $\mathbb{R}_{++}^l$). The Kuhn and Tucker conditions for an interior solution for this program are then the following: $\sum_{i \leq m} x_i^* = e$ and there exist $p \gg 0$ in $\mathbb{R}^l$ and $\alpha \gg 0$ in $\mathbb{R}^m$ such that $\alpha_i \partial_{x_i} u_i(x_i^*) = p$ for all $i \leq m$. The convexity property of Assumption 1(iii)(a), combined with the increasingness of utility in its own ophelimity and the continuity of ophelimity in $\mathbb{R}_+^l$, imply the quasi-concavity of ophelimity functions in $\mathbb{R}_+^l$. These first-order conditions are therefore necessary and sufficient by Arrow and Enthoven (1961, Thm. 1(b) and Thm. 2(a) or 2(b)). ■

LEMMA 4: (i) $\{x \in O | u_j(x_j) = 0\} \subset M_{ij}$ *for all* (i, j). (ii) $\{x \in F | u_i(x_i) = 0$ *and* $u_j(x_j) > 0\} \subset F \backslash M_{ij}$ *for all* (i, j) *such that* $i \neq j$. (iii) M_{ij} *is closed for all* (i, j). (iv) $\bigcup_{j \in N} M_{ij} = O$ *whatever* i. (v) *If* $x^* \in M_{ij}$, *then* $x^* \in F$ *and either* $x_j^* = 0$ *or* $x_j^* \gg 0$ *and there exist* $p \gg 0$ *in* $\mathbb{R}^l$ *and, for any* $k \in \{q \in N | x_q^* \gg 0\}$, *a real number* $\lambda_k > 0$ *such that* $\partial_{x_k} u_k(x_k) = \lambda_k p$ *and* $\partial_{u_j} w_i(u(x)) \lambda_j \geq \partial_{u_k} w_i(u(x)) \lambda_k$.

Proof:

(i) Consider first some x^* in O such that $u_j(x_j^*) = 0$. By Lemma 3(ii), we then have $x_j^* = 0$. Therefore, $\{u(x) \in \mathbb{R}^n | x \in O$ and $u_k(x_k) \geq u_k(x_k^*)$ for all $k \neq j\} = \{u(x^*)\}$ as a consequence of the definition of O. The result follows immediately from the definition of M_{ij}.

(ii) By Assumption 1(i)(c) and 1(iii)(b), we have $x_j^* \gg 0 = x_i^*$ and $w_i(u(x^*)) = 0$ whenever x^* lies in $\{x \in F | u_i(x_i) = 0$ and $u_j(x_j) > 0\}$. Allocation x such that $x_j = 0$ and $x_k = x_k^*$ for all k distinct from j, and allocation $x^{**} \in F$ defined by $x_i^{**} = x_i^* + x_j^*$, $x_j^{**} = 0$, and $x_k^{**} = x_k^*$ for all k distinct from i and j, are then such that $w_i(u(x^{**})) > w_i(u(x))$ (by the increasingness of utility in its own ophelimity) and $w_i(u(x)) = 0 = w_i(u(x^*))$ (Assumption 1(iii)(b)), while $u_k(x_k^{**}) \geq u_k(x_k^*)$ for all $k \neq j$. Therefore, $x^* \notin M_{ij}$.

(iii) Consider some converging sequence $(x^q)_{q \in \mathbb{N}}$ of elements of M_{ij}, denote x^* its limit, and suppose that $x^* \notin M_{ij}$. Since x^* is feasible by closedness of the set of feasible allocations, this means that there exists a feasible x^{**} such that $w_i(u(x^{**})) > w_i(u(x^*))$ and

$u_k(x_k^{**}) \geq u_k(x_k^*)$ for all $k \neq j$. By Lemma 4(i), we must have $x_j^* \gg 0$. The convexity of preferences (Assumption 1(iii)(a)) and resulting quasi-concavity of ophelimity functions (proof of Lemma 3(iii)) allow us to assume, without loss of generality, that $x_j^{**} \gg 0$ (the elements of segment $[x^{**}, x^*]$ being feasible and verifying the inequalities above). Using the continuity of utility and ophelimity functions in $\mathbb{R}_+^n$ and $\mathbb{R}_+^l$ respectively, the increasingness of ophelimity functions in $\mathbb{R}_{++}^l$, and the observation above (proof of Lemma 3(i)) that ophelimity is nonnegative in $\mathbb{R}_+^l$ and strictly positive in $\mathbb{R}_{++}^l$, we can assume, again without loss of generality, that $u_k(x_k^{**}) > u_k(x_k^*)$ for all $k \neq j$. But we must have then, by continuity, $w_i(u(x^q)) > w_i(u(x^*))$ and $u_k(x_k^q) > u_k(x_k^*)$ for all $k \neq j$ and large enough q, a contradiction. Therefore M_{ij} is closed.

(iv) In view of the closedness of sets M_{ij} and O, it suffices to prove, eventually, that $(\bigcup_{j \in N} M_{ij}) \cap \mathbb{R}_{++}^{ln} = O \cap \mathbb{R}_{++}^{ln}$. But this follows readily from Lemmas 2 and 3 and the simple observation that, using the notations of these lemmas, for all x^* in O and all i, there exists j such that $\partial_{u_j} w_i(u(x^*))\lambda_j \geq \partial_{u_k} w_i(u(x^*))\lambda_k$ for all k.

(v) Let $x^* \in M_{ij}$. The definition of M_{ij}, the increasingness of w_i in its own ophelimity, and the monotonicity properties of u_j (increasing in $\mathbb{R}_{++}^l$ by Assumption 1, nonnegative in $\mathbb{R}_+^l$ and positive in $\mathbb{R}_{++}^l$ as established in the proof of Lemma 3(i)) readily imply that $x^* \in F$. Moreover, Lemma 4(iv) implies that $x^* \in O$. Lemma 3(ii) implies then that we have either $x_i^* = 0$ or $x_i^* \gg 0$, whatever i. If $x_j^* = 0$ the proof is completed. Suppose therefore that $x_j^* \gg 0$, and let, without loss of generality, $\{i \in N | x_i \gg 0\} = \{1, \ldots, m\} = I$. If $m = 1$, then $j = 1$, and the result is a simple consequence of Lemma 3(iii). Suppose next that $m>1$. The definition of M_{ij} implies that x^* solves $\text{Max}\{w_i(u(x)) | x \text{ is feasible}, u_k(x_k) \geq u_k(x_k^*)\ \forall\ k \in I \backslash \{j\},\ x_k = 0\ \forall k \in N \backslash I\}$. The conclusion follows then from Arrow and Enthoven (1961: Thm. 2(b)). ■

LEMMA 5: *Consider some price-action vector (p, a) such that $p > 0$; $a_i \in B_i(p, a)$ for all i; and $g(t(a))$ has no directed circuit. Then, for all i and all $j \neq i$, there exist $a_i^* = (z_i^*, t_i^*) \in B_i(p, a)$ such that: $t_{ij}^* \in [0, e]$ for all j; and $p(\omega_i + \sum_{j \in N}(t_{ji}(a) - t_{ij}^*)) = 0$.*

Proof: It suffices to prove that there exists $t_{ij}^* \in [0, e]$ such that $p(\omega_i + \sum_{j \in N} t_{ji}(a)) = pt_{ij}^*$. Denote, for all i, $I(i)$ the set of vertices j of $g(t(a))$ such that there exists a directed path $((i_k, j_k))_{1 \leq k \leq m}$ in $g(t(a))$ with $i_1 = j$ and $j_m = i$. We have, by definition, $I(j) \subset I(i)$ for all $j \in I(i)$, so that the total wealth transferred to agent $i\, p\sum_{j \in N} t_{ji}(a) = p\sum_{j \in I(i)} t_{ji}(a)$ stems, directly or indirectly, from the pool of agents $I(i)$. Moreover, the absence of directed circuits in $g(t(a))$ implies that $i \notin I(i)$ for all i. Therefore, $p\sum_{j \in N} t_{ji}(a)$ is a wealth transfer from the pool of agents $I(i)$ to outside agent i. Aggregating the budget constraints of the elements of $I(i)$, we get $p\sum_{j\in N} t_{ji}(a)) \leq p\sum_{j\in I(i)}(\omega_j + \sum_{k\in N\setminus\{i\}}(t_{kj}(a) - t_{jk}(a))) \leq p\sum_{j\in I(i)}\omega_j$. Hence, $p(\omega_i + \sum_{j\in N} t_{ji}(a)) \leq p\omega_i + p\sum_{j\in I(i)}\omega_j$. Since $p > 0$, we have $p\omega_i + p\sum_{j\in I(i)}\omega_j \leq p\sum_{j\in N}\omega_j = pe$. Therefore, $p(\omega_i + \sum_{j\in N} t_{ji}(a)) \leq pe$ and the result follows. ■

LEMMA 6: *Suppose that $\gamma'(x)$ has no directed circuit whatever $x \in O$. Then, there exists a social equilibrium.*[16]

Proof: This lemma is the main piece of the existence proof. It is built on the pattern of Arrow's and Hahn's (1971) proof of existence of a competitive market equilibrium.

Denote the following: T the set of gift vectors t such that $t_{ij} \in [0, e]$ for all (i, j); D the cartesian product $S_l \times F \times T \times S_n$; and f some homeomorphism from $u(O)$ to the unit simplex S_n of $\mathbb{R}^n$ such that, for all i and all $u^* \in u(O)$, $f_i(u^*) = 0$ if and only if $u^*_i = 0$ (e.g., Arrow

4(iii) implies then that there exists some q_0 such that, for all $q \geq q_0$, $x^q \notin M_{ij}$ whenever $x^* \notin M_{ij}$. Therefore, $t'^* \in \tau(p^*, x^*, t^*, s^*)$ and correspondence τ is upper hemicontinuous.

Denote by ψ the product correspondence defined by $\psi(d) = \pi(d) \times \chi(d) \times \tau(d) \times \sigma(d)$ for all $d \in D$. Its values are nonempty, compact, convex subsets of D, and the correspondence is upper hemicontinuous. Moreover, because D is a nonempty, compact, convex set, ψ has some fixed point (p^*, x^*, t^*, s^*) in this set (Kakutani's fixed-point theorem). To finish, let us establish that (p^*, x^*, t^*, s^*) defines an equilibrium. By Theorem 1, we have to prove that: (i) $p^* x_i^* = p^*(\omega_i + \Delta_i t^*)$ for all i; (ii) $x^* \gg 0$; (iii) $x^* \in M$; and (iv) $t_{ij}^* = 0$ whenever $x^* \notin M_{ij}$. Point (iv) is verified by construction of ψ. Let us establish the other points.

Suppose first that $p^* x_i^* > p^*(\omega_i + \Delta_i t^*)$ for some i. Then $s_i^* = 0$ by definition of σ, and therefore $u_i(x_i^*) = 0$ by definition of f and χ. This, in turn, implies that $x_i^* = 0$, and therefore $p^* x_i^* = 0 \leq p^*(\omega_i + \Delta_i t^*)$ by definition of τ, a contradiction.

Suppose now that $p^* x_i^* \leq p^*(\omega_i + \Delta_i t^*)$, the inequality being strict for at least one i. Adding up over N, we must then have $p^* \sum_{i \in N} x_i^* < p^* \sum_{i \in N}(\omega_i + \Delta_i t^*) = p^* e$, but this is inconsistent with the definition of χ, which implies $x^* \in F$ and therefore $\sum_{i \in N} x_i^* = e$. This establishes (i).

Suppose next that $x_{ih}^* = 0$ for some (i, h). Since $x^* \in O$ by definition of χ, we must have, by Lemma 3, $x_i^* = 0$. It follows then from Lemma 4(ii) that $x^* \notin M_{ij}$ whenever $u_j(x_j^*) > 0$ (that is, whenever $x_j^* \gg 0$). Denote I the set of agents with 0 consumption (nonempty by assumption). By definition of τ and by point (i) established above, we must then have, for all $i \in I$, $0 = p^* x_i^* = p^*(\omega_i + \Delta_i t^*) = p^*(\omega_i + \sum_{j \notin I} t_{ji}^* + \sum_{j \in I}(t_{ji}^* - t_{ij}^*))$. Therefore, adding up over I, we must have $0 = p^* \sum_{i \in I} x_i^* = p^* \sum_{i \in I}(\omega_i + \sum_{i \in I} \sum_{j \notin I} t_{ji}^*) \geq p^* \sum_{i \in I} \omega_i$. But since, eventually, $p^* \gg 0$ by Lemma 3 and the definition of π, and since $\omega_i > 0$ for all i by assumption, we must have $p^* \sum_{i \in I} \omega_i > 0$, a contradiction. This establishes (ii).

To finish, suppose that $x^* \notin M_{ii}$ for some i. The definition of τ and point (i) above imply then $p^* x_i^* = p^*(\omega_i + \Delta_i t^*) = 0$ for such an agent. Therefore, since $p^* \gg 0$, we must have $x_i^* = 0$. But then $x^* \in M_{ii}$ by Lemma 4(i), a contradiction. ■

Proof of Theorem 3: In view of Lemma 6, it suffices to prove that $\gamma'(x)$ has no directed circuit whatever $x \in O$. Suppose the contrary—that is, consider some directed circuit $\Gamma = ((i_k, j_k))_{1 \leq k \leq m}$ of $\gamma'(x)$.

Let I be the set $\{i \in N | \, x_i \gg 0\}$; let $\#I = m$; let $(p, \lambda) \in \mathbb{R}_{++}^l \times \mathbb{R}_{++}^m$ be a vector of market prices and marginal ophelimities of wealth supporting x (Lemma 3(iii)); and let $R_i = px_i$ for all i. From Lemma 3(ii), we have, for all i, either $x_i = 0$, and then $u_i(x_i) = 0 = v_i(p, R_i)$, or $x_i \gg 0$, and then, as a simple consequence of Lemma 3(iii), $u_i(x_i) = v_i(p, R_i)$. Moreover, we have $\partial_{R_i} v_i(p, R_i) = \lambda_{\mathrm{i}}$ for all $i \in I$ by the differentiability of indirect ophelimity functions in $\mathbb{R}^l_{++} \times \mathbb{R}_{++}$ (Assumption 2(i)).

Notice next that $x_i \gg 0$ for all vertex i of circuit Γ as a consequence of Lemma 4(ii). Let (i, j) be a dart of Γ. Then x is (i, j)-maximal and not (i, i)-maximal, by definition of $\gamma'(x)$. Lemmas 3(iii) and 4(v) and the paragraph above together imply then that $\partial_{u_j} w_i(u(x)).\partial_{R_j} v_j(p, R_j) > \partial_{u_i} w_i(u(x)).\partial_{R_i} v_i(p, R_i)$. Assumption 2(ii) implies in turn that $R_j > R_i$ for all dart (i, j) of Γ, which is impossible. ■

A3. Proof of Theorem 4

LEMMA 7: *Let $x \in M \cap \mathbb{R}^{ln}_{++}$. $\Omega(x) - \{x\}$ is a nonempty closed convex cone.*

Proof: Convexity and closedness are simple consequences of the definition of $\Omega(x)$. Nonemptiness follows from Theorem 2, which implies that $x \in \Omega(x)$ whenever $x \in M \cap \mathbb{R}^{ln}_{++}$. Consider now some $\omega^* \in \Omega(\mathrm{x})$, some nonnegative real number α, and let us prove that $\alpha(\omega^* - x) \in \Omega(x) - \{x\}$. Denote $\tau_{ij} = t_{ji} - t_{ij}$; $\tau_i = (\tau_{ij})_{j>i} = (\tau_{ii+1}, \ldots, \tau_{ij}, \ldots, \tau_{in})$. The range of the linear function f, which is $t \to (\tau_1, \ldots, \tau_i, \ldots, \tau_{n\text{-}1})$, defined on $\Pi_{i\in N} T_i$, is $\mathbb{R}^{l(n\text{-}1)(n\text{-}2)/2}$. The rank of the linear function $(\tau_1, \ldots, \tau_i, \ldots, \tau_{n\text{-}1}) \to (\Delta_1 t, \ldots, \Delta_n t)$, where $t \in f^{-1}(\tau)$, is $l(n-1) \leq l(n-1)(n-2)/2$. Therefore, for all $\omega \in \mathbb{R}^{ln}$ such that $\sum_{i\in N} \omega_i = e$ there exists a gift vector $t \geq 0$ such that $t_{ii} = 0$ and $x_i - \omega_i = \Delta_i t$ for all i. Let t^* be such a gift vector for ω^*, and consider gift vector αt^*. We then have $\alpha t^* \geq 0$; $\alpha t^*_{ii} = 0$, and $\alpha(x_i - \omega^*_i) = \Delta_i \alpha t^*$ for all i. Therefore, $\alpha(\omega^* - x) \in \Omega(x) - \{x\}$ and $\Omega(x) - \{x\}$ is a cone. ■

LEMMA 8: *For all $p \in \mathbb{R}^l$ and all $t \in \prod_{i\in N} T_i$, there is a $t' \in \prod_{i\in N} T_i$ such that $g(t')$ is a forest and $p\Delta_i t' = p\Delta_i t$ for all i.*[17]

Proof: Consider a circuit $\Gamma = ((i_k, j_k))_{1\leq k\leq m}$ of $g(t)$.

Suppose without loss of generality that $pt_{i_1 j_1} = \min_k pt_{i_k j_k}$, and define recursively the following two orientation classes of the darts of Γ: (i_1, j_1) has positive orientation; (i_{k+1}, j_{k+1}) has positive (respectively, negative) orientation if either (i_k, j_k) has positive orientation and $j_k = i_{k+1}$ (respectively, $j_k = j_{k+1}$) or (i_k, j_k) has negative orientation and $j_k = j_{k+1}$ (respectively, $j_k = i_{k+1}$) (with the usual convention that $(i_{m+1}, j_{m+1}) = (i_1, j_1)$). The adjacent darts (i_k, j_k) and (i_{k+1}, j_{k+1}) thus have identical (opposite) orientations in the circuit if the head of the former coincides with the tail (head) of the latter. This ori-

[17]This lemma was established in Mercier Ythier (1992), as the first step in the proof of Proposition 3.

entation is well defined, for if a dart had simultaneously a positive and negative orientation, then this should be the case of all darts by the recursive definition above, which would imply in turn that Γ has a single vertex i and a single dart (i, i), which contradicts the definition of $g(t)$.

There exists a gift vector $t^1 \in \prod_{i \in N} T_i$ such that: $pt^1_{i_k j_k} = pt_{i_k j_k} - pt_{i_1 j_1}$ whenever (i_k, j_k) has positive orientation in Γ; $pt^1_{i_k j_k} = pt_{i_k j_k} + pt_{i_1 j_1}$ whenever (i_k, j_k) has negative orientation in Γ; and $t^1_{i_k j_k} = t_{i_1 j_1}$ whenever i or j is not a vertex of Γ. And one verifies readily that $g(t^1)$ does not contain circuit Γ (dart (i_1, j_1) has been deleted). Moreover, $p\Delta_i t^1 = p\Delta_i t$ for all i because of the following: $\Delta_i t^1 = \Delta_i t$ whenever i is not a vertex of Γ; if i is a common vertex of two adjacent darts (j, i) and (i, k) of identical, positive (negative) orientation in Γ, then $p(t^1_{ik} - t^1_{ji}) = pt_{ik} - pt_{i_1 j_1} - pt_{ji} + pt_{i_1 j_1} = pt_{ik} - pt_{ji}$ (respectively, $p(t^1_{ik} - t^1_{ji}) = pt_{ik} + pt_{i_1 j_1} - pt_{ji} - pt_{i_1 j_1} = pt_{ik} - pt_{ji}$); and if i is a common vertex of two adjacent darts (j, i) and (k, i) of opposite orientations in Γ, the orientation of (j, i) being positive (negative), then $p(t^1_{ji} + t^1_{ki}) = pt_{ji} - pt_{i_1 j_1} + pt_{ki} + pt_{i_1 j_1} = pt_{ji} + pt_{ki}$ (respectively, $p(t^1_{ji} + t^1_{ki}) = pt_{ji} + pt_{i_1 j_1} + pt_{ki} - pt_{i_1 j_1} = pt_{ji} + pt_{ki}$). The conclusion then follows from a recursive application of the algorithm above to all circuits of $g(t)$ (in finite number since $g(t)$ is finite). ■

Proof of Theorem 4: Part (i) of Theorem 4 is a simple corollary of Theorems 1 and 2. Let $x \in M \cap \mathbb{R}^{ln}_{++}$. Part (iii) is a straightforward consequence of part (ii). In view of Lemma 7, we only have to prove, therefore, that convex set $\Omega(x)$ has dimension $l(n - c(\gamma(x))$.

Let p be the unique price vector of S_l which supports x (cf. Lemma 2). Denote by Ψ the set of spanning forest subdigraphs of $\gamma(x)$, and, for all $\Gamma \in \Psi$, let $\Omega_\Gamma(x)$ be the convex set $\{\omega \in \mathbb{R}^{ln} | \exists\, t \in \prod_{i \in N} T_i$ such that: $t_{ij} > 0$ if and only if $(i, j) \in \Gamma$; and $px_i = p(\omega_i + \Delta_i t)$ for all $i\}$. We have then $\Omega(x) = \bigcup_{\Gamma \in \Psi} \Omega_\Gamma(x)$ since, by Lemma 8, the wealth transfers associated with any gift vector can be achieved by a gift vector whose associated graph is a forest subgraph of the former. From the definition of a spanning subgraph, we know that $c(\Gamma) \geq c(\gamma(x))$ for all $\Gamma \in \Psi$. And from Tutte (1984, Thm. I.36), there exists a $\Gamma \in \Psi$ such that $c(\Gamma) = c(\gamma(x))$. It suffices to prove, therefore, that convex set $\Omega_\Gamma(x)$ has dimension $l(n - c(\Gamma))$ whenever Γ is a spanning forest subdigraph of $\gamma(x)$ such that $c(\Gamma) = c(\gamma(x))$.

Consider thus, from now on, a $\Gamma \in \Psi$ such that $c(\Gamma) = c(\gamma(x))$. By definition of a spanning graph, the set of vertices of Γ is N. By definition of a forest, we must have $i \neq j$ whenever $(i, j) \in \Gamma$ (loop-darts (i, i) are 1-circuits). The *incidence matrix* of Γ, denoted M_Γ, is defined in the following way: to every dart (i, j) of Γ, ranked lexicographically (as in a gift vector t), there corresponds one column of M_Γ; to every vertex i of Γ, ranked in increasing order, there corresponds one row of M_Γ; the entries of column (i, j) are, respectively, -1 on row i, $+1$ on row j, and 0 on the other rows. A well-known result of graph theory is then that matrix M_Γ has full rank $n - c(\Gamma)$, equal to the number of darts of Γ, if and only if Γ is a forest graph (Berge 1970, Thm. 1).

For any $t \in \prod_{i \in N} T_i$ such that $t_{ij} > 0$ if and only if $(i, j) \in \Gamma$, denote t_Γ the vector obtained from t by deleting its components t_{ij} such that $(i, j) \notin \Gamma$. The product $t_\Gamma .\ M_\Gamma^T$ of the row vector t_Γ by the transpose M_Γ^T of the incidence matrix of Γ is then the vector of net transfers $(\Delta_1 t, \ldots, \Delta_n t)$. Denoting $px = (px_1, \ldots, px_n)$, we have therefore $\Omega_\Gamma(x) = \{\omega \in \mathbb{R}^{ln} | \exists t \in \prod_{i \in N} T_i$ such that: $t_{ij} > 0$ if and only if $(i, j) \in \Gamma$; and $px = p(\omega + t_\Gamma . M_\Gamma^T)\}$.

Since Γ has exactly $n - c(\gamma(x))$ darts the dimension of convex set $\{t \in \prod_{i \in N} T_i | t_{ij} > 0$ if and only if $(i, j) \in \Gamma\}$ is $l(n - c(\gamma(x)))$. From this and the fact that $p \neq 0$ and rank $M_\Gamma^T = n - c(\gamma(x))$, it follows readily that the dimension of $\Omega_\Gamma(x)$ is $l(n - c(\gamma(x)))$. ■

References

ALTIG, D., and S. J. DAVIS (1993) Borrowing constraints and two-sided altruism with an application to social security, *Journal of Economic Dynamics and Control* **17**, 467–494.

ANDREONI, J. (1989) Giving with impure altruism: Applications to charity and Ricardian equivalence, *Journal of Political Economy* **97**, 1447–1458.

ARCHIBALD, G. C., and D. DONALDSON (1976) Non-paternalism and the basic theorems of welfare economics, *Canadian Journal of Economics* **9**, 492–507.

ARROW, K. J., and A. C. ENTHOVEN (1961) Quasi-concave programming, *Econometrica* **29**, 779–800.

ARROW, K. J., and HAHN, F. H. (1971) *General Competitive Analysis.* San Francisco: Holden-Day.

AUTEN, G., and D. JOULFAIAN (1996) Charitable contributions and intergenerational transfers, *Journal of Public Economics* **59**, 55–68.

BARRO, R. J. (1974) Are government bonds net wealth? *Journal of Political Economy* **82**, 1095–1117.

BECKER, G. S. (1974) A theory of social interactions, *Journal of Political Economy* **82**, 1063–1093.

BERGE, C. (1970) *Graphes et Hypergraphes.* Paris: Dunod.

BERGSTROM, T. C. (1970) A "Scandinavian consensus" solution for efficient income distribution among non-malevolent consumers, *Journal of Economic Theory* **2**, 383–398.

BERGSTROM, T. C., L. BLUME, and H. VARIAN (1986) On the private provision of public goods, *Journal of Public Economics* **29**, 25–50.

BERNHEIM, B. D., and K. BAGWELL (1988) Is everything neutral? *Journal of Political Economy* **96**, 308–338.

COX, D., and G. JAKUBSON (1995) The connection between public transfers and private interfamily transfers, *Journal of Public Economics* **57**,129–167.

DEBREU, G. (1952). A social equilibrium existence theorem, *Proceedings of the National Academy of Sciences* **38**, 886–893.

GALE, W. G., and J. K. SCHOLZ (1994). Intergenerational transfers and the accumulation of wealth, *Journal of Economic Perspectives* **8**, 145–160.

HOCHMAN, H. M, and J. D. RODGERS (1969) Pareto optimal redistribution, *American Economic Review* **59**, 542–557.

KOLM, S. C. (1966) La production optimale de justice sociale; in *Economie Publique*, pp. 109–177. Paris: Editions du CNRS, 1968. [The optimal production of social justice; in *Public Economics*, pp. 145–200. London: MacMillan, 1969.]

KOTLIKOFF, L. J., and L. H. SUMMERS (1981) The role of intergenerational transfers in aggregate capital accumulation, *Journal of Political Economy* **89**, 706–732.

KRANICH, L. (1988) Altruism and efficiency: A welfare analysis of the Walrasian mechanism with transfers, *Journal of Public Economics* **36**, 369–386.

LEMCHE, S. Q. (1986) Remarks on non-paternalism and the second theorem of welfare economics, *Canadian Journal of Economics* **19**, 270–280.

MERCIER YTHIER, J. (1989) Equilibre général et don; Thèse de Doctorat en Sciences Economiques, Institut d'Etudes Politiques de Paris.

MERCIER YTHIER, J. (1992) Existence of a general equilibrium of individual gifts; *Document de travail du LAMIA*, Université de Paris (Panthéon-Sorbonne).

MERCIER YTHIER, J. (1993) Equilibre général de dons individuels, *Revue Economique* **44**, 925–950.

MERCIER YTHIER, J. (1998a) The distribution of wealth in the liberal social contract. *European Economic Review* **42**, 329–347.

MERCIER YTHIER, J. (1998b) Existence of an equilibrium of property rights, *Mathematical Social Sciences* **35**, 261–272.

MERCIER YTHIER, J. (1998c) Regular distributive social systems; Document de travail du GREQAM, Centre National de la Recherche Scientifique et Ecole des Hautes Etudes en Sciences Sociales.

MUSGRAVE, R. (1959) *The Theory of Public Finance.* New York: MacGraw-Hill.

NAKAYAMA, M. (1980) Nash equilibria and Pareto optimal income redistribution, *Econometrica* **48**, 1257–1263.

PARETO, V. (1916) *Traité de Sociologie Générale;* in *Oeuvres Complètes*, vol. 12. Genève: Droz, 1966.

POSTLEWAITE, A. (1979) Manipulation via endowments, *Review of Economic Studies* **46**, 255–262.

RAMSEY, F. P. (1928) A mathematical theory of saving, *Economic Journal* **38**, 543–549.

ROBERTS, R. D. (1984) A positive model of private charity and public transfers, *Journal of Political Economy* **92**, 136–148.

STARK, O. (1993) Nonmarket transfers and altruism, *European Economic Review* **37**, 1413–1424.

TUTTE, W. T. (1984) *Graph Theory;* in *Encyclopedia of Mathematics and its Applications,* vol. 21, Gian-Carlo Rota, ed. Reading, MA: Addison-Wesley.

WINTER, S. J. Jr. (1969) A simple remark on the second optimality theorem of welfare economics, *Journal of Economic Theory* **1**, 99–103.