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A limit theorem on the dual core of a distributive social system

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> "Nous sommes conduits (…) à nous demander si divers aspects de la vie sociale (…) ne consistent pas en phénomènes dont la nature rejoint celle même du langage"
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> C. Levi-Strauss, Anthropologie Structurale, 1958

Abstract. I consider abstract social systems where the distribution of wealth is an object of common concern. I study, in particular, the systems where liberal distributive social contracts consist of the Pareto-efficient distributions that are unanimously preferred to the initial distribution. I define a *Dual Distributive Core* from a process of decentralized auction on the budget shares of Lindahl associated with net transfers, operated by coalitions aiming at increasing the value of the public good for their members while maintaining their utility levels. I establish that the dual distributive core converges, as the number of distributive agents becomes large relative to the number of agent types, to a typically finite number of distributive liberal social contracts, which correspond to the Lindahl equilibria that are unanimously preferred to the initial distribution. This process of decentralized auction provides a theoretical foundation for contractual policies of redistribution. The comparison with the usual notion of core with public goods (Foley 1970) yields the following results in this context: the Foley-core is a subset, generally proper, of the set of liberal distributive social contracts; it does *not* contain, in general, distributive Lindahl equilibria.

Introduction

The liberal distributive social contract is a putative contract between all individual members of the society, which consists of the distributions of individual wealth endowments that bear the following two features: they are unanimously preferred to the distribution of wealth that prevailed prior contractual redistribution; and individuals or groups do not want to make gifts anymore, once contractual redistribution has been performed (Kolm 1985, Chap. 19; Mercier Ythier 1998). Such contracts are susceptible to account for benevolent gift-giving like, for instance, redistribution by charities, public assistance or international aid.

When individuals have the same opinions on the desirable orientation of redistribution (for instance, when they think that wealth transfers, if any, should flow from the wealthier to the less wealthy), the liberal distributive social contracts coincide with the Pareto efficient distributions which are unanimously preferred to the original one (Mercier Ythier 2000b, Theorem 4.3, and 1998, Theorem 1). The liberal distributive social contract suffers therefore from the same problem of indeterminacy as Edgeworth's "contract curve" (1881) in the context of market exchange.

The solution elaborated by Edgeworth in his *Mathematical Psychics* (1881) and later on developed by Debreu and Scarf (1963) is well-known: the contract curve shrinks to the set of competitive market equilibria as the number of consumer types becomes small relative to the number of consumers per type. The present article establishes an analogous result for liberal distributive social contracts, which can be formulated synthetically as follows: the *dual distributive core* converges to the set of *social contract equilibria* as the number of distributive types becomes small relative to the number of distributive agents per type.

The essence of the argument is the following. The vector of individual net transfers is viewed as a public good (Kolm 1966). Pareto efficient individual net transfers are associated with *distributive values* which correspond to the shares of individuals in individual net transfers. I imagine that coalitions can block Pareto efficient net transfers by proposing a vector of shares of their members in overall net transfers which increases the value of the public good for all of them, while maintaining their utility levels. The dual distributive core is made of the Pareto efficient vectors of net transfers which are unblocked in that sense by any coalition. It converges to a subset of liberal distributive social contracts as the number of agents per type is increased to infinity. The limit set consists of the social contract equilibria, which correspond to a relevant variant of the distributive equilibrium of Lindahl-Bergstrom (1970). In short, a (typically) finite number of liberal distributive social contracts emerges from an exchange of shares between distributive agents in infinite number, in the same way as a finite number of vectors of competitive market prices emerges from an exchange of endowments between consumers in infinite number.

The exchange process analyzed here is conducted on values. It is, in other words, an exchange of signals, that is, a process of communication between distributive agents. A contractual distribution of property rights (or a finite number of them) is achieved by the sole virtue of this communication process. This article elicits thus theoretical foundations for an operational equilibrium concept which applies to the distributive aspects of the so-called "contractual policy". It resorts to a theory of value and social equilibrium understood as the outcome of a process of social communication among human beings, that is, essentially, as a phenomenon of language.

The paper is organized as follows. Section 1 defines the liberal social contract and states the indeterminacy problem. Section 2 defines the social contract equilibrium and studies its properties of optimality and existence. Section 3 defines and characterizes the dual distributive core. Section 4 specifies the procedure of replication of the root social system. Section 5 establishes the convergence property. The conclusion interprets the results.

1 Liberal distributive social contract

1.1 Pure distributive social system

I consider the following abstract social system ("pure distributive social system": Mercier Ythier 1993, 1998a). It is made of individuals, who are private owners of money wealth. These agents enjoy a full right of private property (jus utendi et abutendi) over their own wealth. They are liable, in particular, to transfer it to others by means of gifts.

Gift-giving consists of benevolent wealth transfers, revealing an altruistic concern of the donor for the beneficiary. It can be individual or collective. Collective gift-giving is performed by coalitions of agents pooling their individual ownerships (the endowments of members plus the gifts received by them) and deciding on members' consumption and gifts. The collective decision rule of coalitions is the weak unanimous preference: gifts are performed by a coalition if and only if the resulting distribution of wealth is preferred by some members at least and is vetoed by none.

The interactions of these individuals and/or coalitions of benevolent donors are modelled in the simplest conceivable way: any set of agents takes the gifts of its complement as fixed, which means that individuals and coalitions ignore or neglect the possibility of a strategic dependence of the gifts of others on their own gifts; agreements are nonbinding, which implies that only self-sustainable agreements are allowed to survive; and there is no limit on communication between agents whatsoever.

Formally, individuals are denoted by an index i running in $N = \{1,\ldots,n\}$. Wealth is divisible, and its aggregate amount is assumed independent of individual consumption and transfer decisions. The share $\omega_i \in [0,1]$ of total wealth owned by individual i prior consumption or transfer is his *initial endowment* or *right*. A *consumption* x_i of individual i is the money value of his consumption of commodities. A *gift* t_{ij} *from individual i to individual j* ($j \neq i$) is a nonnegative money transfer from individual i's estate (his initial ownership plus the gifts he received from others) to individual j's. A *gift-vector of individual i* is a vector $t_i = (t_{ij})_{j \in N \setminus \{i\}}$ of $\mathbb{R}_+^{n-1}$. We ignore alternative uses of wealth, like disposal or production, so that the following accounting identity is verified for all i: $x_i + \sum_{j:j \neq i} t_{ij} = \omega_i + \sum_{j:j \neq i} t_{ji}$.

Individuals have preferences on the final distribution of wealth, that is, on the vectors of individual consumption expenditures. Denoting $x = (x_1, \ldots, x_n)$ such vectors, individual i is endowed with a *distributive utility function* $w_i : x \rightarrow w_i(x)$, defined on the space of consumption distributions $\mathbb{R}^n$.

A distribution of initial rights $(\omega_1, \ldots, \omega_n)$ is denoted by ω. It is an element of the unit simplex $S_n = \{x \in \mathbb{R}_+^n \mid \sum_i x_i = 1\}$ of $\mathbb{R}^n$. The elements of S_n are the *feasible* distributions of wealth. A *social system w* is a vector $(w_1, \ldots, w_n)$ of individual utility functions. A *social system of private property* is a pair (w, ω).

A distribution x is *strongly Pareto efficient in w* (is a *strong distributive optimum of w*) if it is feasible and if there is no feasible distribution x' such that $w_i(x') \geq w_i(x)$ for all i and $w_i(x') > w_i(x)$ for at least one i.

A *gift-vector t* is a vector $(t_1, \ldots, t_n)$. For all nonempty subset I of N: t_I (resp. $t_{\setminus I}$) is the vector of gifts obtained from t by deleting t_i for all $i \notin I$ (resp. for all $i \in I$) ;$(t_{\setminus I}, t_I^*)$ is the gift-vector obtained from t and t^* by substituting t_i^* for t_i in gift-vector t for all $i \in I$. $\Delta_i t$ is the net transfer $\sum_{j:j \neq i}(t_{ji} - t_{ij})$ accruing to individual i when t is the gift-vector. $x(\omega, t)$ is the vector of individual consumption expenditures $(\omega_1 + \Delta_1 t, \ldots, \omega_n + \Delta_n t)$, that is, given the accounting identity above, the unique consumption distribution associated with the distribution of rights ω and the gift-vector t. $x_i(\omega, t)$ is the i-th projection $pr_i x(\omega, t) = \omega_i + \Delta_i t$.

The gift-vector t is *weakly blocked* by coalition I in the social system of private property (w, ω) if there exists a t_I^* such that, for all $i \in I$: (i) $w_i(x(\omega, (t_{\setminus I}, t_I^*))) \geq w_i(x(\omega, t))$, with a strict inequality for at least one i; (ii) and $x_i(\omega, (t_{\setminus I}, t_I^*)) \geq 0$.

Finally, the liberal distributive social contract of a pure distributive social system of private property (w, ω^0) is defined as follows:

Definition 1. (i) *ω is in the strong distributive core of w if* 0 *is weakly blocked by no coalition in* (w, ω). (ii) *ω is a strong liberal distributive social contract* of (w, ω^0) *if*: (*a*) *ω is in the strong distributive core of w* ; (*b*) *and* $w_i(\omega) \geq w_i(\omega^0)$ *for all i.*

All subsequent references to the distributive optimum, distributive core and distributive social contract use the strong versions of these concepts as stated in the definitions above[1]. The adjective *strong* appending to them will therefore be dropped in the sequel for the sake of brevity.

[1] Mercier Ythier 1998a, 1998b define the liberal distributive social contract and the distributive core from the following stronger notion of a blocking coalition, requiring that deviations benefit *all of* the members of the deviating coalitions (as in Aumann 1959): gift-vector t is *blocked* by coalition I in the social system of private property (w, ω) if there exists a t_I^* such that, for all $i \in I$: (i) $w_i(x(\omega, (t_{\setminus I}, t_I^*))) > w_i(x(\omega, t))$; (ii) and $x_i(\omega, (t_{\setminus I}, t_I^*)) \geq 0$.

I consider from now on a fixed social system of private property (w,ω^0), and denote by: P^* the set of distributive optima of w ; C^* the distributive core of w ; L^* its set of liberal distributive social contracts. It follows readily from definitions that $L^* \subset C^* \subset P^*$.

1.2 The liberal social contract and usual notions of core

The distributive core is defined above, somewhat unconventionally, as a set of *distributions of rights* (i.e., vectors of individual endowments) that are unblocked by any subset of agents. A more conventional notion of core would define it as a set of unblocked *outcomes*, that is, in the present context, of unblocked *consumption distributions*.

The notion of blocking coalition or action can be defined in several different ways also. Two of them appear particularly suitable in the present setting. They are embodied respectively in the definition of the strong distributive equilibrium and core and Foley distributive equilibrium and core of a social system *of private property*:

Definition 2. (i) *t is a strong distributive equilibrium* of (w,ω) *if it is weakly blocked by no coalition in* (w,ω). (ii) *The Aumann distributive core of* (w,ω) *is* $\{x(\omega,t)$:*t is a strong distributive equilibrium of* $(w,\omega)\}$. (iii) *t is Foley-blocked by coalition I in* (w,ω) *if there exists a t_I^* such that, for all $i \in I$:* $w_i(x(\omega,(0_{\setminus I},t_I^*))) \geq w_i(x(\omega,t))$, *with a strict inequality for at least one i; and* $x_i(\omega,(0_{\setminus I},t_I^*)) \geq 0$. (iv) *$t$ is a Foley distributive equilibrium of* (w,ω) *if it is Foley-blocked by no coalition in* (w,ω). (*v*) *The Foley distributive core of* (w,ω) *is* $\{x(\omega,t)$:*t is a Foley distributive equilibrium of* $(w,\omega)\}$.

The strong distributive equilibrium and the notion of core derived from it are applications of the general notion of strong Nash equilibrium (Aumann 1959) to the context of the pure distributive social systems. A strong distributive equilibrium is a gift-vector such that no coalition, taking the gifts of its complement as fixed, can cooperatively deviate in a way that benefits all of its members[2]. The application of the strong equilibrium to distributive social systems raises the same type of difficulties as in many other contexts: a strong distributive equilibrium such that *not all*

[2] It is required here, more precisely, as in Debreu and Scarf (1963) that the deviation makes no member worse off, and benefits at least one of them. This notion of strong equilibrium is stronger than the notion of Aumann (1959). The same comment holds for the notions of Foley-blocking coalition and Foley equilibrium defined here, as compared to their definitions in Foley (1970). Foley requires that a deviation benefits all members of the coalition, while I only require here that it benefits some of these members and makes none of them worse off.

individual gifts are null almost never exists (Mercier Ythier, 2000b, Theorems 4.1 and 4.2)[3].

The Foley distributive core translates, likewise, the notion of core with public goods of Foley (1970) into the context of pure distributive social systems. It embodies a restriction on admissible deviations, stating essentially that the provision of public goods by a deviating coalition must be feasible from the sole endowments of its members. Foley's notion of core was originally conceived in a context where public goods are essentially distinct from private ones, and notably produced from the latter. Its application to pure distributive social systems raises basic conceptual issues: all "goods" (individual consumption or transfers) are simultaneously public (as objects of common concerns) and private (as objects of private property) in pure distributive social systems. In particular, the right of private property allows individuals to consider the gifts they receive from others as their own wealth, and use them freely for their own consumption or gifts. This implies notably that the gifts received by the members of a coalition make a part of the resources that the coalition can factually but also legitimately use to deviate, a possibility that the notion of core of Foley assumes away by construction.

The following two theorems establish simple relations between these notions of equilibrium and core and the set of liberal social contracts:

Theorem 1. (*i*) *0 is a Foley distributive equilibrium of* (w,ω) *if and only if 0 is a strong distributive equilibrium of* (w,ω). (*ii*) *If* w_i *is quasi-concave for all i and strictly quasi-concave*[4] *for some i, and if* 0 *is a Foley equilibrium of* (w,ω), *then the Foley core of* (w,ω) *is* $\{\omega\}$.

Proof. (i) is an immediate consequence of definitions.

(ii) Let 0 and $t\neq 0$ be Foley equilibria of (w,ω). Then $w_i(x(\omega,t)) \geq w_i(\omega)$ whatever i, for coalition $\{i\}$, playing 0, Foley-blocks t if $w_i(x(\omega,t)) < w_i(\omega)$. But then $w_i(x(\omega,t)) = w_i(\omega)$ whatever i, for otherwise the grand coalition $N = \{1,\ldots,n\}$, playing t, would Foley-block 0. The quasi-concavity assumption, and the fact that $t \rightarrow x(\omega,t)$ is affine, jointly imply then that $x(\omega,t) = \omega$: otherwise, t and 0 would be Foley-blocked by the grand coalition N playing any strictly convex combination of them, that is, any λt such that $\lambda \in]0,1[$. ■

[3] A notable exception is the type of situation considered by Becker (1974) where the equilibrium distribution coincides with the distribution that maximizes the utility of the single donor in the whole set of feasible distributions. Becker's analysis can be appropriate in the context of family redistribution, but will generally not be so in the context of charitable redistribution, notably because the beneficiaries of charity usually receive supports from multiple donors (see for instance Musgrave 1969, or Mercier Ythier 2000a: [10]).

[4] w_i is quasi-concave if for all real number $\lambda \in [0,1]$, and all (x,x'), $w_i(x') \geq w_i(x)$ implies $w_i(\lambda x + (1-\lambda)x') \geq w_i(x)$. It is strictly quasi-concave if for all real number $\lambda \in]0,1[$, and all (x,x') such that $x \neq x'$, $w_i(x') \geq w_i(x)$ implies $w_i(\lambda x' + (1-\lambda)x) > w_i(x)$.

Theorem 2. *Let w be fixed, and C^* and P^* denote its distributive core and its set of distributive optima respectively.* (i) $C^* = \cup_{\omega \in S_n}\{x(\omega,t)$:*t is a strong distributive equilibrium of* $(w,\omega)\}$. (ii) $C^* = \{\omega \in S_n$:0 *is a Foley distributive equilibrium of* $(w,\omega)\}$; *moreover, if* w_i *is quasi-concave for all i and strictly quasi-concave for some i, then* $C^* = \cup_{\omega \in C^*}\{x(\omega,t)$:*t is a Foley distributive equilibrium of* $(w,\omega)\}$. (*iii*) *If* $C^* = P^*$, *then: for all* $\omega \in S_n$, *the Foley distributive core of* (w,ω) *is contained in the set of liberal distributive social contracts of* (w,ω).

Proof. (i) C^* is contained in $\cup_{\omega \in S_n}\{x(\omega,t)$:$t$ is a strong distributive equilibrium of $(w,\omega)\}$ by definition. Note that, as a simple consequence of definitions: if t is a strong equilibrium of (w,ω), then 0 is a strong equilibrium of $(w,x(\omega,t))$.Therefore $\cup_{\omega \in S_n}\{x(\omega,t)$:$t$ is a strong distributive equilibrium of $(w,\omega)\} \subset C^*$.

(ii) is a straightforward consequence of Theorem 1.

(iii) Suppose now that $C^* = P^*$, let ω^0 be a fixed element of S_n and let L^* denote the set of liberal social contracts of (w,ω^0). We have $L^* = \{\omega \in P^*: w_i(\omega) \geq w_i(\omega^0)$ for all $i\}$. Let t be a Foley equilibrium of (w,ω^0). Then $x(\omega,t)$ is in P^*, for t is not Foley-blocked by coalition N. And any distribution ω such that $w_i(\omega) < w_i(\omega^0)$ is Foley-blocked by coalition $\{i\}$. Hence the conclusion. ■

Status quo strong equilibrium is generically unique as a consequence of the Theorem 4 of Mercier Ythier (2004) that is: if 0 is a strong equilibrium of (w,ω), then, generically, 0 is its sole strong equilibrium. Theorem 1 states therefore, essentially, that the Aumann and Foley distributive cores of a social system of private property (w,ω) that has status quo as an equilibrium reduce both to the initial distribution ω.

The distributive core of a social system w is identical with the union $\cup_{\omega \in S_n}\{x(\omega,t)$:$t$ is a strong distributive equilibrium of $(w,\omega)\}$ of the family of Aumann distributive cores of the social systems of private property (w,ω) associated with all feasible initial distributions (Theorem 2(i)). The distributive core of a social system w coincides, in other words, with the set of wealth distributions that are in the Aumann distributive core of (w,ω) for some distribution of rights ω.

The distributive core of w is identical, also, with the set of initial distributions that are not Foley-blocked by any coalition (Theorem 2(ii)). Moreover, if status quo is a Foley equilibrium of (w,ω), the Foley core of (w,ω) reduces to the distribution of rights ω (Theorem 1(ii)). The distributive core of w can be viewed therefore, alternatively, as the union of the family of Foley cores of the social systems of private property (w,ω) where the Foley core reduces to the initial distribution.

In the social systems where the distributive core coincides with the set of distributive optima, the set of liberal social contracts of any associate system of private property is characterized as the set of Pareto efficient distributions unanimously preferred to the initial distribution, and contains therefore the Foley core of this system (Theorem 2(iii)). The Foley core is then, generally, a *proper* subset of the set of liberal social contracts, as demonstrated in the example below.

Example 1. Consider the social system of three agents with Cobb-Douglas utility functions defined by: $w_i(x) = x_1^{\beta_{i1}} x_2^{\beta_{i2}} x_3^{\beta_{i3}}$, with $\beta_{ii} = 1/2$ and $\beta_{ij} = 1/4$ for all $i = 1, 2, 3$, and all $j \neq i$. Let $\beta^i = (\beta^{i1}, \beta^{i2}, \beta^{i3})$. From Mercier Ythier (1998b) Theorems 4 and 5: $C^* = P^* = \mathrm{co}\{\beta^1, \beta^2, \beta^3\}$, the convex hull of $\{\beta^1, \beta^2, \beta^3\}$ (cf. Fig. 1).

Let $\omega^0 = (0.02,0.49,0.49)$. Let L^* denote the set of liberal social contracts of (w,ω^0). $L^* = P^*$ since $C^* = P^*$ and $w_i(x) \geq w_i(\beta^j) = 2^{-1.75} > w_i(\omega^0)$ for all i, all $j \neq i$ and all $x \in P^*$.

Let F^* denote the Foley core of (w,ω^0). Since $\omega_1^0 < 1/4$ and $F^* \subset P^* \subset \{x \in S_3 : x_1 \geq 1/4\}$, agent 1 must receive a net transfer from agent 2 or 3 in the Foley core. Therefore, F^* coincides with the set of Pareto efficient distributions that are unblocked by coalitions {2}, {3} and {2,3}.

The set of distributions unblocked by {2,3} is the set of solutions of $\mathrm{Max}\{(w_2(\omega^0+\theta), w_3(\omega^0+\theta)) : \theta_1+\theta_2+\theta_3 = 0,\ \omega^0+\theta \geq 0\}$, that is, segment $[\beta^2, \beta^3]$. A distribution is unblocked by {2}if and only if the corresponding utility is $\geq \mathrm{Max}\{w_2(\omega^0+\theta):\ \theta_1+\theta_2+\theta_3 = 0,\ \theta_3 = 0,\ \omega^0+\theta \geq 0\}$, that is, $\geq w_2(0.17,0.34,0.49)$. Likewise, by symmetry, x is unblocked by {3} if and only if $w_3(x) \geq w_3(0.17,0.49,0.34)$. Let ω^2 and ω^3 denote the elements of $[\beta^2,\beta^3]$ such that $w_2(\omega^2) = w_2(0.17,0.34,0.49)$ and $w_3(\omega^3) = w_3(0.17,0.49,0.34)$ (one gets: $\omega^2 \simeq (0.25,0.303,0.447)$ and $\omega^3 \simeq (0.25,0.447,0.303)$).The utility of both

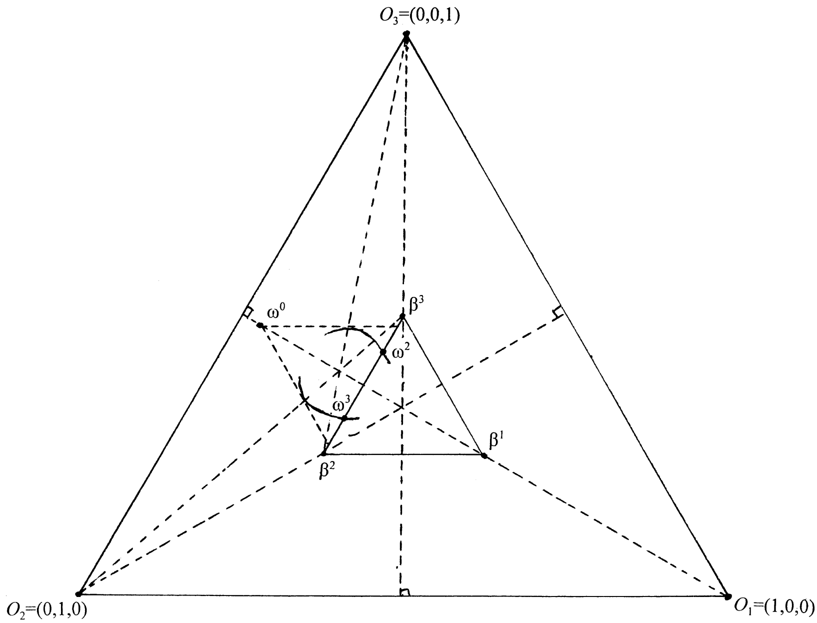

Fig. 1. Foley core and liberal social contracts

agent 2 and agent 3 is $\geq w_2(0.17,0.34,0.49) = w_3(0.17,0.49,0.34)$ everywhere in $[\omega^2,\omega^3]$. Therefore $F^* = [\omega^2,\omega^3]$ and is strictly contained in L^*.

1.3 Existence and indeterminacy of the distributive liberal social contract.

The condition $C^* = P^*$ is essentially equivalent to the existence of a liberal social contract of (w,ω^0) for all $\omega^0 \in S_n$ (Mercier Ythier 2000b, Theorem 4.3). It means that, whenever a distribution of rights is blocked by some coalition, it must be blocked by the grand coalition as well. There is, in other words, a unanimous agreement on the desirable direction of redistribution (while agents might disagree on its desirable magnitude).

This is verified, notably, in a context, particularly suitable for the analysis of voluntary redistribution, where individuals agree that redistributive transfers, if any, should flow downward, from the wealthier to the less wealthy (Mercier Ythier 1998a). Formally, let e_{ij} denote the element of $\mathbb{R}^n$ whose components are all equal to 0 except the i-th one, equal to -1, and j-th one, equal to $+1$, and suppose that:

Assumption 1. (i) $w_i(x+\tau e_{ij}) \leq w_i(x)$ for all $\tau \in \mathbb{R}_+$ whenever $x_j \geq x_i$. (ii) $w_i(x+\tau e_{jk}) \geq w_i(x)$ for all $\tau \in [0,\frac{1}{2}(x_j - x_k)]$ whenever j and k are distinct from i and $x_j \geq x_k$.

A simple consequence of the Theorem 4 of Mercier Ythier (1998b) is that $C^* = P^*$ whenever w verifies assumption 1[5].

When the distributive core coincides with the set of distributive optima, the set of liberal social contracts of a social system of private property is simply characterized as the set of Pareto efficient distributions unanimously preferred to the initial distribution of rights. The typical situation with respect to determinacy is then the following: either L^* reduces to the initial distribution of rights (status quo), or it is a continuum of distributions[6].

[5] This Theorem establishes that, with the assumption above, the (weak) distributive core C contains P^*. The proof of the theorem can be adapted straightforwardly to establish that P^* is contained in C^*. The result follows then from the fact, already noticed in Sect. 1 above, that C^* is contained in P^* by definition.

[6] With assumption 1, we have: $L^* = P^* \cap \{\omega \in S_n w_i(\omega) \geq w_i(\omega^0)$ for all $i\}$. P^* is diffeomorphic to the interior of the n-simplex when distributive preferences verify suitable convexity and regularity assumptions (Mercier Ythier 1997, Theorem 5). $\{\omega \in S_n | w_i(\omega) \geq w_i(\omega^0)$ for all $i\}$ is a convex set of dimension $n-1$, provided that its interior is nonempty and that preferences are convex. L^* is nonempty when utility functions are continuous and verify Assumption 1 (cf. the proof of Theorem 1 in Mercier Ythier 1998a). Combining the assumptions of these two theorems, the interior of L^*, therefore, is either empty or a differentiable manifold of dimension $n-1$.

2 Social contract equilibrium

The following notion of distributive equilibrium is a variant of the Lindahl distributive equilibrium (Bergstrom 1970). The *social contract equilibrium* can be viewed, informally, as the outcome of a collective decision process for wealth redistribution as a public good. The process consists, more precisely, of a coordination scheme for Pareto-efficient redistribution, which respects individual views on the initial distribution of property rights. It solves, first, the public good problem raised by utility interdependence in the following familiar way: the public good is the vector of net transfers $(x_1-\omega^0{}_1 ..,x_n-\omega^0{}_n)$; an auctioneer sets everybody's shares in individual net transfers, so as to maximize the social value of the public good; each individual chooses the vector of net transfers that maximizes his utility, subject to a "budget constraint" involving his "expenditure" in the public good (computed from his shares in net transfers as set by the auctioneer); equilibrium is a vector of net transfers that solves simultaneously the auctioneer's and individuals' programs. But the coordination scheme embodies, second, the possibility for every individual to veto any departure from the initial distribution ω^0: budget constraints are specified in such a way that status quo is an accessible choice for everybody, whatever the shares picked by the auctioneer.

Denote: θ_i the net transfer accruing to individual i (which can be a positive or negative number); $\theta = (\theta_1,\ldots,\theta_n)$; π_{ij} the "share" of individual i in θ_j (which can take on negative values); $\pi_i = (\pi_{i1},\ldots,\pi_{in})$; $\pi = (\pi_1,\ldots,\pi_n)$. Let $A = \{\pi\in \mathbb{R}^{n^2} | \sum_i \pi_i = e\}$, where e denotes the vector of $\mathbb{R}^n$ the components of which are all equal to 1, be the set of *vectors of shares*. The *social contract equilibrium* is then defined as follows:

Definition 3. (π,θ) *is a social contract equilibrium of* (w,ω^0) *if :* (*i*) $(\pi,\omega^0+\theta)\in A\times S_n$; (*ii*) *and, for all i,* θ *is a maximum of* $z\to w_i(\omega^0+z)$ *in* $B_i(\pi) = \{z\in \mathbb{R}^n : \omega_i^0+z_i \geq 0$ and $\pi_i.z \leq 0\}$.

The set of social contract equilibrium distributions of (w,ω^0) will be denoted by E^*.

2.1 Optimality of the social contract equilibrium

Theorem 3. (i) *Suppose that for all i, all* $x\in S_n$*, and all neighborhood V of x in* S_n*, there is an* $x'\in V$ *such that* $w_i(x') > w_i(x)$ (*local nonsatiation in* S_n)*. Then* $E^*\subset P^*$. (*ii*) *Suppose moreover that w verifies Assumption 1. Then* $E^*\subset L^*$.

Proof. (i) The proof of the first part uses a variant of the familiar argument (Debreu 1954, Theorem 1).

Suppose that (π,θ) is a social contract equilibrium of (w,ω^0) and that $\omega^0+\theta\notin P^*$. There is then a θ' such that $\omega^0+\theta'\in S_n$ and $w_i(\omega^0+\theta') \geq w_i(\omega^0+\theta)$ for all i, with a strict inequality for at least one i.

$w_i(\omega^0 + \theta') > w_i(\omega^0 + \theta)$ implies $\pi_i.\theta' > 0 \geq \pi_i.\theta$, since θ is a maximum of $z \to w_i(\omega^0 + z)$ in $B_i(\pi)$ by definition of an equilibrium.

Suppose that $w_i(\omega^0 + \theta') = w_i(\omega^0 + \theta)$. There is a sequence $\theta^q \to \theta'$ such that, for all q: $\omega^0 + \theta^q \in S_n$ and $w_i(\omega^0 + \theta^q) > w_i(\omega^0 + \theta') = w_i(\omega^0 + \theta)$ (local nonsatiation in S_n); hence $\pi_i.\theta^q > \pi_i.\theta$. Letting q tend to infinity, one gets $\pi_i.\theta' \geq \pi_i.\theta$.

Therefore $\sum_i \pi_i \theta' > \sum_i \pi_i.\theta = 0$. But $\sum_i \pi_i.\theta' = \sum_i \theta'_i = 0$ by definition of π and θ', a contradiction. ■

(ii) $C^* = P^*$ by Assumption 1 and the proof of the Theorem 4 of Mercier Ythier (1998b). $w_i(\omega^0 + \theta^*) \geq w_i(\omega^0)$ for all i by construction of the social contract equilibrium. The conclusion follows from Theorem 3(i) and the definition of the liberal distributive social contract. ■

The Pareto-efficiency of equilibrium distributions obtains easily under a suitable variant of the usual assumption of local nonsatiation of preferences (Theorem 3(i)). Moreover, the equilibrium distribution must be unanimously preferred to ω^0, as a consequence of the specification of budget constraints.[7] The optimality property of Theorem 3(ii) should come therefore as no surprise: the social contract equilibria of a social system of private property belong to its set of liberal social contracts whenever the social system verifies Assumption 1, as, more generally, whenever its distributive core coincides with its set of distributive optima.

There exists no such relation between the distributive equilibrium of Bergstrom and liberal social contracts, or between Lindahl equilibria of both types and the Foley core. The examples below display variants of the social system of benevolent Cobb-Douglas agents of Example 1 which verify Assumption 1 and where: the Foley core contains neither the social contract equilibrium nor the Bergstrom equilibrium (Example 2); the Foley core contains the social contract equilibrium but not the Bergstrom equilibrium (Example 3); L^* does not contain the Bergstrom equilibrium and, in particular, a majority prefers the initial distribution to the Bergstrom equilibrium (Example 3).

These examples imply that Foley's statement that Lindahl equilibria are in the core (Foley 1970, p. 6) is not verified in the context of pure distributive social systems. His notion of Lindahl equilibrium translates, in this context, into the distributive equilibrium of Bergstrom, while his notion of core translates into our definition 2(v). The public good of the distributive social

[7] This is not true, in general, with Bergstrom's version of the distributive equilibrium. His budget constraints read $\pi_i.x = \omega_i^0$, where π_{ij} denotes the share of individual i in j's consumption expenditure. The accessibility of ω^0 to individual i depends then on π_i, and equilibrium is generally not unanimously preferred to the initial distribution of rights, as established by the Example 1 of Mercier Ythier (1998a) and by the Example 3 below. Note that constraints $\pi_i.\theta = 0$ can receive the following natural interpretation at equilibrium: letting $t_{ij} = \pi_{ij}\theta_j$ for all (i,j) such that $i \neq j$ and remembering that $\pi \in A$, one can rewrite them equivalently as $\theta_i = \Sigma_{j,j\neq i}(t_{ji} - t_{ij})$, which can be viewed as the accounting identity defining the net transfer of agent i.

system is the distribution of wealth, or equivalently the vector of net transfers, which cannot be produced from private goods, for the simple reason that all "private" goods (individual wealth and transfers) are public (objects of common concern) in this setup. The assumption that public goods are produced from"pure" private goods does not hold, in general, in the distributive equilibrium of Lindahl-Bergstrom, while this condition is essential for the proof of Foley's theorem. We end up here again with the conclusion that Foley's setting does not fit in the general analysis of redistribution as a public good pretreated here.

Example 2. Let (w,ω^0) be the social system of private property of Example 1.

Let ω be a Bergstrom equilibrium distribution of (w,ω^0). From Bergstrom (1970, p. 387) one gets: $\omega_j = \sum_{i\in N} \beta_{ij}\omega_i^0$ for all j, hence $\omega = (0.255,0.3725,0.3725)$, so that ω is not in the Foley core $F^* = [\omega^2,\omega^3]$ computed in Example 1.

Let (π,θ) be a social contract equilibrium of (w,ω^0), and denote by ω' the corresponding equilibrium distribution $\omega^0+\theta$. $\omega'\in P^*$ by Theorem 3(i), hence is »0 (i.e. has all its components positive). The first-order conditions for a maximum of $z\rightarrow w_i(\omega^0+z)$ in $B_i(\pi) = \{z\in \mathbb{R}^n : \omega_i^0 + z_i \geq 0 \text{ and } \pi_i.z \leq 0\}$ imply then that, for all i, there exists a positive real number λ_i such that: $\partial w_i(\omega').\theta = \lambda_i\pi_i\theta = 0$. Moreover $\theta_1 = -(\theta_2+\theta_3) > 0$ since $\omega'_1 > 0 = \omega_1{}^0$. The symmetry of utility functions of 2 and 3 in the first-order conditions readily implies that $\theta_2 = \theta_3 < 0$, and therefore $\omega'_2 = 0.49 + \theta_2 = \omega'_3$. Suppose that ω' is in the Foley core and let us derive a contradiction. Since $\omega'_2 = \omega'_3$, we must have $\omega' = (1/4,3/8,3/8)$. And one verifies by direct computation that then $\partial w_1(\omega').\theta = (-(3/2)^{0.5}+(1/6)^{0.5})\theta_2 > 0$, the contradiction we were looking for.

Both the Bergstrom equilibrium and the social contract equilibrium imply, here, more redistribution from agents 2 and 3 to agent 1 than is compatible with their belonging to the Foley core.

Example 3. We maintain the social system w of Examples 1 and 2 but let now ω^0 be any element of P^*.

The strict quasi-concavity of utility functions in $\mathbb{R}^3_{++}$ readily implies that $\{x\in S_3 : w_i(x) \geq w_i(\omega^0)\} = \{\omega^0\}$. Therefore: $L^* = \{\omega^0\}$ as a consequence of Mercier Ythier, 1998b, Theorems 4 and 5; $F^* = \{\omega^0\}$ as a consequence of Theorem 2; and $E^* = \{\omega^0\}$ as a consequence of Theorem 3(i) and Lemmas 2 and 4.

Bergstrom's equilibrium distribution of (w,ω^0) is $\left(\sum_{i=1}^3 \beta_{i1}\omega_i^0, \sum_{i=1}^3 \beta_{i2}\omega_i^0\right.$ $\left.\sum_{i=1}^3 \beta_{i3}\omega_i^0\right)$. One verifies easily that this distribution is $\neq\omega^0$, hence not in L^* nor in F^* unless $\omega^0 = (1/3,1/3,1/3) = \omega^c$. More precisely, let ω^{ij} be the element of P^* such that $\omega_i^{ij} = 1/3$, $\omega_j^{ij} = 5/12$ and $\omega_k^{ij} = 1/4$ for all i, all $j\neq i$ and all $k\neq i$, j (cf. Fig. 2). We have then the following:

(i) exactly two agents i and j strictly prefer the initial distribution ω^0 to the corresponding equilibrium distribution of Bergstrom if and only if their initial endowments ω_i^0 and ω_j^0 are both $> 1/3$, that is, if and only if ω^0 is in the relative

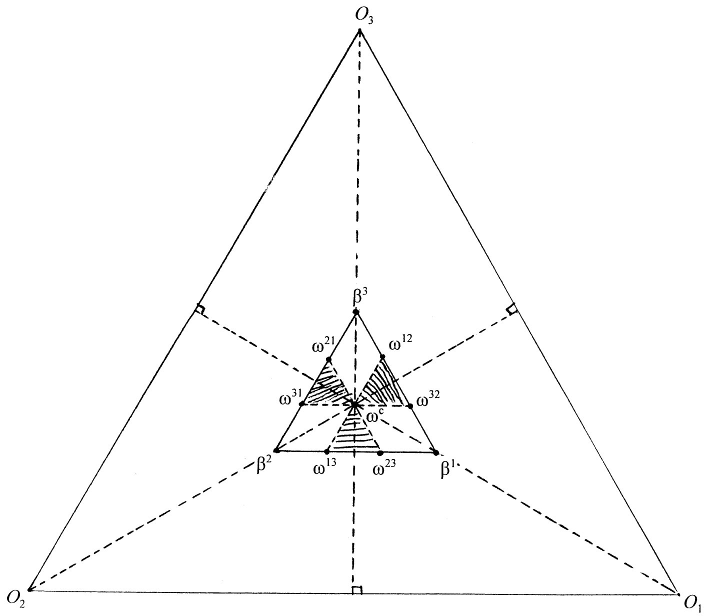

Fig. 2. Lindahl equilibrium and the liberal social contract

interior of co$\{\omega^{ij}, \omega^{ji}, \omega^{c}\}$ in P^{*}. In Fig. 2 therefore, the initial distribution is preferred to the Bergstrom distribution by a majority of agents if and only if it belongs to the interior of the hatched area (relative interior in P^{*}).

(ii) Exactly one agent strictly prefers the initial distribution to the Bergstrom distribution if and only if the endowments of others are both $< 1/3$, that is, if and only if the initial distribution belongs to the complement in $P^{*}\backslash\{\omega^{c}\}$ of the union of relative interiors in P^{*} of sets co$\{\omega^{ij}, \omega^{ji}, \omega^{c}\}$.

(iii) The Bergstrom distribution coincides with the social contract equilibrium distribution if and only if the initial distribution of all agents is 1/3, that is, if and only if $\omega^{0} = \omega^{c}$.

2.2 Existence of social contract equilibrium

The existence property presented in this section is established in the manner of the proof of existence of a competitive market equilibrium of Arrow and Debreu (1954) which makes use of an explicit representation of the auctioneer, described as a fictitious agent which chooses prices so as to maximize the value of the equilibrium outcome.

This paves the way to the design of the process of decentralized auction embodied in the notion of dual distributive core defined in the next section.

The existence of a social contract equilibrium raises but two specific difficulties. Individual vectors of shares π_i, first, should be bounded away from 0, in order to get "sufficiently" continuous individual reaction correspondences. The second problem stems from the possible presence of malevolence in distributive preferences. An individual is said *distributively malevolent* to another when his distributive utility is decreasing in the latter's consumption expenditure. At equilibrium, the distributive malevolence of an individual (say, i's malevolence to j) translates into a negative value of the corresponding share π_{ij}, the latter being then the money equivalent of i's marginal utility of the wealth transfer to j (Appendix: Lemma 1(i)). One needs, consequently, an assumption bounding the set inside which the auctioneer is allowed to pick the vector of shares. In view of the definition of the set of admissible shares, it will be sufficient to put, so to speak, a "lower bound" on malevolence.

The definition of bounded malevolence is set in the technically convenient framework of differentiable utility. The derivative of w_i (resp. partial derivative of w_i with respect to x_j) at x is denoted by $\partial w_i(x)$ (resp. $\partial_{x_j} w_i(x)$).

Definition 4. *Suppose that:* w_i *is twice differentiable for all* i; *for all* $\lambda\in]0,1[$, $w_i(x') \geq w_i(x)$ *implies* $w_i(\lambda x'+(1-\lambda)x) \geq w_i(x)$ (*quasi-concavity*). *Then,* (w,ω^0) *verifies bounded distributive malevolence if:* (*i*) *there exist two real numbers* $v>0$ *and* $\varepsilon>0$ *such that, for all solution* (θ,λ_i) *to* $\mathrm{Max}\{w_i(\omega^0+z):z\in B_i(\pi)\}$ (*Lemma 1*) *such that* $\lambda_i>0$: $(1/\lambda_i)\partial_{x_i} w_i(\omega^0+\theta) \geq v$; *and* $(1/\lambda_i)\partial_{x_j} w_i(\omega^0+ ta) \geq -\varepsilon$ *for all* $j\neq i$; (*ii*) *and there exists a compact set* K *such that, for all* $i\in N$ *and all* $\pi\in\{\pi\in A:\pi_{jj} \geq v$ *for all* $j\}$, *the set of solutions* (θ,λ_i) *to* $\mathrm{Max}\{w_i(\omega^0+z):z\in B_i(\pi)\}$ *is nonempty and contained in* $K\times\mathbb{R}_{++}$.

Note that the lower bound ε can be arbitrarily large. This assumption is compatible, therefore, with the presence of intense malevolent feelings in distributive preferences.

Theorem 4. *Suppose that* w_i *is twice differentiable and quasi-concave for all* i, *that* (w,ω^0) *verifies bounded distributive malevolence, and that* $\omega^0\in\mathbb{R}^n_{++}$ (*the interior of* $\mathbb{R}^n_+$ *in* $\mathbb{R}^n$). *Then, there exists a social contract equilibrium of* (w,ω^0).

Proof. Let $A' = \{\pi\in A:\pi_{ii} \geq v$ for all $i\in N$ and $\pi_{ij} \geq -2\varepsilon$ for all $(i,j)\in N\times N\}$: it is clearly compact and convex; and it is nonempty since $\varepsilon>0$. A' is the set where the auctioneer will pick the vector of shares. Let Θ be a compact convex subset of $\mathbb{R}^n$ containing 0 and K in its interior. The vectors of transfers chosen by individuals will belong to Θ by bounded malevolence. The Cartesian product $\prod_{i\in N}\Theta = \{(\theta^1,\ldots,\theta^n):\theta^i\in\Theta$ for all $i\in N\}$ is denoted by Θ^n; it is, of course, a compact set that contains $\prod_{i\in N}\mathrm{K}$ in its interior.

For any given $(\theta^1,\ldots,\theta^n)\in\Theta^n$, the auctioneer maximizes $\pi\to\sum_i \pi_i\theta^i$ in A'. Let $\Pi(\theta^1,\ldots,\theta^n)$ denote the corresponding set of maxima. It is nonempty since A' is nonempty and compact, and $\pi\to\sum_i \pi_i\theta^i$ is continuous. $\Pi:(\theta^1,\ldots,\theta^n)\to\Pi(\theta^1,\ldots,\theta^n)$ is therefore a well-defined correspondence on Θ^n. This correspondence is upper hemicontinuous by continuity of $\pi\to\sum_i \pi_i\theta^i$ (as an application of Berge 1963, VI.3, or Debreu 1982, Lemma 1). Its values are closed (by continuity of $\pi\to\sum_i \pi_i\theta^i$) and therefore compact subsets of A'; they are convex by linearity of $\pi\to\sum_i \pi_i\theta^i$.

From bounded malevolence, it is equivalent for individual i to maximize $z \rightarrow w_i(\omega^0 + z)$ in $B_i(\pi)$ or in $B'_i(\pi) = B_i(\pi) \cap \Theta$ when $\pi_{jj} \geq \nu$ for all j. Let the corresponding set of maxima be denoted by $T_i(\pi)$. This set is nonempty by assumption. T_i: $\pi \rightarrow T_i(\pi)$ is therefore a well-defined correspondence on A'. Its values are closed (by continuity of w_i) and therefore compact subsets of Θ; they are convex by quasi-concavity of w_i. Correspondence $B'_i: \pi \rightarrow B'_i(\pi)$ is well-defined in A'; and it is continuous as a consequence of Debreu (1982, Lemma 3) (Θ is nonempty, compact and convex; and $0 > \text{Min}\{\pi_i.\theta{:}\theta \in \Theta\}$ for all $\pi \in A$ such that $\pi_i \neq 0$, since 0 is in the interior of Θ). The continuity of B'_i in $\{\pi \in A'{:}\pi_i \neq 0\}$ and of w_i in $\mathbb{R}^n$ imply that correspondence T_i is upper hemi-continuous in A' (Berge 1963, VI.3).

Define the following product correspondences: $T = \prod_{i \in N} T_i{:}A' \rightarrow \Theta^n$, the values of which are the Cartesian products $T(\pi) = \prod_{i \in N} T_i(\pi)$; Φ: $A' \times \Theta^n \rightarrow A' \times \Theta^n$, the values of which are the Cartesian products $\Phi(\pi,(\theta^1,\ldots,\theta^n)) = \Pi(\theta^1,\ldots,\theta^n) \times T(\pi)$. From the three paragraphs above, Φ is defined on and takes its values in the same nonempty compact convex set; it is upper hemicontinuous, compact-and convex-valued. It has, therefore, a fixed point $(\pi^*,(\theta^1,\ldots,\theta^n))$ in $A' \times \Theta^n$ by Kakutani's fixed point theorem. In view of the definition of a social contract equilibrium, it will be sufficient to establish, to finish with, that $\theta^1 = \ldots = \theta^n = \theta^*$ and that $\sum_i \theta_i^* = 0$.

Suppose, without loss of generality, that $\theta_1^1 > \theta_1^2$, and let us derive a contradiction. $\pi^* \in A'$, so that $\pi_{11}^* \geq \nu$ and $\pi_{22}^* \geq \nu$. Bounded malevolence and Lemma 1(i) imply then that there exists a real number $\alpha > 0$ such that: $\alpha \leq \pi_{21}^* + 2\varepsilon$. Let π' be the vector of shares such that: $\pi'_{11} = \pi_{11}^* + \alpha$; $\pi'_{22} = \pi_{21}^* - \alpha$; $\pi'_{ij} = \pi_{ij}^*$ for all other (i,j). Then: $\pi' \in A'$; and $\sum_i \pi'_i.\theta^i = \alpha(\theta_1^1 - \theta_1^2) + \sum_i \pi_i^*.\theta^i > \sum_i \pi_i^*.\theta^i$. But this contradicts the definition of correspondence Π. Therefore $\theta^1 = \ldots = \theta^n = \theta^*$. Moreover $\pi_i^* \theta^* = 0$ for all i by bounded malevolence and Lemma 1(i). Since $\pi^* \in A$ by definition of Π, we have, to finish with, $\sum_i \pi_i^*.\theta^* = \sum_i \theta_i^* = 0$. ■

3 Dual distributive core

The remainder of this article develops a notion of decentralized auction relative to voluntary redistribution and explores its connections with the social contract equilibrium.

The process of decentralized auction considered here and in Sect. 4 and 5 below is carried on the distributive values π_{ij} that support a distributive optimum ω. The distributive agents are allowed to form *representative coalitions*, which choose the individual shares of their members. A coalition is representative if the number of its members (Sects. 1–3) or members'types (Sects. 4 and 5) exceeds an *a priori given* threshold $m \in \{1,\ldots,n\}$. A coalition blocks the Pareto efficient distribution of rights ω if it can announce a vector of individual shares of its members that increases the minimum value $\pi_i(\omega - \omega^0)$ compatible with the maintenance of i's utility level for all member i. Coalitions can be viewed as decentralized auctioneers tending to the joint maximization of the values of the public good for their members while defending, simultaneously, the individual utility levels of members.

The dual distributive core consists of the efficient distributions of rights that are not blocked in the sense above.

The formal definition of the dual distributive core that follows makes use of the familiar fact that if utility functions are continuous and verify convexity, there is, for all $\theta \in \mathbb{R}^n$ which is not a local minimum of w_i, a vector of distributive values of i that supports θ, that is, a nonzero π_i in $\mathbb{R}^n$ such that $\pi_i.\theta = \text{Min}\{\pi_i.z{:}w_i(\omega^0 + z) \geq w_i(\omega)\}$ (Appendix: Lemma 3(i)-(a)). We say that the vector of distributive values $\pi = (\pi_1,\ldots,\pi_n) \in \mathbb{R}^{n^2}$ *supports* ω-ω^0 if: π_i supports ω-ω^0 for all i; and there exists $\alpha \in \mathbb{R}_{++}$ such that $\sum_i \pi_i = \alpha e$. Supporting distributive values are thus defined up to a positive multiplicative constant, in the same way as market prices, and essentially for the same reason, namely, the homogeneity property of support functions (Lemmas 3(i)-(b)). Supportability by a *unique vector of shares* (i.e., by a unique $\pi \in A$) characterizes, nevertheless, distributive efficiency, except for the coincidental occurrence of binding nonnegativity constraints, kinks in indifference loci or linear dependencies in the family of gradients of utility functions (Lemma 4 (iii)).

Definition 5. (i) *A distributive optimum ω is dual-blocked by coalition I in (w, ω^0) if there exists a vector of distributive values π^* supporting $\omega - \omega^0$, and a family of individual vectors of distributive values $\{\pi_i{:}i \in I\}$ such that: $\sum_{i \in I} \pi_i = e$; and $\text{Min}\{\pi_i.\theta{:}w_i(\omega^0 + \theta) \geq w_i(\omega)\} \geq \pi_i^*.(\omega - \omega^0)$ for all $i \in I$, with a strict inequality for at least one i.* (*ii*) *ω is in the dual distributive core of (w, ω^0) if it is a distributive optimum which is not dual-blocked by any representative coalition, that is, by any coalition of at least m members.*

The dual distributive core of (w, ω^0) will be denoted by D^*.

The next theorem states that the dual core: (i)–(a) contains the set of interior social contract equilibria for all m; (i)–(b) is essentially identical with the latter if $m = 1$; (ii) is essentially identical with the set of distributive optima if $m = n$. In other words, the dual core is equivalent to the social contract equilibrium when the condition of representativeness induces no restriction on the set of admissible coalitions, and to the distributive Pareto optimum when the grand coalition is the only admissible coalition.

Theorem 5. (i) *Suppose that utility functions are twice differentiable, verify convexity, and that $\partial w_i(x) \neq 0$ for all $i \in N$ and all $x \in S_n$ (differentiable nonsatiation of utility functions in S_n). Then*: (*a*) *$E^* \cap \mathbb{R}^n_{++}$ is contained in D^**; *(b) if $m = 1$ and if $\omega^0 \in \mathbb{R}^n_{++}$, $D^* \cap \mathbb{R}^n_{++}$ is contained in E^*. (ii) Suppose moreover that: for all $\lambda \in]0, 1[$ and all (x, x') such that $x \neq x'$, $w_i(x') \geq w_i(x)$ implies $w_i(\lambda x' + (1 - \lambda)x) > w_i(x)$ (strict quasi-concavity*[8]*); and $\sum_i \mu_i \partial w_i(x) \neq 0$ for all*

[8] Strict quasi-concavity implies convexity, which implies quasi-concavity.

nonzero $\mu \in \mathbb{R}^n_+$ *and all* $x \in \mathbb{R}^n_+$ *such that* $\sum_i x_i \leq 1$ (*differentiable nonsatiation of the weak Paretian preordering*[9]). *If* $m = n$, *if* $\omega \in P^* \cap \mathbb{R}^n_{++}$, *and if* rank $\{\partial w_1(\omega), \ldots, \partial w_n(\omega)\} = n$, *then* $\omega \in D^*$.

Proof. (i)–(a) Let (π, θ) be a social contract equilibrium of (w, ω^0) such that $\omega = \omega^0 + \theta \in \mathbb{R}^n_{++}$.

Then θ is a maximum of $z \rightarrow w_i(\omega^0 + z)$ in $B_i(\pi) = \{z \in \mathbb{R}^n : \omega_i^0 + z_i \geq 0$ and $\pi_i.z \leq 0\}$ for all i by definition of equilibrium. Differentiable nonsatiation implies that $\pi_i \neq 0$ (Lemma 1(i)) and that $\pi_i.\theta = \text{Min}\{\pi_i.z : w_i(\omega^0 + z) \geq w_i(\omega^0 + \theta)\} = 0$ (Lemma 1(i) and Lemma 2(i)) for all i. In particular, π supports θ.

If π' is, now, any supporting vector of θ, there is for all i, an $\alpha_i \in \mathbb{R}^n_{++}$ such that $\pi'_i = \alpha_i \pi_i$ (remember that $\omega \in \mathbb{R}^n_{++}$ by assumption, and apply Lemma 2(i)). And again therefore: $\pi_i.\theta = \text{Min}\{\pi_i.z : w_i(\omega^0 + z) \geq w_i(\omega^0 + \theta)\} = 0$ for all i.

Suppose that $\omega \notin D^*$. From the paragraph above, the definition of the dual distributive core and Theorem 3(i), there exist π' supporting θ, $\pi'' \in A$ and a nonempty $I \subset N$ such that $\pi''_i.\theta \geq \pi'_i.\theta$, with a strict inequality for at least one i. But then: $0 = (\sum_{i \in I} \pi''_I).\theta > \sum_{i \in I} \pi'_i.\theta = 0$, a contradiction. ■

(i)-(b) Suppose that $m = 1$ and $\omega^0 \in \mathbb{R}^n_{++}$, and consider some $\omega \in D^* \cap \mathbb{R}^n_{++}$, and any supporting π of $\theta = \omega - \omega^0$. As a simple consequence of the definition of the dual distributive core when $m = 1$: $\pi_i.\theta \geq 0$ for all i. There is an $\alpha \in \mathbb{R}_{++}$ such that $\alpha\pi \in A$ by definition of π. Hence $\sum_i \pi_i.\theta = (\sum_i \pi_i).\theta = (1/\alpha)e.\theta = 0$, and therefore $\pi_i.\theta = 0$ for all i. One concludes by Lemma 2(ii). ■

(ii) Suppose that $m = n$, consider some $\omega \in P^* \cap \mathbb{R}^n_{++}$ such that $\text{rank}\{\partial w_1(\omega), \ldots, \partial w_n(\omega)\} = n$, and let π^* be the unique element of A that supports $\theta = \omega - \omega^0$ (Lemma 4(iii)).

Let $c_i : \mathbb{R}^n \times \mathbb{R} \rightarrow [-\infty, +\infty[$ denote the function $(\pi_i, u) \rightarrow \inf\{\pi_i.z : w_i(\omega^0 + z) \geq u\}$. Let $u_i^* = w_i(\omega)$, and consider the programs $\text{Max}\{c_j(\pi_j, u_j^*) : \sum_i \pi_i \leq e$ (resp. $\sum_i \pi_i \geq e$) and $c_k(\pi_k, u_k^*) \geq \pi_k^*.\ \theta$ for all $k \neq j\}$, $j = 1, \ldots n$. The Kuhn-Tucker first-order conditions associated with the j-th programs can be stated as follows: there exists $(\delta_j, \gamma_j) \in \mathbb{R}^n \times \mathbb{R}^n_+$, with $\gamma_{jj} = 1$, such that, for all $k \in N$, $\gamma_{jk} \partial_{\pi_k} c_k(\pi_k, u_k^*) \leq \delta_j$ (resp. $\gamma_{jk} \partial_{\pi_k} c_k(\pi_k, u_k^*) \geq \delta_j$), $(\gamma_{jk} \partial_{\pi_k} c_k(\pi_k, u_k^*) - \delta_j).(e - \sum_i \pi_i) = 0$, and $\gamma_{jk}(c_k(\pi_k, u_k^*) - \pi_k^*.\theta^*) = 0$. These conditions are well-defined by Lemma 3(ii), and are sufficient by Lemma 3(i)–(b) and Arrow and Enthoven, 1961, Theorem 3(c). They are verified by $(\delta, \gamma, \pi) = (\theta, e, \pi^*)$, since $\partial_{\pi_i} c_i(\pi_i^*, u_i^*) = \theta^*$ for all i (Lemma 3(ii)).Therefore π^* is a simultaneous solution to these programs.

From Lemma 4(iii), the supporting vectors of θ are the elements of $\{\alpha\pi^* : \alpha \in \mathbb{R}^n_{++}\}$. Suppose that ω is dual-blocked, i.e. that there exist $\alpha \in \mathbb{R}^n_{++}$ and

[9] The differentiable nonsatiation of the weak Paretian preordering implies the differentiable nonsatiation of utility functions in S_n.

$\pi \in A$ such that $c_i(\pi_i, u_k^*) \geq \alpha \pi_i^*.\theta$ for all i, with a strict inequality for at least one i. The homogeneity property of support functions (Lemma 3(i)–(b)) implies that $c_i((1/\alpha)\pi_i, u_k^*) \geq \pi_i^*.\theta$ for all i, with a strict inequality for at least one i. Moreover, $\sum_i (1/\alpha)\pi_i \leq e$ if $\alpha \geq 1$ and $\sum_i (1/\alpha)\pi_i \geq e$ if $\alpha \leq 1$. But we have then a contradiction with the conclusion of the former paragraph. ■

4 Replicating distributive social systems

This section applies to distributive theory the procedure imagined by Edgeworth (1881) and generalized by Debreu and Scarf (1963) to increase the number of agents of an economy while preserving its fundamental structure of preferences and endowments.

The procedure consists essentially in distinguishing a fixed finite number of types of agents, and increasing evenly the number of agents of each type. Two difficulties appear immediately in application to distributive social systems.

Firstly, the number of arguments of utility functions coincides with the number of individuals, and expands therefore with the latter. We have to assume, consequently, that distributive preferences are essentially invariant to such increases in the dimension of the space of distributions. This points to conditions of separability. The solution adopted below combines additive separability relative to individuals with weak separability relative to types.

Secondly, two agents having identical distributive utility functions are not identical by such from the viewpoint of distributive theory. Suppose for instance that agents 1 and 2's distributive preferences are both represented by the first projection $x \rightarrow x_1$. Individual 1 is then unsympathetically isolated, and consequently never wants to give, while individual 2 only cares about individual 1's well-being, and thus wants to transfer his whole wealth to him. The solution below relies on the natural requirement that the consumptions of any two agents of the same type be considered by all as interchangeable, which is tantamount to a condition of partial anonimity (anonimity inside types).

From now on, the index i will be reinterpreted as referring to the agents' *type*. The root social system (w, ω^0) of Sect. 1 is thus made of n types of agents, with a single individual in each type. The number of individuals of type i is denoted by r (the same number for all types). For $r \geq 2$, agents are indexed by a pair (i, q), where $q = 1, \ldots, r$ distinguishes the individuals of type i. Former notations and definitions are then adapted in the obvious way. In particular, the utility function of individual (i, q) is denoted by w^r_{iq}, his initial endowment by $\omega^{0,r}_{iq}$, his consumption by x^r_{iq} and his individual distributive values by π^r_{iq}. We let: $w^1_{i1} = w_i$; $\omega^{0,1}_{i1} = \omega^0_i$; $x^1_{i1} = x_i$; $w^r = (w^r_{11}, \ldots, w^r_{1r}, \ldots, w^r_{n1}, \ldots, w^r_{nr})$; $\omega^{0,r} = (\omega^{0,r}_{11}, \ldots, \omega^{0,r}_{1r}, \ldots, \omega^{0,r}_{n1}, \ldots, \omega^{0,r}_{nr})$; $x^r = (x^r_{11}, \ldots, x^r_{1r}, \ldots, x^r_{n1}, \ldots, x^r_{nr})$; $\pi^r_i = (\pi^r_{i1}, \ldots, \pi^r_{ir})$; $\pi^r = (\pi^r_1, \ldots, \pi^r_n)$. The vector of $\mathbb{R}^{nr}$ whose components are all equal to 1 is denoted by e^r. The following defines replicas and extends accordingly to the latter the definition of the dual distributive core.

Definition 6. $((w^r, \omega^{0,\, r}))_{r \geq 1}$ *is a sequence of r-replicas of the distributive social system* (w, ω^0) *if*: (i) $\omega_{iq}^{0,r} = \omega_i^0$ *for all* $(i, q) \in N \times \{1, \ldots, r\}$; (*ii*) *for all* $r \geq 1$ *and all* $i \in N$ *there exist* $v_i^r:\mathbb{R} \to \mathbb{R}$ *and* $f_i^r:\mathbb{R}^n \to \mathbb{R}$ *such that*: (a) $f_i^r(v_i^r(x_1), \ldots, v_i^r(x_n)) = w_i(x_1, \ldots, x_n)$ *for all* $(x_1, \ldots, x_n) \in \mathbb{R}^n$; (b) *and* w_{iq}^r: $x^r \to f_i^r((1/r)\sum_{q=1}^r v_i(x_{1_q}^r), \ldots, (1/r)\sum_{q=1}^r v_i^r(x_{nq}^r))$ *for all* $q \in \{1, \ldots, r\}$.

Definition 7. (*i*) *A distributive optimum* ω^r *is dual-blocked* by *coalition I in* $(w^r, \omega^{0,\, r})$ *if: there exist a vector of distributive values* π^{r*} *supporting* ω^r-$\omega^{0,r}$ *and a family of individual vectors of distributive values* $\{\pi_{iq}^r:(i, q) \in I\}$ *such that*: $\sum_{(i,q)\in I} \pi_{iq}^r = e^r$; *and* $\inf\{\pi_{iq}^r.\theta^r:w_{iq}^r(\omega^{0,r} + \theta^r) \geq w_{iq}^r(\omega^r)\} \geq \pi_{iq}^{r*}.(\omega^{r*} - \omega^{0,r})$ *for all* $(i, q) \in I$, *with a strict inequality for at least one* (i, q). (*ii*) ω^r *is in the dual distributive core of* $(w^r, \omega^{0,\, r})$ *if it is a distributive optimum of the latter and is not dual-blocked by any representative coalition, that is, by any coalition I such that* $\#\{i:\exists q$ *such that* $(i, q) \in I\} \geq m$.

The distributive utility functions of Definition 6(ii) combine two elements: an additively separable and anonymous index of welfare measurement, which aggregates the wealth of all individuals of the same type; and a utility function on the distributions of the values per capita of this index across types[10]. They replicate the utility functions of the root social system for distributions that assign the same consumption to all individuals of the same type.

Definition 7 extends Definition 5 to the replicas of the root social system. It requires that blocking coalitions contain representatives of at least m distributive types. As r becomes large relative to m, coalitions draw their representativeness, and subsequent ability to express claims on the distribution of wealth, from the fact that they reflect a significant fraction of the spectrum of distributive characteristics. I assume, in particular, from now on, that m is large enough to imply that the dual distributive core of the root social system strictly contains the set of social contract equilibria (Theorem 5).

I establish below that all individuals of the same type have the same consumption at any distributive optimum when utility functions are quasi-concave (or verify Assumption 1) and exhibit inequality aversion inside types.

[10] There is clearly no loss of generality in defining preferences on the index per capita. This specification has the advantage of making sense asymptotically for a wide class of functions v. An axiomatic characterization of indexes $(1/r)\sum_{q=1}^r v(x_{iq}^r)$ involves, essentially, two principles: anonimity (inside types); and separability (Blackorby et al. 1996).

Theorem 6. *Consider a sequence* $((w^r, \omega^{0, r}))_{r \geq 1}$ *of replicas of* (w, ω^0), *and suppose that: for all* $r \geq 1$, *and all* $(i, q) \in N \times \{1, \ldots, r\}$, w^r_{iq} *either verifies assumption 1 or is quasi-concave; and* $w^r_{iq}(x^r) \geq w^r_{iq}(z^r)$ *implies* $w^r_{iq}(\lambda x^r + (1 - \lambda) z^r) > w^r_{iq}(z^r)$ *for all real number* $\lambda \in]0, 1[$ *and all pair of consumption distributions* (x^r, z^r) *such that* $x^r_{is} \neq z_{is}$ *for some s (inequality aversion inside types). Then, the distributive optima of any replica assign the same consumptions to the individuals of the same type.*

Proof. Let x^r be a feasible distribution such that $x^r_{iq} > x^r_{is}$, and denote by σ: $\mathbb{R}^{nr} \to \mathbb{R}^{nr}$ the permutation of components $(i-1)r + q$ and $(i-1)r + s$ of a consumption distribution of $\mathbb{R}^{nr}$ (e.g. of x^r_{iq} and x^r_{is} in x^r). By Definition 6: $w^r_{ju}(\sigma(x^r)) = w^r_{ju}(x^r)$ for all $(j, u) \in N \times \{1, \ldots, r\}$. And the feasibility of x^r clearly implies that $\lambda\sigma(x^r) + (1 - \lambda)x^r$ is feasible for all $\lambda \in [0, 1]$. Let $\lambda \in]0, (1/2)[$. The move from x^r to the convex combination $\lambda\sigma(x^r) + (1 - \lambda)x^r$ consists then of a redistribution from (i, q) to (i, s) that leaves the former richer than the latter. Since utility functions verify either assumption 1 or quasi-concavity, we have therefore $w^r_{ju}(\lambda\sigma(x^r) + (1 - \lambda)x^r) \geq w^r_{ju}(x^r)$ for all $(j, u) \neq (i, q)$. Inequality aversion and $w^r_{iq}(\sigma(x^r)) = w^r_{iq}(x^r)$imply moreover that $w^r_{iq}(\lambda\sigma(x^r) + (1 - \lambda)x^r) > w^r_{iq}(x^r)$. Hence the conclusion. ■

5 Convergence of the dual distributive core

Theorem 6 is analogous to the Theorem 2 of Debreu and Scarf (1963). It allows us to consider efficient distributions as elements of $\mathbb{R}^n$. More precisely, let the *r*-replica of distribution $\omega \in \mathbb{R}^n$ be the distribution ω^r which assigns the right ω_j to all individuals of type *j* when *j* runs over *N*. Theorem 6 states that the set of distributive optima of the *r*-replica of the root social system consists of the *r*-replicas of the distributions of the latter. The condition (ii)-(a) of Definition 6 readily implies then that the set of distributive optima of any *r*-replica is the set of *r*-replicas of the elements of P^*. Likewise, the distributive core and the set of liberal distributive social contracts of any replica may be identified with C^* and L^* respectively, under Assumption 1. And the set of equilibrium distributions of any replica can be identified with E^* under the assumption of Theorem 3(i) (local nonsatiation in S_n). In short, replication leaves all these sets essentially unchanged.

This is not true for the dual distributive core. The dual distributive core D_r of an *r*-replica ($D_1 = D^*$) may be identified with a subset of P^* (Theorem 6) that contains $E^* \cap \mathbb{R}^n_{++}$ (Theorem 5(i)). Our main result, analogous to the Theorem 3 of Debreu and Scarf, 1963, is that the sequence of dual distributive cores converges, essentially, to E^* as *r* grows to infinity (the qualification stemming from potential difficulties associated with the coincidental occurrence of binding nonnegativity constraints or linear dependencies in the family of gradients of utility functions).

Theorem 7. *Consider a sequence* $((w^r, \omega^{0,\,r}))_{r\,\geq\,1}$ *of replicas of* (w, ω^0). *Suppose that*: $\omega^0 \in \mathbb{R}^n_{++}$; w_i *is twice differentiable and strictly quasi-concave for all i*; *and the weak Paretian preordering of w verifies differentiable nonsatiation in* $\{x \in \mathbb{R}^n_{++}\colon \sum_i x_i \leq 1\}$. *Suppose moreover that* w^r_{iq} *is quasi-concave and verifies inequality aversion inside types for all* $r \geq 1$ *and all* $(i, q) \in N \times \{1, \ldots r\}$. *If* $\omega \in (\cap_{r\,\geq\,1} D_r) \cap \mathbb{R}^n_{++}$ *is such that rank* $\{\partial w_1(\omega), \ldots, \partial w_n(\omega)\} = n$, *then* $\omega \in E^*$.

Proof. The following argument reproduces, with only minor differences, the proof of the Theorem 3 of Debreu and Scarf (1963).

Suppose first that $\omega^* = \omega^0$. Let $\theta^* = \omega_i{}^* - \omega^0 = 0$ be supported by $\pi^* \in A$ (Lemma 4(iii)). We have then $\pi^*_i.\theta^* = 0$ for all i. And this implies that $\omega^* \in E^*$ by Lemma 2(ii).

Suppose next that $\omega^* \neq \omega^0$. Let $\theta^* = \omega^* - \omega^0$ be supported by $\pi^* \in A$ (Lemma 4(iii)). Denote $u_i{}^* = w_i(\omega^*)$, and let $\Gamma_i = \{\pi_i\colon c_i(\pi_i, u_i{}^*) > \pi_i{}^*.\theta^*\}$. Γ_i is a convex set for all i by the concavity of support function $c_i(., u_i{}^*)$ (Lemma 3(i)–(b)); and its interior in $\mathbb{R}^n$ is nonempty since $c_i(., u_i{}^*)$ is C^1 and $\partial\pi_i c_i(\pi^*_i, u^*) = \theta^* \neq 0$ (Lemma 3(ii)). Let Γ denote $\{\sum_i \alpha_i\pi_i\colon \alpha \in S_n \cap \mathbb{R}^n_{++}$ and $\pi \in \Pi_i\Gamma_i\}$. Γ is clearly convex and has a nonempty interior in $\mathbb{R}^n$.

Suppose that $e \in \Gamma$. There is $\alpha \in S_n \cap \mathbb{R}^n_{++}$ and $\pi \in \Pi_i\Gamma_i$ such that $\sum_i \alpha_i\pi_i = e$. For all $k \in \mathbb{N}$, let a^k_i be the smallest integer greater than or equal to $k\alpha_i$. For each i and k, let $\pi^k_i = [(k\alpha_i/a^k_i)]\pi_i$ and observe that $\pi^k_i \in [0, \pi_i]$ and tends to π_i as k tends to infinity. The continuity of support functions implies that $c_i(\pi_i, u_i{}^*) > \pi_i{}^*.\theta^*$ for sufficiently large k. Let k be such an integer, and notice that $\sum_i a^k_i\pi^k_i = k\sum_i \alpha_i\pi_i = ke$. Let $r = \text{Max}_i\, a^k_i$ and consider, in the corresponding r-replica, a coalition I made of a^k_i individuals of type i, where i runs over N. This coalition is representative. Denote by: $\pi^{k,r}_i$ the r-replica of π^k_i, that is, a vector of distributive values of an individual of type i of $(w^r, \omega^{0,\,r})$ which assigns π^k_{ij} to all individuals of type j when j runs over N; $\pi^{*,r}_i$ the r-replica of π^*_i. Consider next the vector of distributive values π^r such that $\pi^r_{iq} = (1/k)\pi^{k,r}_i$ for all $(i, q) \in I$. We have $\sum_{(i,q)\in I} \pi^r_{iq} = (k/k)e^r = e^r$. Moreover, the homogeneity property of support functions (Lemma 3(i)–(b)) and Lemma 5 yield: $c^r_{iq}(\pi^r_{iq}, u^*_i) = rc_i((1/k)\pi^k_i, u^*_i) = (r/k)c_i(\pi^k_i, u^*_i) > (r/k)c_i(\pi^*_i, u^*_i) = rc_i((1/k)\pi^*_i, u^*_i) = c^r_{iq}((1/k)\pi^{*,r}_i, u^*_i)$ for all $(i, q) \in I$. Since$((1/k)\pi^{*,r}_1, \ldots, (1/k)\pi^{*,r}_n)$ is a vector of distributive values which supports the r-replica of θ^* (as a consequence of Lemma 5), coalition I dual-blocks the latter in $(w^r, \omega^{0,\,r})$, a contradiction.

Therefore $e \notin \Gamma$. The separating hyperplane theorem implies that there exists, consequently, a nonzero p in $\mathbb{R}^n$ such that $z.p \geq e.p$ for all $z \in \Gamma$. Continuous function $z \to z.p$ is therefore bounded below in the closure $\text{cl}\Gamma_i = \{\pi_i\colon c_i(\pi_i, u_i{}^*) \geq \pi_i{}^*.\theta^*\} (\subset \text{cl}\Gamma = \{\sum_i \alpha_i\pi_i\colon \alpha \in S_n$ and $\pi \in \Pi_i \text{cl}\Gamma_i\})$. This implies in turn that it has a minimum $z_i{}^*$ in $\text{cl}\Gamma_i$. Since $\text{cl}\Gamma_i$ has a nonempty interior in $\mathbb{R}^n$, $z_i{}^*$ verifies the following necessary first-order conditions: there exists $\gamma_I \in \mathbb{R}_+$ such that $p = \gamma_i\partial_{\pi_i}c_i(z^*_i, u^*_i)$ and $\gamma_i(c_i(z^*_i, u^*_i) - \pi^*_i.\theta^*) = 0$ (Arrow and Enthoven, 1961, last § of Sect. 4(1)). $p \neq 0$ implies then that γ_i is nonzero, hence that $c_i(z^*_i, u^*_i) = \pi^*_i.\theta^*$, and therefore that $z^*_i = \pi^*_i$ and $p = \gamma_i\theta^*$. Since the same is true for all i, there exists in fact a $\gamma \in \mathbb{R}_{++}$ such that $p = \gamma\theta^*$. We can therefore let $p = \theta^*$. But then $z.\theta^* \geq e.\theta^* = 0$ for all $z \in \text{cl}\Gamma_i$, which implies that $\pi_i^*.\theta^* \geq 0$ for all i. Since moreover $\sum_i \pi_i^*.\theta^* = (\sum_i \pi_i^*).\theta^* = e.\theta^* = 0$, we get $\pi_i^*.\theta^* = 0$ for all i. Lemma 2(ii) implies then that $\omega^* \in E^*$. ■

Definition 7 does not imply that the sequence (D_r) of the dual distributive cores of replicated social systems is nonincreasing. But the following variant does:

Definition 7′. *(i) A distributive optimum ω^r is dual-blocked by coalition I in $(w^r, \omega^{0,r})$ if: there exist a vector of distributive values π^{r*}* supporting $\omega^r - \omega^{0,r}$, *a family of individual vectors of distributive values $\{\pi^r_{iq}:(i, q) \in I\}$, and a family $\{\pi_i:(i, q) \in I\}$ of vectors of $\mathbb{R}^n$ such that: $\sum_{(i,q)\in I} \pi^r_{iq} = e^r$; $\pi^r_{iq,ju} = \pi_{ij}$ for all $(i, q) \in I$ and all $(j, u) \in N \times \{1, \ldots, r\}$; and* $\inf\{\pi^r_{iq}.\theta^r : w^r_{iq}(\omega^{0,r} + \theta^r) \geq w^r_{iq}(\omega^r)\} \geq \pi^{r*}_{iq}.(\omega^{r*} - \omega^{0,r})$ *for all $(i, q) \in I$, with a strict inequality for at least one (i, q). (ii) ω^r is in the dual distributive core of $(w^r, \omega^{0,r})$ if it is a distributive optimum of the latter and is not dual-blocked by any representative coalition, that is, by any coalition I such that $\#\{i:\exists q$ such that $(i, q) \in I\} \geq m$.*

The notion of dual distributive core of definition 7’ is slightly weaker than the notion of definition 7, for it is now required that members of blocking coalitions of same generic type i assign the same shares π_{ij} to all individuals of same generic type j. Letting D'_r denote the dual distributive core of the r-replica of (w, ω^0) so defined, we have $D_r \subset D'_r$ for all r. The statement and proof of Theorem 7 apply identically to the sequence (D'_r). The next theorem establishes that the latter is nonincreasing.

Theorem 8. *Suppose that, for all $r \geq 1$ and all $(i, q) \in N \times \{1, \ldots r\}$, w^r_{iq} is C^1, quasi-concave, and verifies inequality aversion inside types. Then: $D'_r \subset D'_s$ whenever $r \leq s$.*

Proof. I prove that if a distributive optimum is dual-blocked in the sense of Definition 7′ in a replica it must be dual-blocked in the sense of Definition 7′ by the same coalition in any larger replica.

Let ω^r be a distributive optimum dual-blocked by I in $(w^r, \omega^{0,r})$ in the sense of Definition 7′. There exist a vector of distributive values π^{r*} supporting ω^r-$\omega^{0,r}$, a family of individual vectors of distributive values $\{\pi^r_{iq}:(i, q) \in I\}$, and a family $\{\pi_i:(i, q) \in I\}$ of vectors of $\mathbb{R}^n$ such that: $\sum_{(i,q)\in I} \pi^r_{iq} = e^r$; $\pi^r_{iq,js} = \pi_{ij}$ for all $(i, q) \in I$ and all $(j, s) \in N \times \{1,\ldots,r\}$; and $\inf\{\pi^r_{iq}.\theta^r : w^r_{iq}(\omega^{0,r} + \theta^r) \geq w^r_{iq}(\omega^r)\} \geq \pi^{r*}_{iq}.(\omega^{r*} - \omega^{0,r})$ for all $(i, q) \in I$, with a strict inequality for at least one (i, q).

From Theorem 6, ω^r-$\omega^{0,r}$ is the r-replica of a Pareto efficient vector of net transfers ω-ω^0 of the root social system. Lemma 2(i) applied to utility functions w_i and their replicas w^r_{iq}, and Definition 6, imply then that π^{r*} is the r-replica of a vector π^* of $\mathbb{R}^{n^2}$ that supports ω-ω^0; in particular,

$\pi^{r*}_{iq}.(\omega^{r*} - \omega^{0,r}) = r\pi^*_i.(\omega^* - \omega^0)$ for all $(i, q)\in I$. And family $\{\pi^r_{iq}:(i, q)\in I\}$ is the r-replica of family $\{\pi_i:(i, q)\in I\}$ by Definition 7′.

Let $s \geq r$, and denote by ω^s, π^{s*} and $\{\pi^s_{iq}:(i, q)\in I\}$ the s-replicas of ω, π^*, and $\{\pi_i:(i, q)\in I\}$ respectively. By construction: $\pi^s_{iq,ju} = \pi_{ij}$ for all $(i, q)\in I$ and all $(j, u)\in N\times\{1,\ldots,s\}$, and therefore $\sum_{(i,q)\in I}\pi^s_{iq} = e^s$. From Lemma 5: $\inf\{\pi^s_{iq}.\theta^{r}:w^s_{iq}(\omega^{0,s}+\theta^s) \geq w^s_{iq}(\omega^s)\} = sc_i(\pi_i,w_i(\omega)) = (s/r)\inf\{\pi^r_{iq}.\theta^r: w^r_{iq}(\omega^{0,r}+\theta^r) \geq w^r_{iq}(\omega^r)\}$. Moreover, $\pi^{s*}_{iq}.(\omega^{s*} - \omega^{0,s}) = s\pi^*_i.(\omega^* - \omega^0) = s/r)\pi^{r*}_{iq}.(\omega^{r*} - \omega^{0,r})$ for all $(i, q)\in I$. Hence $\inf\{\pi^s_{iq}.\theta^r:w^s_{iq}(\omega^{0,s}+\theta^s) \geq w^s_{iq}(\omega^s) \geq \pi^{s*}_{iq}.(\omega^{s*} - \omega^{0,s})$ for all $(i, q)\in I$, with a strict inequality for at least one (i, q). Therefore ω^s is dual-blocked by I in $(w^s, \omega^{0,s})$. ■

6 Conclusion

The process of decentralized auction embodied in the notion of dual distributive core is a device for producing and exchanging information on the distributive preferences of individuals, which determines a distribution of wealth that is both Pareto efficient and unanimously preferred to the initial distribution.

The normative properties of Pareto efficiency and unanimous preference jointly characterize the distributive liberal social contract. In that sense, the dual core participates in a well-defined conception of distributive justice.

Symmetrically, the selection of a determinate outcome, the social contract equilibrium, that this process of communication operates inside the set of liberal social contracts, is empty of any content of distributive justice: it relies simply on the joint maximization of the value of the public good and of individual utilities subject to the normative prescription above, that status quo be an alternative always accessible to all individuals. Its normative content, if any, consists of implicit prescriptions on the good practice of social communication, involving notably the effective and honest (though indirect) expression by individuals of their true preferences relative to the public good. The spontaneous fulfilment of such prescriptions does not seem unlikely for the public good and the social contract under consideration, consisting of benevolent redistributions of wealth unanimously preferred to the initial distribution.

The comparison of this property of determinacy with the analogous property obtained by Debreu and Scarf for market economies elicits a salient difference, besides the obvious specificities associated with the use of the concepts and tools of duality theory: the coincidence of the dual core with social contract equilibrium obtains more easily, under conditions which permit the introduction of severe restrictions on the set of admissible coalitions. The condition of representativeness of coalitions that enters the definition of the dual core is a product of this enlargement of the logical possibilities relative to determinacy.

This article opens on three lines of research.

The notion of representativeness that I use here ("scope representativeness") is narrowly conditioned by the technique of replication of Debreu and Scarf. The tools of measure theory already used to dispense with the specificities of this technique in the study of the convergence of the core of an economy (Hildenbrand 1974) will find interesting applications in the context of distributive social systems also, notably by permitting the definition of more realistic notions of representativeness of a coalition, such as the requirement that the statistical distribution of its members' types be "not too skewed" as compared to the same distribution in the population as a whole. Besides a substantial gain in descriptive accuracy, valuable per se, the introduction of more suitable notions of replication and representativeness would permit to treat also in a more satisfactory way, in terms of generality of the analysis, questions of great theoretical interest concerning the speed of convergence of the dual core and its relation with the nature and strength of the requirements of representativeness of coalitions.

A basic implication of the present analysis is that *direct democracy*, understood as a process of communication where all the subgroups of society, including individuals, can express their views on the public good, and *representative democracy*, where the right of a group to participate in the public debate stems from the public rules that determine its representativeness, yield essentially the same results concerning voluntary transfers of individual property rights in societies where the number of social types is small relative to the population as a whole. This conclusion relies upon the fundamental though implicit hypothesis that the social communication relative to the public good is free and honest. It raises a set of questions for future research, concerning the degree of adequacy of this hypothesis to the reality of social communication, and the implications of its observed violations for the theory of its functioning, and notably for the explanation of the genesis, shape and behavior of representative groups.

The distribution of wealth is distinguished, as a public good, by the combination of extended common concerns and extended property rights of individuals: its character of public good stems from common concerns, while the property right of individuals on their own wealth makes the private provision (gifts) both possible and in some sense legitimate. In such a context, the public good problem, that is, the Pareto inefficiency of social equilibrium, follows from a legitimate use by individuals of their right to give, which is motivated by their altruistic common concerns, and which induces inefficient externalities through the common concerns of others. The distributive liberal social contract and social contract equilibrium reconcile Pareto efficiency with private property in this context. A third set of questions for future research concerns the possibility to extend these solutions to other goods involving the same type of combination of extended common concerns and individual property rights, such as, notably, human wealth and social insurances.

Appendix

Lemma 1. (*i*) *Suppose that w_i is differentiable and quasi-concave and that $\omega_i^0 > 0$. If θ is a maximum of $z \rightarrow w_i(\omega^0 + z)$ in $B_i(\pi) = \{z \in \mathbb{R}^n : \omega_i^0 + z_i \geq 0$ and $\pi_i.z \leq 0\}$, there exists then $\lambda_i \in \mathbb{R}_+$ such that: $\lambda_i \pi_i.\theta = 0$; $\partial_{x_i} w_i(\omega^0 + \theta) \leq \lambda_i \pi_{ii}$ and $(\lambda_i \pi_{ii} - \partial_{x_i} w_i(\omega^0 + \theta)).(\omega_i^0 + \theta_i) = 0$; $\partial_{x_j} w_i(\omega^0 + \theta) = \lambda_i \pi_{ij}$ for all $j \neq i$.* (*ii*) *Suppose that w_i is twice differentiable and quasi-concave and that $\partial w_i(\omega^0 + \theta) \neq 0$. If $\omega_i^0 + \theta_i \geq 0$, if $\pi_i.\theta = 0$, and if there exists $\lambda_i \in \mathbb{R}_{++}$ such that $\partial_{x_i} w_i(\omega^0 + \theta) \leq \lambda_i \pi_{ii}$, $(\lambda_i \pi_{ii} - \partial_{x_i} w_i(\omega^0 + \theta)).(\omega_i^0 + \theta_i) = 0$, and $\partial_{x_j} w_i(\omega^0 + \theta) = \lambda_i \pi_{ij}$ for all $j \neq i$, then θ is a maximum of $z \rightarrow w_i(\omega^0 + z)$ in $B_i(\pi)$.*

Proof. (i) and (ii) are simple consequences of Arrow-Enthoven (1961, Sect. 4(1)) (respectively: of the last § of this section, and of Theorem 3(b)). ■

Lemma 2. (*i*) *Suppose that w_i is C^1 and quasi-concave and that $\pi_i \neq 0$. Then θ* is a *minimum of $z \rightarrow \pi_i.z$ in $\{z \in \mathbb{R}^n : w_i(\omega^0 + z) \geq u\}$ if and only if: $w_i(\omega^0 + \theta) = u$; and there exists $\mu_i \in \mathbb{R}_{++}$ such that $\pi_i = \mu_i \partial w_i(\omega^0 + \theta)$.* (*ii*) *Suppose moreover that w_i is twice differentiable and that $\omega_i^0 > 0$. The following three statements are then equivalent*: (*a*) *$\omega_i^0 + \theta_i > 0$, $\pi_i.\theta = 0$, and θ is a minimum of $z \rightarrow \pi_i.z$ in $\{z \in \mathbb{R}^n : w_i(\omega^0 + z) \geq u\}$*; (*b*) *$\omega_i^0 + \theta_i > 0$, $\partial w_i(\omega^0 + \theta) \neq 0$, and θ is a maximum of $z \rightarrow w_i(\omega^0 + z)$* in *$B_i(\pi)$*; (*c*) *$\omega_i^0 + \theta_i > 0$, $\pi_i.\theta = 0$, $w_i(\omega^0 + \theta) = u$, and there exists $\lambda_i \in \mathbb{R}_{++}$ such that $\partial w_i(\omega^0 + \theta) = \lambda_i \pi_i$.*

Proof. (i) The first-order condition is necessary (e.g., Mas-Colell, 1985, D.1). It is sufficient by Arrow-Enthoven (1961, Theorem 3 (b)). ■

(ii) is a simple consequence of Lemma 1 and Lemma 2(i). ■

Lemma 3. *(i) Suppose that w_i is continuous, and that, for all real number $\lambda \in]0, 1[$, $w_i(x') > w_i(x)$ implies $w_i(\lambda x' + (1 - \lambda)x) > w_i(x)$* (*convexity*). *Then:* (*a*) *for all $\theta \in \mathbb{R}^n$ which is not a local minimum of w_i, there exists a nonzero $\pi_i \in \mathbb{R}^n$ such that $\pi_i \theta = \text{Min}\{\pi_i.z : z \in \mathbb{R}^n$ and $w_i(\omega^0 + z) \geq w_i(\omega^0 + \theta)\}$*; (*b*) *for all $u \in w_i(\mathbb{R}^n)$, the support function $c_i(., u) : \pi_i \rightarrow \inf\{\pi_i.z : z \in \mathbb{R}^n$ and $w_i(\omega^0 + z) \geq u\}$, defined on $\{\pi_i \in \mathbb{R}^n : \inf\{\pi_i.z : z \in \mathbb{R}^n$ and $w_i(\omega^0 + z) \geq u\} > -\infty\}$, is positively homogeneous of degree one*[11] *and concave.* (*ii*) *Suppose that for all $\lambda \in]0, 1[$, and all (x, x') such that $x \neq x'$, $w_i(x') \geq w_i(x)$ implies $w_i(\lambda x' + (1 - \lambda)x) > w_i(x)$* (*strict quasi-concavity*). *Then, for all $u \in w_i(\mathbb{R}^n)$, $c_i(., u)$ is C^1 and such that $\partial \pi_i c_i(\pi_i, u) = z$ if and only if $z \in \mathbb{R}^n$, $w_i(\omega^0 + z) \geq u$ and $\pi_i.z = c_i(\pi_i, u)$.*

Proof. (i)-(a) is a simple consequence of the supporting hyperplane theorem (Mas-Colell 1985, F.2.1), and (i)-(b) is an application of Mas-Colell (1985, F.3.2). ■

[11] i.e., $c_i(\lambda \pi_i, u) = \lambda\, c_i(\pi_i, u)$ *for all* $\lambda > 0$.

(ii) The set $\{z\in \mathbb{R}^n{:}w_i(\omega^0+z) \geq u\}$ is closed by continuity of w_i and strictly convex by strict quasi-concavity of w_i. Function $z\to\pi_i.z$ has therefore a unique minimum in this set for any $u\in w_i(\mathbb{R}^n)$ and any $\pi_i\in \mathbb{R}^n$ such that $\inf\{\pi_i.z{:}z\in \mathbb{R}^n$ and $w_i(\omega^0+z) \geq u\} > -\infty$. Let this minimum be denoted by $m_i(\pi_i, u)$. In view of Mas-Colell, 1985, F.3.2, it suffices to prove that functions $m_i(., u)$ are continuous on their domains $\{\pi_i\in\mathbb{R}^n{:}\inf\{\pi_i.z{:}z\in\mathbb{R}^n$ and $w_i(\omega^0+z) \geq u\} > -\infty\}$. But this is a simple consequence of Berge (1963, VI.3). ■

Lemma 4. *Suppose that:* w_i *is* C^1 *and quasi-concave for all i; and* $\sum_i \mu_i\partial w_i(x) \neq 0$ *for all nonzero* $\mu\in\mathbb{R}^n_+$ *and all* $x\in \mathbb{R}^n_+$ *such that* $\sum_i x_i \leq 1$ (*differentiable nonsatiation of the weak Paretian preordering*). (*i*) *If* $\omega\in P^*\cap \mathbb{R}_{++}$, *there exist* $\alpha\in\mathbb{R}_{++}$ *and* π *such that*: $\sum_i \pi_i = \alpha e$; *and for all i, either* $\pi_i = 0$, *or* π_i *supports* θ (*i.e.* $\pi_i\neq 0$ *and* $\pi_i.\theta = \mathrm{Min}\{\pi_i.z{:}z\in \mathbb{R}^n$ *and* $w_i(\omega^0+z) \geq w_i(\omega)\}$. (*ii*) *If* $\omega\in S_n\cap \mathbb{R}_n$ *is such that* $\mathrm{rank}\{\partial w_1(\omega), \ldots, \partial w_n(\omega), e\} = n$, *then*: $\omega\in P^*$ *if and only if there exists* π *that supports* $\theta = \omega - \omega^0$ (*i.e. there exist* $\alpha\in \mathbb{R}_{++}$ *and* π *such that* $\sum_i \pi_i = \alpha e$ *and* π_i *supports* θ *for all i*). (*iii*) *If* $\omega\in S_n\cap \mathbb{R}^n_{++}$ *is such that* $\mathrm{rank}\ \{\partial w_1(\omega), \ldots, \partial w_n(\omega)\} = n$, *then:* $\omega\in P^*$ *if and only if there exists a unique* $\pi^*\in A$ *that supports* $\theta = \omega - \omega^0$; *and the set of vectors* π *which support* θ *is the open half line* $\{\alpha\pi^*{:}\alpha\in \mathbb{R}_{++}\}$.

Proof. (i) Let $\omega \in \mathbb{R}^n_{++}$ be a weak maximum of $x\to(w_1(x),\ldots, w_n(x))$ in $\{x\in \mathbb{R}^n_+{:}\sum_i x_i \leq 1\}$. It verifies then the following necessary first-order conditions (Mas-Colell 1985, D.1): there exists a nonzero $(\mu, \alpha) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ such that $\sum_i \mu_i\partial w_i(\omega) = \alpha e$ and $\alpha(1 - \sum_i \omega_i) = 0$. There is a nonzero μ_i ($\mu = 0$ implies $\sum_i \mu_i\partial w_i(\omega) = \alpha e = 0$ and therefore $(\mu, \alpha) = 0$). The differentiable nonsatiation of the weak Paretian preordering implies in turn that $\alpha > 0$, hence that $\sum_i \omega_i = 1$. In particular, ω is a weak maximum of $x\to(w_1(x),\ldots, w_n(x))$ in $\{x\in \mathbb{R}^n_+{:}\sum_i x_i = 1\}$.

Let now $\omega\in P^*\cap \mathbb{R}^n_{++}$. From the paragraph above and the definition of a distributive optimum, ω is then a weak maximum of $x\to(w_1(x),\ldots, w_n(x))$ in $\{x\in \mathbb{R}^n_+{:}\sum_i x_i = 1\}$ and verifies the following necessary first-order condition: there exist a nonzero $\mu\in\mathbb{R}^n_+$ and $\alpha\in\mathbb{R}^n_{++}$ such that $\sum_i \mu_i\partial w_i(\omega) = \alpha e$. Let $\pi_i = \mu_i\partial w_i(\omega)$ for all i. Differentiable nonsatiation and Lemma 2 (i) imply then that either μ_i (hence π_i) $= 0$ or μ_i (hence π_i) $\neq 0$ and π_i supports ω-ω^0. ■

(ii) Let $\omega\in S_n\cap\mathbb{R}^n_{++}$ be such that $\mathrm{rank}\{\partial w_1(\omega), \ldots, \partial w_n(\omega), e\} = n$.

If $\omega\in P^*$, there is, by Lemma 4(i), a nonzero $\mu\in\mathbb{R}^n_+$ and $\alpha \in \mathbb{R}_{++}$ such that $\sum_i \mu_i\partial w_i(\omega) = \alpha e$. But $\mathrm{rank}\{\partial w_1(\omega), \ldots, \partial w_n(\omega), e\} = n$ readily implies then that $\mu\in\mathbb{R}^n_{++}$. Therefore $(\pi_1, \ldots \pi_n)$ supports ω-ω^0 (Lemma 2(i)).

Suppose conversely that $\omega - \omega^0$ is supported by a vector π. ω verifies then the following set of conditions: $\omega\in S_n$; and there exist $\mu \in \mathbb{R}^n_{++}$ and $\alpha \in \mathbb{R}_{++}$ such that $\sum_i \mu_i\partial w_i(\omega) = \alpha e$ (Lemma 2(i)). But then $w_i(\omega) = \mathrm{Max}\{w_i(x){:}x\in S_n$ and $w_j(x) \geq w_j(\omega)$ for all $j\neq i\}$ for all i by differentiable nonsatiation and Arrow-Enthoven (1961, Theorem 3(b) and Sect. 4(2)). Therefore $\omega \in P^*$ by definition of a distributive optimum. ■

(iii) Let $\omega \in S_n \cap \mathbb{R}^n_{++}$ be such that $\text{rank}\{\partial w_1(\omega), \ldots, \partial w_n(\omega)\} = n$. In view of Lemmas 2(i) and 4(ii), it will suffice to prove that, for all $\alpha \in \mathbb{R}_{++}$, there exists $\mu \in \mathbb{R}^n_{++}$ such that $\sum_i \mu_i \partial w_i(\omega) = \alpha e$ (or, equivalently, such that $\partial w(\omega).\mu^T = \alpha e^T$, where $\partial w(\omega)$ denotes the Jacobian matrix of $x \to (w_1(x), \ldots, w_n(x))$ at ω, and μ^T and e^T are the transposes of row vectors μ and e). The following two statements are clearly equivalent: there exist $\mu \in \mathbb{R}^n_{++}$ and $\alpha \in \mathbb{R}_{++}$ such that $\partial w(\omega).\mu^T = \alpha e^T$; and for all $\alpha \in \mathbb{R}_{++}$, there exists $\mu \in \mathbb{R}^n_{++}$ such that $\partial w(\omega).\mu^T = \alpha e^T$. Moreover, the (n, n)-matrix $\partial w(\omega)$ is nonsingular by assumption, so that, for any given $\alpha \in \mathbb{R}_{++}$, $\partial w(\omega).\mu^T = \alpha e^T$ holds for one and only one $\mu \in \mathbb{R}^n$. Hence the conclusion. ■

Lemma 5. *Consider the r-replica* $(w^r, \omega^{0,\,r})$, *suppose that* w^r_{iq} *verifies inequality aversion inside types, let* $\pi^*_i \in \mathbb{R}^n$ *support* $\theta^* \in \mathbb{R}^n$, *and denote by* $\pi^{*,r}_i$ (*resp.* $\theta^{*,r}$) *the r-replica of* π^*_i (*resp.* θ^*) (*i e. the vector of distributive values of an individual of type i* (*resp. the vector of net transfers*) *of* $(w^r, \omega^{0,\,r})$ *which assigns value* π^*_{ij} (*resp. net transfer* θ^*_j) *to all individuals of type j when j runs over N*). *Then:* $\inf\{\pi^{*,r}_i.\theta^r : w^r_{iq}(\omega^{0,r} + \theta^r) \geq w^r_{iq}(\omega^{0,r} + \theta^{*,r})\} = rc_i(\pi^*_i, w_i(\omega^0 + \theta^*))$ *for all* $q = 1, \ldots, r$.

Proof. Let $u = w^r_{iq}(\omega^{0,r} + \theta^{*,r})$, and denote by $c^r_{iq}(., u)$ the support function $\pi^r_i \to \inf\{\pi^r_i.\theta^r : w^r_{iq}(\omega^{0,r} + \theta^r) \geq u\}.w^r_{iq}(\omega^{0,r} + \theta^{*,r}) = w_i(\omega^0 + \theta^*)$ by Definition 6, so that $c^r_{iq}(\pi^{*,r}_i, u) \leq \pi^{*,r}_i.\theta^{*,r} = rc(\pi^*_i, u)$. Suppose that $c^r_{iq}(\pi^{*,r}_i, u) < rc_i(\pi^*_i, u)$. There is then a θ^r such that $w^r_{iq}(\omega^{0,r} + \theta^r) \geq u$ and $\pi^{*,r}_i.\theta^r < rc_i(\pi^*_i, u)$. Let $\theta^{**} \in \mathbb{R}^n$ assign the average net transfer $\sum_{q=1}^r (1/r)\theta^r_{iq}$ to individual i when i runs over N, and denote by $\theta^{**,r}$ its r-replica. The anonimity condition embodied in Definition 6, combined with inequality aversion, readily implies that $w^r_{iq}(\omega^{0,r} + \theta^{**,r}) = w_i(\omega^0 + \theta^{**}) > u$. Moreover: $r\pi^*_i.\theta^{**} = \pi^{*,r}_i.\theta^{**,r} = \pi^{*,r}_i.\theta^r < rc_i(\pi^*_i, u)$. But we have then both $w_i(\omega^0 + \theta^{**}) > u$ and $\pi^*_i.\theta^{**} < c_i(\pi^*_i, u)$, which contradicts the definition of $c_i(., u)$. ■

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