Optimal redistribution in the distributive liberal social contract

Jean Mercier Ythier*

December 2007†

Abstract

We consider abstract social systems of private property, made of n individuals endowed with non-paternalistic interdependent preferences, who interact through exchanges on competitive markets and Pareto-efficient lump-sum transfers. The transfers follow from Kolm's distributive liberal social contract, here characterized as a redistribution of initial endowments such that the resulting market equilibrium allocation is both Pareto-efficient relative to individual interdependent preferences, and unanimously weakly preferred to the initial market equilibrium. We establish the existence of such cooperative solutions to the public good problem of redistribution. The market equilibrium allocations associated with the transfers of the distributive liberal social contract maximize weighted sums of individual interdependent utilities in the set of attainable allocations. In-kind and monetary transfers are essentially equivalent, for social contract redistribution. Finally, we compare the distributive liberal social contract solutions with the alternative Pareto-efficient solutions of the Lindahl equilibrium and the core with public good.

Keywords: Walrasian equilibrium; Pareto-efficiency; liberal social contract; interdependent preferences; public goods; distribution.

1 Distribution in the liberal social contract

The liberal social contract (Kolm, 1985, 1987ab, 1996: 5, and 2004: Chap.3) is a normative reference, corresponding to the unanimous agreement of individuals derived from the sole consideration of their preferences and rights by abstracting away all conceivable impediments to the achievement of this agreement or implementation of its contents, that is, notably, informational and other obstacles to the elaboration of the clauses of the social contract, and difficulties with

*Université de Metz, BETA and IDEP

†This paper has benefited from the comments of Serge Kolm, Marc Fleurbaey, an anonymous referee and the participants to the conference in honor of Serge Kolm at the University of Caen, 18-19 May 2007.

their enforcement[1].

It differs from alternative normative theories such as Harsanyi's derivation of utilitarianism (1955) or Rawl's *Theory of Justice* (1971) by deducing the normative reference from *actual* individual preferences and rights.

Harsanyi and Rawls use the fiction of the veil of ignorance of the original position for abstracting away all possible sources of alteration of the impartiality of individual judgment that may follow from individual's actual position in society, his "interests" in an all-inclusive sense, comprehending not only material wealth (the rich and the poor), but also human wealth (the sick and the healthy, the smart and the dull), distinctions, set of interpersonal relations etc. Individuals, so abstractly placed in a position of objectivity, form their impartial judgment over social states by means of acts of imaginative sympathy, which consist of imagining themselves successively occupying all actual positions in society. The norms of justice are unanimous agreements of such individual impartial judgments, obtained from rational deliberation and bargaining in the construct of Rawls, and from the axioms of rational decision under uncertainty in the construct of Harsanyi.

The liberal social contract, by contrast, is a unanimous agreement of individuals in their actual position in society. It is a *positive* theory in that respect. It becomes normative, hence a theory of justice, only insofar as the process of contracting, and subsequent implementation of clauses, are concerned. The operation of abstraction that is performed at this level extends to the whole space of social contracting, as an "as if" or ceteris paribus proviso, the abstract characteristics of perfectly competitive market exchange, notably costless and immediate information, bargaining and enforcement. The norm of justice is the unanimous agreement that obtains in these ideal conditions of perfect social contracting. It defines the (ideal) objective of collective action.

Actual collective action inspired by the liberal social contract fills in the gap between the norm of the social contract and the reality of society by means of actual contractual arrangements, or institutional substitutes for them, which permit the achievement, partial or complete, of some of its ideal objectives subject to the constraints associated with the actual costs of corresponding action. These modalities of collective action include state intervention, but do not reduce to the latter, in principle at least. In other words, the liberal social contract is mute, by construction, on the modalities of its implementation, as the costs of the latter proceed from the circumstances (information, transaction and enforcement costs) that are assumed away for its derivation.

We are specifically interested, in this article, in the *distributive* aspects of the liberal social contract (Kolm, 1985: Chap.19). We provide a formal interpretation of the notion and analyze some of the latter's basic properties within the framework of the theory of Pareto-optimal redistribution developed from

[1] Kolm's liberal social contract and the construct of Nozick, 1974, share some important common features, notably in the method of derivation of a consistent system of individual rights, including property right as a central piece. They also differ in several important respects, and notably in their respective emphasis on market failures and other contract failures, which are essential in the former and almost absent from the latter.

the contributions of Kolm, 1966 and Hochman and Rodgers, 1969.[2] The latter appears peculiarly suitable for these purposes, notably for the following three reasons.

It considers, first, individual preferences over the distribution of wealth (in short, distributive preferences). An individual concern about another's wealth can be of the benevolent type, also called altruistic concern, or of the malevolent type, in situations of envy or of ill-intended gift.[3] Altruistic concerns, in particular, if they are strong or widespread enough, induce willingness to give. If the gifts are properly oriented and intended, they can be accepted by beneficiaries. Subject to the same condition, they may also arouse no frustration and cause no objection from those who do not take part in the gift-giving relationship as donor or beneficiary. They may, therefore, meet unanimous agreement (the latter understood in the wide, or *weak*, sense that includes indifference as a case of agreement), that is, produce legitimate (Pareto-improving) redistribution according to the ethical principle of the liberal social contract.

Individual distributive preferences, second, make distribution a public good, as an object of common concern of the individual members of society (Kolm, 1966). Any individual wealth, likewise, is a public good (or bad) for the set of individuals who feel concerned about it, whenever this set does not reduce to the wealth owner himself. Pareto-improving redistribution confronts, consequently, the general problems of individual and collective action in the presence of public goods, such as free-riding or preference revelation problems (e.g. Musgrave, 1970) and difficulties of coordination (Warr, 1983). The distributive liberal social contract yields a first-best Pareto-optimal solution to these problems, that is, a Pareto optimum relative to individual distributive preferences determined in the ideal conditions of perfect social contracting (null costs of information, transaction and enforcement).

The theory of Pareto-optimal redistribution suitably articulates, third, competitive markets and Pareto-optimal redistribution. Competitive markets are in some sense implied by the normative reference to perfect contracting in the definition of the (norm of the) liberal social contract. And the first and second fundamental theorems of welfare economics extend to the case of non-paternalistically interdependent utilities, provided that malevolence, if any, is not so strong as to imply Pareto-optimal disposal of aggregate resources (see notably Winter, 1969, Archibald and Donaldson, 1976, and Rader, 1980). That is: Pareto-optimal distribution achieves competitive market equilibrium in this

[2] See Mercier Ythier, 2006, notably 4.1 and 6.1, for a comprehensive review of this literature.

[3] Envy is defined by economic theory as a situation where an individual prefers another's position (here, another's wealth) to his own. Envy in this sense does not imply malevolence; nor does malevolence imply envy in this sense. They can be associated, though, in the psychological attitudes of some relative to the wealthy, when the consideration of wealthy positions creates both dissatisfaction with one's own and subsequent resentment for the source of painful comparison. Malevolent distributive concern does not reduce to the case of envy, although the latter certainly is of great practical importance. Another important case is ill-intended gift (see Kolm, 2006: 4.2, for a comprehensive classification of gift motives, including the types of malevolent gift-giving).

setup.

This article fits the liberal social contract in the theory of Pareto-optimal redistribution in the following way. We first retain the basic assumptions of the latter theory, that is, essentially, non-paternalistic utility interdependence, competitive markets and private property. We next define an original position of the social system, which consists of the actual (pre-transfer) distribution of individual endowments and associate Walrasian equilibrium. In other words, the original position corresponds to the allocation of resources by the market prior social contract redistribution.[4]

The (norm of the) distributive liberal social contract relative to an original position then consists of *a set of lump-sum transfers achieved from the endowment distribution of the original position, and of some associate Walrasian equilibrium, such that the latter is a strong Pareto optimum relative to individual non-paternalistic preferences, and is unanimously weakly preferred to the Walrasian equilibrium of the original position.*

This definition embodies two sets of voluntary transfers: The transfers of market exchange, resulting in an allocation which is Pareto-efficient relative to individual *consumption* preferences (e.g. Debreu, 1954: Theorem 1), and also is unanimously preferred to the *endowment distribution* relative to these same preferences (Debreu and Scarf, 1963: Theorem 1); and the transfers of the social contract, resulting in an allocation which is Pareto-efficient relative to individual *distributive* preferences, and unanimously preferred to the *market equilibrium distribution* of the original position relative to the latter preferences.[5]

It defers, in this second respect (unanimous preference relative to initial market equilibrium distribution), from the Pareto-efficient solutions of Foley (1970) and Bergstrom (1970): The core solution of Foley implies unanimous preference relative to initial endowment distribution, but does not imply, in general, unanimous preference relative to market equilibrium with null provision of public goods (see the calculated example of section 5 below); and the Lindahl equilibrium of Bergstrom neither implies unanimous preference relative to initial endowment distribution[6], nor unanimous preference relative to market equilibrium with null redistribution (see section 5).[7]

[4] Note that the anteriority of the original position relative to the social contract solution is *logical*, not *chronological*. Time is abstracted in this rational reconstruction of the distribution institution. The redistributive transfers of the social contract, as, more generally, any individual or collective acts derived in the norm of the liberal distributive social contract "before" or "after" social contract redistribution, are imaginary by construction, hence reversible.

[5] Although atemporal as noted above ([4]), the logical sequence of market transfers improving upon the endowment distribution and social contract transfers improving upon market equilibrium distribution somehow matches the historical sequence of the economic revolution from the late eighteenth century and welfare state revolution from the late nineteenth century. The first type of argument synthesizes the economists' view on the contribution of market exchange to economic development. The second one is the liberal social contract reason for the building up of a redistributive welfare state in developed market economies.

[6] As established by Example 3 of Mercier Ythier: 2004. See pp. 311-314 of Mercier Ythier: 2006, for a general analysis of the relations of the distributive liberal social contract, Bergstrom's Lindahl equilibrium and Foley's core solution when there is only one consumption commodity (and therefore no market exchange).

[7] Note that, symmetrically, the distributive liberal social contract solutions need not, in

To sum up, the distributive liberal social contract considers Pareto-optimal redistribution *within an already constituted and functioning market economy*. Consequently and consistently, it appreciates the Pareto-improvement of wealth distribution relative to a hypothetical original position of market equilibrium with null redistribution.

It should be noted, to conclude this introductory section, that the notion defined in the statement above captures only one part of the general notion of Kolm (1985: Chap. 19) as it ignores the process of bargaining and social communication which is supposed, in Kolm's construct, to yield a *unique* social contract solution (that is, a unique allocation and associate distribution of wealth) from any pre-contractual social state. The social contract notion, such as specified above, typically covers a large number of solutions from a given pre-contractual social state (see the Figure in section 4 below). One and only one of these numerous solutions is *the* distributive liberal social contract in the sense of Kolm. The definition above should be viewed, in other words, as a set of fundamental *necessary* conditions for the distribution aspects of the liberal social contract, rather than as a full characterization of it.

In the remainder of this article, we provide a formal definition of the notions and assumptions above (section 2); set and interpret the working assumptions of differentiability and convexity (section 3); derive and analyze the fundamental property of separability of allocation and distribution (section 4); and situate the distributive liberal social contract relative to the comparable solutions of Bergstrom and Foley (section 5). An appendix recalls some useful fundamental properties of differentiable Walrasian economies and collects the proofs of the theorems of section 4.

2 Formal definitions and fundamental assumptions

We consider the following simple society of individual owners, consuming, exchanging and redistributing commodities.[8]

general, be unanimously preferred to the (pre- or even post-social contract) endowment distribution for individual *distributive* preferences. Following the interpretation in historical terms above ([5]), it is conceivable that some individuals express, for some reasons, sentimental or else, a preference for autarky (if the latter corresponds to an egalitarian state of nature à la Jean-Jacques Rousseau, for example), relative to the distributive and market efficiency of the equilibrium allocation of the liberal social contract.

[8] We abstract from production for simplicity. The introduction of privately owned, price-taking, profit-maximizing firms with well-behaved (notably convex) production sets does not imply any significant change for the analysis below. "Private utility"-maximizing owners of firms unanimously wish, in particular, that the firms they own maximize their profits. This holds true also for "social utility"-maximizing owners endowed with non-paternalistic interdependent utilities (because social utility maximization supposes private utility maximization for such individuals). This conformity of views of any individual in his different economic and social positions and roles of firm owner, consumer and (potential) donor supposes perfect competitive exchange, that is, price-taking behavior of individuals and firms, and complete markets (with or without uncertainty). It does not hold true anymore, in general, in cases

There are n individuals denoted by an index i running in $N = \{1, ..., n\}$, and l goods and services, denoted by an index h running in $L = \{1, ..., l\}$. We let $n \geq 2$ and $l \geq 1$ in the sequel, that is, we consider social systems with at least two agents and at least one commodity [9].

The final destination of goods and services is individual consumption. A consumption of individual i is a vector $(x_{i1}, ..., x_{il})$ of quantities of his consumption of commodities, denoted by x_i. The entries of x_i are nonnegative by convention, corresponding to demands in the abstract exchange economy outlined below. An allocation is a vector $(x_1, ..., x_n)$, denoted by x.

Individuals exchange commodities on a complete system of perfectly competitive markets. There is, consequently, for each commodity h, a unique market price, denoted by p_h, which agents take as given (that is, as independent from their consumption, exchange or transfer decisions, including their collective transfer decisions if any). We let $p = (p_1, \ldots, p_l)$.

Transfer decisions are made by coalitions, formally defined as any nonempty subset I of N, which may possibly be reduced to a single individual. A transfer of commodity h from individual i to individual j is a nonnegative quantity t_{ijh}. We let: $t_{ij} = (t_{ij1}, \ldots, t_{ijl})$ denote i's commodity transfers to j; $t_i = (t_{ij})_{j:j \neq i}$ denote the collection of i's transfers to others (viewed as a row-vector of $\mathbb{R}_+^{l(n-1)}$). A collection of transfers of the grand coalition N is denoted by t, that is: $t = (t_1, \ldots, t_n)$.

We make the following assumptions on commodity quantities: (i) they are perfectly *divisible*; (ii) the total quantity of each commodity is given once and for all (*exchange economy* with *fixed total resources*) and equal to 1 (the latter is a simple choice of units of measurement of commodities); (iii) an allocation x is attainable if it verifies the aggregate resource constraint of the economy, specified as follows: $\sum_{i \in N} x_{ih} \leq 1$ for all h (this definition of attainability implies *free disposal*).

The vector of total initial resources of the economy, that is, the diagonal vector (1,...,1) of $\mathbb{R}^l$, is denoted by ρ. The set of attainable allocations $\{x \in \mathbb{R}_+^{ln} : \sum_{i \in N} x_i \leq \rho\}$ is denoted by A.

The society is a *society of private property.* In particular, the total resources of the economy are owned by its individual members. The initial ownership or endowment of individual i in commodity h is a nonnegative quantity ω_{ih}. The vector $(\omega_{i1}, ..., \omega_{il})$ of i's initial endowments is denoted by ω_i. We have $\sum_{i \in N} \omega_i = \rho$ by assumption. The initial distribution $(\omega_1, \ldots, \omega_n)$ is denoted by ω.

---

of imperfect competition or incomplete markets. But, in the latter case, we are outside the enchanted world of Arrow-Debreu economy which, we argued in section 1, is an essential part of the more general notion of perfect social contracting that underlies the norm of the distributive liberal social contract. Note, finally, that the types of activities that are really essential for the functioning of the distributive liberal social contract are the transfer activities of market exchange and social contract gift-giving. Production, consumption and disposal activities are only subsidiary in this respect.

[9] the special case $l = 1$ is studied in Mercier Ythier, 1997, several results of which are subsumed in the results of the present study, and notably in Theorem 2

Individuals have preference preorderings over allocation, which are well defined (that is, reflexive and transitive) and complete. The allocation preferences of every individual i are assumed *separable* in his own consumption, that is, i's preference preordering induces a unique preordering on i's consumption set for all i. We suppose that preferences can be represented by utility functions. In particular, the preferences of individual i over his own consumption, as induced by his allocation preferences, are represented by the ("private", or "market") utility function $u_i : \mathbb{R}^l_+ \to \mathbb{R}$, which we will sometimes also name ophelimity function by reference to Pareto, 1913 and 1916. The product function $(u_1 \circ \mathsf{pr}_1, \ldots, u_n \circ \mathsf{pr}_n) : (x_1, ..., x_n) \to (u_1(x_1), ..., u_n(x_n))$, where pr_i denotes the i-th canonical projection $(x_1, ..., x_n) \to x_i$, is denoted by u. Finally, we suppose that individual allocation preferences verify the following hypothesis of *non-paternalistic utility interdependence*: For all i, there exists a ("social", or "distributive") utility function $w_i : u(\mathbb{R}^n_+) \to \mathbb{R}$, increasing in its i-th argument, such that the product function $w_i \circ u : (x_1, ..., x_n) \to w_i(u_1(x_1), ..., u_n(x_n))$ represents i's allocation preferences. Whenever i's distributive utility is increasing in j's ophelimity, this means that individual i endorses j's consumption preferences within his own allocation preferences ("non-paternalism"). Note, nevertheless, that non-paternalistic utility interdependence does not imply *distributive benevolence*, in the sense of individual distributive utilities increasing in some others' ophelimities. It is compatible, in particular, with the *distributive indifference* of an individual i relative to any other individual j, that is, the constancy of i's distributive utility in j's ophelimity in some open subset of domain $u(\mathbb{R}^n_+)$ ("local" distributive indifference of i relative to j) or in the whole of it ("global" indifference). It is compatible, also: With local or global *distributive malevolence*, in the sense of individual distributive utilities decreasing in some others' ophelimities; and, naturally, with any possible combination of local benevolence, indifference or malevolence of any individual relative to any other. For the sake of clarity, we reserve the terms "individual distributive utility function" for functions of the type w_i and "individual social utility function" for functions of the type $w_i \circ u$. The terms "individual distributive preferences" and "individual social preferences", on the contrary, are used as synonymous, and designate individual preference relations over allocation, in short, individual allocation preferences.

Individual private utilities are normalized so that $u_i(0) = 0$ for all i. Naturally, this can be done without loss of generality, due to the ordinal character of allocation preferences.

We let w denote the product function $(w_1, \ldots, w_n) : \hat{u} \to (w_1(\hat{u}), ..., w_n(\hat{u}))$, defined on $u(\mathbb{R}^n_+)$.

We use as synonymous the following pairs of properties of the preference preordering and its utility representations: C^1 preordering, and C^1 utility representations; *monotonic* (resp. *strictly monotonic*) preordering, and *increasing* (resp. *strictly increasing*) utility representations; *convex* (resp. *strictly convex*) preordering, and *quasi-concave* (resp. *strictly quasi-concave*) utility representations. Their definitions are recalled, for the sole utility representations, in footnote [11] below.

A social system is a list (w, u, ρ) of distributive and private utility functions of individuals, and aggregate initial resources in consumption commodities. A social system of private property is a list (w, u, ω), that is, a social system where the total resources of society are owned by individuals and initially distributed between them according to distribution ω.

It will not be necessary, for the definite purposes of this article, to develop a fully explicit concept of social interactions, synthesized in a formal notion of social equilibrium, such as those of Debreu, 1952, Becker, 1974 or Mercier Ythier, 1993 or 1998a for example.[10] The following informal description, and set of partial definitions, will suffice.

Market exchange is operated by individuals, who interact "asympathetically" (Edgeworth, 1881) or "nontuistically" (Wicksteed, 1913) on anonymous markets, through ophelimity-maximizing demands determined on the sole basis of market prices and individual wealth.

Sympathetic or altruistic interactions take place in redistribution. They may proceed, in principle, from a whole range of moral sentiments of individuals, from individual sentiments of affection between relatives to individual moral sentiments of a more universal kind such as philanthropy or individual sense of distributive justice. They may, likewise, find their expression in a large variety of actions, from individual gift-giving to family transfers, charity donations, or public transfers. We concentrate, in this article, on *lump-sum redistribution which meets the (weak) unanimous agreement of the grand coalition*, that is, redistribution of initial endowments that is approved by some individual members of society (one of them at least) and is disapproved by none. Note that, due to distributive indifference, any bilateral transfer so (weakly) preferred by the unanimity of individuals may be an object of effective concern for only a very limited number of persons, possibly reduced to the donor and the beneficiary of transfer. In other words, the abstract notion of altruistic transfer that we use here covers a wide spectrum of possibilities of voluntary redistribution, such as individual gifts, or collective transfers within groups of any possible size from families to society as a whole.

These elements of social functioning are summarized in the formal definitions below, of a *competitive market equilibrium*, and a *distributive liberal social contract*. They are complemented by the two notions of Pareto efficiency naturally associated with them, that is, respectively, the Pareto-efficiency relative to individual private utilities (in short, *market efficiency*, or *market optimum*), and the Pareto-efficiency relative to individual social utilities (in short, *distributive efficiency*, or *distributive optimum*).

Definition 1: A pair (p, x) such that $p \geq 0$ is a *competitive market equilibrium* with free disposal of the social system of private property (w, u, ω) if: (i) x is attainable; (ii) $p_h(1 - \sum_{i \in N} x_{ih}) = 0$ for all h; (iii) and x_i maximizes u_i in $\{z_i \in \mathbb{R}^l_+ : \sum_{h \in L} p_h z_{ih} \leq \sum_{h \in L} p_h \omega_{ih}\}$ for all i.

Definition 2: An allocation x is a *strong* (resp. *weak*) *market optimum* of

[10] See Mercier Ythier, 2006: 3.1.1, 4.2.1 and 6.1.1 for a review of such notions.

the social system (w, u, ρ) if it is attainable and if there exists no attainable allocation x' such that $u_i(x'_i) \geq u_i(x_i)$ for all i, with a strict inequality for at least one i (resp., $u_i(x'_i) > u_i(x_i)$ for all i). The set of weak (resp. strong) market optima of (w, u, ρ) is denoted by P_u (resp. $P^*_u \subset P_u$).

Definition 3: An allocation x is a *strong* (resp. *weak*) *distributive optimum* of the social system (w, u, ρ) if it is attainable and if there exists no attainable allocation x' such that $w_i(u(x')) \geq w_i(u(x))$ for all i, with a strict inequality for at least one i (resp., $w_i(u(x') > w_i(u(x))$ for all i). The set of weak (resp. strong) distributive optima of (w, u, ρ) is denoted by P_w (resp. $P^*_w \subset P_w$).

Definition 4: Let (p, x) be a competitive market equilibrium with free disposal of the social system of private property (w, u, ω). Pair $(\omega', (p', x'))$ is a *distributive liberal social contract* of (w, u, ω) relative to market equilibrium (p, x) if (p', x') is a competitive market equilibrium with free disposal of (w, u, ω') such that: (i) x' is a strong distributive optimum of (w, u, ρ); (ii) and $w_i(u(x')) \geq w_i(u(x))$ for all i.

For the sake of brevity, the competitive market equilibrium with free disposal of Definition 1 will often be referred to as Walrasian equilibrium or even simply as "market equilibrium" in the sequel. Likewise, we will often refer to the distributive liberal social contract simply as the "social contract".

Whenever a pair $(\omega', (p', x'))$ is a distributive liberal social contract of (w, u, ω) relative to market equilibrium (p, x), we also refer to ω' as a distributive liberal social contract of (w, u, ω) relative to (p, x), and to x' as a *distributive liberal social contract solution* of (w, u, ω) relative to (p, x).

3 Differentiable, convex social systems

In this section, we first present the working hypotheses of convexity and differentiability, summarized in Assumption 1 below. The definitions of corresponding standard properties of utility functions, such as differentiability, quasi-concavity, strict quasi-concavity and other, are recalled in the associate footnote.

We next discuss the general significance and justifications of the non-technical aspects of parts (ii) and (iii) of the hypothesis, which apply to individual social preferences. We omit a similar discussion of part (i) of the assumption, as the latter corresponds to a set of conditions on private preferences which has become standard in the study of differentiable exchange economies.

We use the following standard notations. Let $z = (z_1, \ldots, z_m)$ and $z' = (z'_1, \ldots, z'_m)$ be elements of $\mathbb{R}^m$, $m \geq 1$: $z \geq z'$ if $z_i \geq z'_i$ for any i; $z > z'$ if $z \geq z'$ and $z \neq z'$; $z >> z'$ if $z_i > z'_i$ for any i ; $z.z'$ is the inner product $\sum_{i=1}^m z_i z'_i$; z^T is the transpose (column-) vector of z.

Let $f = (f_1, \ldots, f_q) : V \to \mathbb{R}^q$, defined on open set $V \subset \mathbb{R}^m$, be the Cartesian product of the C^1 real-valued functions $f_i : V \to \mathbb{R}$: ∂f denotes its first derivative ; $\partial f(x)$, viewed in matrix form, is the $q \times m$ (Jacobian) matrix whose generic entry $(\partial f_i / \partial x_j)(x)$, also denoted by $\partial_j f_i(x)$ (or, sometimes, by $\partial_{x_j} f_i(x)$), is the first partial derivative of f_i with respect to its j-th argument at

x; finally, the transpose $[\partial f_i(x)]^T$ of the i-th row of $\partial f(x)$ is the gradient vector of f_i at x.

Assumption 1[11] **: Differentiable convex social system:** (i) For all i, u_i is: (a) continuous, strictly increasing, and unbounded above ; (b) C^1 in $\mathbb{R}^l_{++}$; (c) strictly quasi-concave in $\mathbb{R}^l_{++}$; (d) and such that $x_i >> 0$ whenever $u_i(x_i) > 0$ $(= u_i(0))$.(ii) For all i, w_i is: (a) increasing in its i-th argument and continuous; (b) C^1 in $\mathbb{R}^n_{++}$; (c) quasi-concave; (d) and such that $w_i(\hat{u}) > w_i(0)$ if and only if $\hat{u} >> 0$. (iii) For all i, $w_i \circ u$ is quasi-concave.

Assumption 1 will be maintained throughout the sequel.

The convexity of individual social preferences admits a natural interpretation and justification in terms of inequality aversion, as it implies a "preference for averaging" (in the sense that, if z and z' are indifferent for the preference relation, then $\alpha z + (1-\alpha)z'$ is weakly preferred to both z and z' for any α in $[0, 1]$).

The boundary condition 1-(ii)-(d) on distributive utilities is a substantial assumption. Associated with 1-(i)-(d) (the standard, technically convenient analogue for private utilities), it implies that all individuals strictly prefer allocations where every individual is enjoying a positive wealth and welfare, to allocations where any individual is starving or freezing to death.

The monotonicity and convexity assumptions on individual social preferences are narrowly conditioned by the object of these preferences, and notably by its large-scale character (the allocation of resources in society as a whole).

We only assume that an individual's distributive utility is increasing in his own private utility (see section 2). The latter follows from the basic hypothesis of separability of individual allocation preferences in own consumption, and interprets as a simple consistency requirement, stipulating that an individual's "social" view on his own consumption, as induced by his allocation preferences, must coincide with his "private" view on the same object, as represented by his private utility function.

[11] Recall that u_i is defined on $\mathbb{R}^l_+$, the nonnegative orthant of $\mathbb{R}^l$. We say that such a function is *increasing* (resp. *strictly increasing*) if $x_i >> x'_i$ (resp. $x_i > x'_i$) implies $u_i(x_i) > u_i(x'_i)$. It is *continuously differentiable* (or C^1) on $\mathbb{R}^l_{++}$ if it is differentiable and has a continuous first derivative on this domain. It is: *quasi-concave* if $u_i(x_i) \geq u_i(x'_i)$ implies $u_i(\alpha x_i + (1\text{-}\alpha)x'_i) \geq u_i(x'_i)$ for any $1 \geq \alpha \geq 0$; *strictly quasi-concave* if $u_i(x_i) \geq u_i(x'_i)$, $x_i \neq x'_i$ implies $u_i(\alpha x_i + (1\text{-}\alpha)x'_i) > u_i(x'_i)$ for any $1 > \alpha > 0$. Note that in the special case of a single market commodity (that is, $l = 1$), we can let $u_i(x_i) = \log(1 + x_i)$ without loss of generality (as "C^1 strictly quasi-concave" degenerates, in this simple case, into "C^1 strictly increasing").

Suppose, next, that utility representation u_i is bounded above and verifies all other assumptions 1-(i). Let $\sup u_i(\mathbb{R}^l_+) = b > a > u_i(\rho)$. Note that $a \in u_i(\mathbb{R}^l_+)=[0,b)$, since u_i is continuous and increasing. Define ξ: $[0,b) \to \mathbb{R}_+$ by: $\xi(t) = t$ if $t \in [0,a)$; and $\xi(t) = t + (t-a)^3 \exp(1/(b-t))$ if $t \in [a,b)$. One verifies by simple calculations that ξ is strictly increasing, and that $\xi \circ u_i$ is C^1, unbounded above, and therefore represents the same preordering as u_i and verifies assumption 1-(i). That is, there is no loss of generality in supposing u_i unbounded above.

Assumption 1-(i) notably implies that $u : \mathbb{R}^{ln}_+ \to \mathbb{R}^n_+$ is onto (since u_i is a continuous, increasing, unbounded above function $\mathbb{R}^l_+ \to [0, \infty)$ for all i), so that the domain $u(\mathbb{R}^l_+)$ of individual distributive utility functions coincides with the nonnegative orthant of $\mathbb{R}^n$. The definitions above extend readily to functions w_i and $w_i \circ u$.

We mentioned in section 2 that our formulation of the hypothesis of non-paternalistic utility interdependence was compatible with the distributive malevolence or indifference of any individual relative to any other, in a local or in a global sense. The casual observation of social life suggests that none of such psychological attitudes can be excluded on a priori grounds. It is also a commonplace of the stylized psychological theory of economists, elaborately expressed in Adam Smith's *Theory of Moral Sentiments* (1759), that individuals should, in most circumstances of ordinary life, be more sensitive to their own welfare (in the sense of their ophelimity) than to the welfare of others (at least "distant" others), notably because the psychological perception of others' welfare proceeds, to a large extent, from acts of imaginative sympathy (imagining oneself in the other's skin), which tend to be associated with less powerful affects in terms of frequency and average intensity, hence to produce less vivid and enduring perceptions, than the perception of one's own welfare through one's own senses[12]. Considered from this elaborate theoretical perspective, or from flat factual evidence, individual social preferences should notably exhibit wide ranges of indifference, distributive or else, due to the large-scale character of their object. It seems natural to expect, for example, that an individual will ordinarily feel indifferent relative to reallocations between individuals of close observable characteristics, such as similar ways of life for instance, if these characteristics are very different from his own and if he has no personal acquaintance with these persons. Such indifference is inconsistent, in general, with strictly monotonic or strictly convex preferences.

We chose, therefore, to keep to a minimum the monotonicity and convexity assumptions on social preferences at the individual level.

4 The separability of allocation and distribution

A fundamental property of the abstract social systems outlined in section 2 is the separability of allocation and distribution. The property states, essentially, that the redistribution of the social contract does not alter the fundamental features of the allocation of resources through the market, which follow from the role of market prices in the coordination of individual supplies and demands, namely, the existence of market equilibrium, the Pareto-efficiency of equilibrium allocations relative to private utilities ("market-efficiency") and the price-supportability of market optima.

The existence of market equilibrium, and the so-called first and second fundamental theorems of welfare economics (that is, respectively, in our terms, the market-efficiency of equilibrium allocation and the price-supportability of market optima), are well-known consequences of Assumption 1-(i). Social contract redistribution was characterized, in section 2, as a redistribution of individual endowments yielding a market equilibrium that is both Pareto-efficient relative

[12] See Lévy-Gargoua et alii, 2006, for a comprehensive review of the literature, and also for original views, on the formation of the social preferences of individuals, developed notably (but not only) from the economists' perspective.

to individual social utilities and unanimously (weakly) preferred to the initial market equilibrium. The separability property readily follows, therefore, from the notion of distributive liberal social contract itself, provided that the latter is consistently defined, that is, provided that there always exists a market equilibrium which is a distributive optimum unanimously preferred to the initial market equilibrium for individual social preferences.

The section is organized as follows. We first establish the inner consistency of the definition of the distributive liberal social contract and characterize the set of social contract solutions in Theorem 1. We next provide a useful characterization of distributive optima as the maxima of averages of individual social utility functions (Theorem 2). We then proceed to the elicitation of an important consequence of separability, namely, the equivalence of cash and in-kind transfers for Pareto-efficient redistribution (Theorem 3). And we conclude with an analysis of the significance and scope of separability.

The proofs of theorems are developed in the appendix.

4.1 Existence of a distributive liberal social contract

The inner consistency of the definition of the distributive liberal social contract is a simple consequence of the well-known fact that distributive optima are necessarily also market optima, provided that: (i) utility interdependence is non-paternalistic; (ii) and the partial preordering of Pareto associated with distributive utilities verifies some suitable property of non-satiation (see notably Rader, 1980 and Lemche, 1986).[13]

The theorem below first fits this basic property into the differentiable setup of the present article, and next draws its consequences for the existence and characterization of distributive liberal social contract solutions.

The strong (resp. weak) partial preordering of Pareto relative to distributive utilities (in short, strong (resp. weak) distributive preordering of Pareto), denoted by $\succ_w$ (resp. $\succ_w^*$), is defined on the set $u(\mathbb{R}_+^n)$ of ophelimity distributions by: $\hat{u} \succ_w \hat{u}'$ (resp. $\hat{u} \succ_w^* \hat{u}'$) if $w(\hat{u}) >> w(\hat{u}')$ (resp. $w(\hat{u}) > w(\hat{u}')$). The weak (resp. strong) ophelimity distributions associated with the distributive optima of (w, u, ρ) are, by definition, the maximal elements of $\succ_w$ (resp. $\succ_w^*$) in the set $u(A)$ of attainable ophelimity distributions, that is, the elements $\hat{u}$ of $u(A)$ such that there exists no $\hat{u}'$ in $u(A)$ such that $\hat{u}' \succ_w \hat{u}$ (resp. $\hat{u}' \succ_w^* \hat{u}$).

Note that weak and strong distributive efficiency are not equivalent, in general, under Assumption 1. We will therefore maintain the distinction between the weak and strong notions of distributive optimum throughout this article. On the contrary, as is well-known, weak and strong market efficiency are equivalent under Assumption 1-(i) (see the Proposition of the Appendix). Therefore we shall not distinguish between them anymore in the sequel.

For any integer $m \geq 2$, we denote by S_m the unit-simplex of $\mathbb{R}^m$, that is, set $\{z = (z_1, \ldots, z_m) \in \mathbb{R}_+^m : \sum_{i=1}^m z_i = 1\}$.

[13] A detailed account of this literature is provided in Mercier Ythier, 2006: 4.1.2.

The following assumption of *differentiable non-satiation of the weak distributive preordering of Pareto* is maintained throughout the sequel:

Assumption 2: For all $\mu \in S_n$ and all $\hat{u} \in u(A) \cap \mathbb{R}^n_{++}$, $\sum_{i \in N} \mu_i \partial w_i(\hat{u}) \neq 0$

Theorem 1 *Let (w, u, ρ) verify assumptions 1 and 2. Then: (i) any distributive optimum is a market optimum; (ii) there exists a distributive liberal social contract for any initial distribution ω, relative to any market equilibrium of (w, u, ω); (iii) the set of distributive liberal social contract solutions of (w, u, ω) relative to Walrasian equilibrium (p, x^0) is the set $\{x \in P^*_w : w(u(x)) \geq w(u(x^0))\}$.*

The Figure below provides a graphical illustration of the intuitions underlying Theorem 1. We consider a 3-agents social system, and denote by $\hat{u}^i$ the maximum of w_i in the set $u(P_u)$ of ophelimity distributions corresponding to the market optima of the social system, and by $\hat{u}^0$ the ophelimity distribution associated with some market equilibrium allocation $x^0 \notin P_w$. We suppose that $P_w = P^*_w$ (this will necessarily be the case, for example, if w_i is strictly quasi-concave for all i). The set $u(P_w)$ of ophelimity distributions corresponding to the distributive optima of the social system is the subarea of surface $u(P_u)$ delimited by the continuous curves $\hat{u}^i\hat{u}^j = \arg\max\{(w_i(\hat{u}), w_j(\hat{u})) : \hat{u} \in P_u\}$ for all pairs $\{i, j\}$ of distinct individuals of $N = \{1, 2, 3\}$. Finally, the ophelimity distributions of $\{\hat{u} \in u(P^*_w) : w(\hat{u}) \geq w(u^0)\}$, which correspond to the distributive liberal social contract solutions relative to x^0, make the subarea of the $u(P_w)$ delimited by the indifference curves of w_2 and w_3 through $\hat{u}^0$.

4.2 Distributive efficiency as aggregation of individual social preferences

An important by-product of the proof of Theorem 1 is the characterization of distributive optima as maxima of weighted averages of individual social utilities (see Theorem 2 below). The latter extends to distributive optima and utilities, with similar arguments, the familiar characterization of market optima as maxima of weighted averages of individual private utilities.[14]

The Pareto-efficient redistribution of the distributive liberal social contract, in particular, implicitly supposes a process of identification of socially desirable allocations by: (i) aggregation, first, of "individual-social" utilities into a "social-social" utility function $\sum_{i \in N} \mu_i (w_i \circ u)$ by means of arbitrary vectors of weights $\mu \in S_n$; (ii) and maximization, second, of these "social-social" utility functions in the set of attainable allocations unanimously weakly preferred to some original equilibrium position (see our constructive proof of the existence of a distributive liberal social contract, in part (ii) of the proof of Theorem

[14] A related property is the supportability of distributive optima by systems of Lindahl and market prices. Bergstrom (1970: lemmas 3 and 5) establishes the latter for non-malevolent convex preferences. Mercier Ythier (2007)develops the same property for the social systems of Assumption 1, which are differentiable and allow for malevolence.

Figure 1: Separability and the set of liberal social contracts

1). Note, nevertheless, that the distributive liberal social contract, such as defined in section 2, does not itself implement the distributive optimum. It only redistributes endowments, and leaves to the market the task of achieving the equilibrium allocation[15].

Theorem 2 *Let (w, u, ρ) verify assumptions 1 and 2. The following two propositions are then equivalent: (i) x is a weak distributive optimum (w, u, ρ); (ii) there exists $\mu \in S_n$ such that x maximizes $\sum_{i \in N} \mu_i (w_i \circ u)$ in A.*

[15] Except, of course, in the special case, where, as in part (ii) of the proof of Theorem 1, endowment redistribution achieves market equilibrium. This special case is theoretically interesting, because it is always accessible in theory (by the second fundamental theorem of welfare economics), and therefore provides an easy and simple way for establishing the existence of a distributive liberal social contract. The corresponding market equilibrium is the autarkic equilibrium, that is, a market equilibrium where each individual demands and consumes his own endowment. This equilibrium is unique in regular differentiable exchange economies (Balasko, 1988: 3.4.4). This implies, in particular, that social contract redistribution fully crowds out market exchange in this case, which therefore appears empty on practical grounds, as actual economies hardly reach or even approach any state of reasonable economic efficiency without large market exchanges.

4.3 Equivalence of cash and in-kind Pareto-efficient redistribution

The third aspect of separability outlined in this section is the equivalence of cash and in-kind Pareto-efficient redistribution.[16]

We first introduce a notion of price-wealth distributive optimum, on a pattern similar to the price-wealth equilibrium of market equilibrium theory, and next establish its equivalence with distributive efficiency.

We use standard definitions and properties of demand and indirect utility functions, which hold true under Assumption 1-(i). Notably, there exists, for each individual i, a continuous demand function $f_i : \mathbb{R}^l_{++} \times \mathbb{R}_+ \to \mathbb{R}^l_+$, that is, a continuous function such that, for any price-wealth vector $(p, r_i) \in \mathbb{R}^l_{++} \times \mathbb{R}_+$, $f_i(p, r_i)$ is the (unique) consumption bundle that maximizes the private utility of individual i subject to this individual's budget constraint $p.x_i \leq r_i$. The (private) indirect utility function of individual i, defined as $v_i = u_i \circ f_i$, also is a continuous function $\mathbb{R}^l_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$. Demand functions are: positively homogeneous of degree 0 (that is, $f_i(\alpha p, \alpha r_i) = f_i(p, r_i)$ for all $(p, r_i) \in \mathbb{R}^l_{++} \times \mathbb{R}_+$ and all $\alpha \in \mathbb{R}_{++}$); and such that $p.f_i(p, r_i) = r_i$ for all $(p, r_i) \in \mathbb{R}^l_{++} \times \mathbb{R}_+$(the so-called additivity property of Walrasian demand). Indirect utility functions are positively homogeneous of degree 0, and strictly increasing with respect to wealth. Since the money wealth of an individual reduces, in our setup, to the market value of his endowment $r_i = p.\omega_i$, we get $\sum_{i \in N} p.f_i(p, p.\omega_i) = \sum_{i \in N} p.\omega_i = p.\rho$ as the expression of Walras Law for aggregate demand, verified for any system of positive market prices $p >> 0$ and any distribution of initial endowments $\omega \in \{z \in \mathbb{R}^{ln}_+ : \sum_{i \in N} z_i = \rho\}$. From Walras Law and the homogeneity properties of individual demands, a system of equilibrium market prices is defined only up to an arbitrary positive multiplicative constant. In the sequel, market prices are normalized so that $p \in S_l$ (that is, we replace p by the equivalent $p/\sum_{i \in L} p_i$; this always is possible since $\sum_{i \in L} p_i$ necessarily is > 0 at equilibrium with our definitions and assumptions). With this normalization, we get $p.\rho = 1$ for any p, which means that the market value of the aggregate resources of the economy is constant relative to normalized market prices, equal to 1. We let: the distribution of money wealth $(r_1, \ldots, r_n)$ be denoted by r; the product function $(p, r) \to (f_1(p, r_1), \ldots, f_n(p, r_n))$ be denoted by f; the product function $(p, r) \to (v_1(p, r_1), \ldots, v_n(p, r_n))$ be denoted by v.

There is a well-known one-to-one correspondence, in differentiable economies, between market optima $x \in P_u$ and the systems of prices and wealth distribution (p, r) such that $\sum_{i \in N} f_i(p, r_i) = \rho$ (price-wealth market equilibria). Precisely, under Assumption 1-(i): For any $x \in P_u$, there exists a unique $p \in S_l$ such that the pair $(p, r)=(p, (p.x_1, \ldots, p.x_n))$ is a price-wealth market equilibrium (and the equilibrium p is then $>> 0$); conversely, if (p, r) is a price-wealth market equilibrium, then $x = f(p, r)$ is a market optimum, p is $>> 0$ and $r = (p.x_1, \ldots, p.x_n)$ (see the Appendix: Proposition). The notion of price-wealth market equilibrium

[16] For an analogous property of equivalence of cash and in-kind transfers in a general equilibrium setup with non-cooperative altruistic transfers, see Mercier Ythier, 2006: Theorem 5, p. 279.

yields a natural alternative definition of distributive optimum as a price-wealth market equilibrium which is not Pareto-dominated, relative to individual social utilities, by any other price-wealth market equilibrium. Formally:

Definition 5: A *price-wealth market equilibrium* of social system (w, u, ρ) is a pair $(p, r) \in S_l \times S_n$ such that $\sum_{i \in N} f_i(p, r_i) = \rho$.

Definition 6: A pair $(p, r) \in S_l \times S_n$ is a (weak) *price-wealth distributive optimum* of social system (w, u, ρ) if: (i) (p, r) is a price-wealth equilibrium of (w, u, ρ) ; (ii) and there exists no price-wealth equilibrium (p', r') of (w, u, ρ) such that $w(v(p', r')) >> w(v(p, r))$.

Theorem 3 *Let (w, u, ρ) verify assumptions 1 and 2. Let x be a market optimum of (w, u, ρ) and $(p, r) \in S_l \times S_n$ be the unique price-wealth market equilibrium such that $x = f(p, r)$. The following two propositions are then equivalent: (i) x is a weak distributive optimum; (ii) (p, r) is a weak price-wealth distributive optimum.*

4.4 Meaning and scope of the separability of allocation and distribution

We very briefly return, to conclude this section, to the meaning and scope of the separability of allocation and distribution in this setup.[17]

Separability states, essentially, that the redistribution of endowments by the distributive liberal social contract, and the allocation of resources by competitive markets, are two autonomous processes, and that these autonomous processes articulate consistently in the sense that the allocation that they jointly produce (they do produce some, which is unanimously preferred to the initial market equilibrium) is Pareto-efficient relative to both the private and the social preferences of individuals.

The separability property relies upon a set of four main conditions: (i) Walrasian equilibrium; (ii) non-paternalistic utility interdependence; (iii) lump-sum endowment transfers; (iv) and non-satiation of the distributive Pareto preordering.

Each of them can be considered as essential for the property, independently of the three others; and they together delineate the scope and the limits of the property.

This set of conditions analyzes as follows: a basic hypothesis on the (ideal) organization and functioning of market exchange (condition (i)); the design of a redistribution institution exactly compatible with the former (conditions (ii) and (iii)); and the hypothesis of civil peace as a common foundation, and/or joint consequence, of market exchange and social contract redistribution (condition (iv)).

[17] See also Mercier Ythier, 2006: 2.2, on the same object.

5 Relation of the distributive liberal social contract to alternative solutions for the Pareto-efficient provision of public goods: A calculated example.

The example calculated below supports the following claim of section 1: Foley's core solutions and Bergstrom's Lindahl equilibrium solutions are not, in general, unanimously preferred to Walrasian equilibrium with null provision of public goods, for individual distributive preferences.

The example illustrates the following general ideas. The building up of a public sector from an initial Walrasian equilibrium with null provision of public goods generally induces changes in the system of equilibrium market prices. These changes in the terms of trade might produce adverse consequences on the wealth and private welfare of some individuals. Public good provision is vetoed by an individual whenever the gains it implies in terms of her social welfare do not compensate for her loss in private welfare (more precisely, for the loss in terms of her individual social welfare that the latter implies). This is precisely what happens with the "egoistic rich" in the example below: She has no taste for the public good (redistribution to the poor) and suffers adverse consequences, in terms of her private welfare, from any departure from the market equilibrium of the original position.

Example: We consider the following simple social system, compatible with the present framework and those of Bergstrom (1970) and Foley (1970) simultaneously.

There are two market commodities ($l = 2$) and four individuals ($n = 4$). The first commodity is the numeraire ($p_1 = 1$). Individuals have identical quasi-linear private utility functions of the type: $u_i(x_i) = x_{i1} + \log x_{i2}$. Individual 4 is "poor": His initial endowment is null ($\omega_4 = 0$). Individuals 1, 2 and 3 are "rich": Individual 1 owns the economy's endowment in numeraire ($\omega_{11} = \rho_1 = 1$); and each of individuals 2 and 3 owns one half of the economy's endowment in the second commodity ($\omega_{22} = \omega_{32} = (1/2)\rho_2 = 1/2$). Rich individual 1 and poor individual 4 are egoistic, that is, $w_i(u(x)) = u_i(x_i)$ for all x, for $i = 1, 4$. Rich individuals 2 and 3 have identical distributive utility functions of the type $w_i(u(x)) = u_i(x_i) + u_4(x_4)$, which imply indifference to the private welfare of the other rich and non-paternalistic altruism to the poor. The private welfare or individual wealth of the poor is a public good in this social system, as object of common concern for the poor and the altruistic rich. This public good is "produced" from transfers of market commodities by means of the concave transfer "technology" $\rho - \sum_{i:i \neq 4} x_i \to u_4(\rho - \sum_{i:i \neq 4} x_i)$.

Straightforward calculations from first-order conditions for market equilibrium yield the following results concerning the Walrasian equilibrium of the original position: $f_i(p, p.\omega_i) = (\omega_{i1} + p_2\omega_{i2} - 1, 1/p_2)$ for all $i < 4$. And naturally $f_4(p, p.\omega_4) = 0$. In particular, the rich have equal demands for commodity

2, which must therefore be equal to 1/3 by the market equilibrium condition. The associate equilibrium price is $p_2 = 3$. The resulting equilibrium demands of the rich are (0,1/3) for individual 1 and (1/2,1/3) for the other two. Equilibrium distributive utility levels are: $-\log 3$ for $i = 1$; and $-\infty$ for the other three individuals.

Any market optimum that induces social utility levels of the altruistic rich $> -\infty$ implies a > 0 consumption of commodity 2 by the poor. The same type of calculations as above then yield a supporting market price =4 for commodity 2, and associate equal individual demands =1/4. Individual 1's distributive utility level is $-\log 4 < -\log 3$. That is, the egoistic rich strictly prefers the Walrasian equilibrium of the original position to any market optimum that yields a social utility level $> -\infty$ for all. In particular, the set of distributive liberal social contracts of this social system reduces to the Walrasian equilibrium of the original position.

Each of the two altruistic individuals acting alone from her endowment can reach $\max\{u_i(x_i) + u_4(x_4) : x \geq 0, x_i + x_4 \leq (0, 1/2)\}$, which is attained at $x_i = x_4 = (0, 1/4)$ and yields distributive utility level $-2\log 4 > -\infty$. Any allocation that involves a null consumption of commodity 2 by the poor is therefore blocked, in the sense of Foley (1970), by coalitions $\{2\}$, $\{3\}$ and $\{2, 3\}$. Consequently, individual 1 strictly prefers the Walrasian equilibrium of the original position to any allocation of the Foley-core. This notably applies to Bergstrom's Lindahl equilibrium, since the latter here belongs to the Foley-core as a consequence of Foley, 1970: 6.[18]

[18] The following minor adjustments must be made, in order to make the frameworks of this example, Foley (1970) and Bergstrom (1970) exactly comparable.

We must first account for the fact that the public good of the example is the private welfare of an individual who can, in principle, be included in the blocking coalitions of Foley, while there is no such possibility in Foley's setup. This difficulty resolves very easily by noticing that: The Foley-core expands, as an immediate consequence of definitions, if admissible coalitions are restricted to the non-empty subsets of {1,2,3} (in the place of $N = \{1, 2, 3, 4\}$); conversely, if an allocation is unblocked by coalition $I \subset \{1, 2, 3\}$, then it is unblocked by $I \cup \{4\}$, since $\omega_4 = 0$ (adding individual 4 leaves unchanged the coalition's set of alternatives). That is, the Foley-core remains the same, whether admissible coalitions are the non-empty subsets of $\{1, 2, 3\}$ or the non-empty subsets of entire N.

There is a slight and actually purely apparent difference, second, in the formal definitions of Lindahl equilibrium by Bergstrom and Foley, namely, the introduction in the latter of value-maximizing firms for the production of public goods. Foley's definition implies, in the example above, the introduction of a firm (or a charity) maximizing the net value $(\sum_{i:i\neq 4} \pi_i)u_4(x_4) - p.x_4$, where π_i denotes the Lindahl-price of the public good for individual i. This introduction can be made, actually, without changing equilibrium, because the equilibrium value of $\sum_{i:i\neq 4} \pi_i$ coincides with the inverse of the marginal utility of wealth of individual 4, so that the f.o.c. for the maximization of charity value (that is: $(\sum_{i:i\neq 4} \pi_i)\partial u_4(x_4) = p$ is automatically verified at equilibrium.

The Lindahl equilibrium of the example can be computed very easily by noticing that it implies a positive consumption of commodity 2 by the poor, and therefore an equilibrium market price of commodity 2 $p_2 = 4$, and equal individual demands of 1/4 for this commodity. We have $\pi_1 = 0$, so that the equilibrium demand of egoistic individual 1 for market commodities is (0,1/4). In particular, she exchanges her endowment in numeraire for commodity 2 purchased to the altruistic rich. Since the distributive utility of an altruistic rich is invariant in numeraire transfers to the poor, strong distributive efficiency requires, then, that the total numeraire endowment of the economy be transferred to the poor and consumed by her. The

6 Conclusion

Kolm's distributive liberal social contract is a normative reference for the Pareto-optimal redistribution of wealth within a functioning market economy. It derives the norm of wealth redistribution from an ideal assumption of perfect private and social contracting.

We develop the analysis of the notion within the formal framework of the theory of Pareto-optimal redistribution.

The distributive liberal social contract consists of a competitive market equilibrium which is both Pareto-efficient relative to individual distributive preferences, and Pareto-improving relative to initial competitive market equilibrium (that is, competitive market equilibrium with null redistribution) for these preferences.

These conditions are essential necessary characteristics of the distributive liberal social contract in this framework. They are not sufficient, in general, for a full characterization of it.

The distributive liberal social contract implies a fundamental property of separability of allocation and distribution. Social contract redistribution, performed by means of lump-sum transfers from initial individual endowments, preserves the basic existence and efficiency properties of competitive market equilibrium. Such in-kind social transfers are essentially equivalent to monetary transfers. Social contract wealth distribution maximizes a weighted average of individual social utility functions, in the set of market equilibria that Pareto-dominate the initial market equilibrium, relative to individuals' social preferences.

7 Appendix

In this appendix, we first summarize in a Proposition some useful standard results relative to the competitive equilibrium of differentiable exchange economies that verify Assumption 1-(i), and next develop the proofs of the theorems.

7.1 Differentiable Walrasian exchange economies

Proposition: Let (u, ρ) verify Assumption 1-(i). The following five propositions are then equivalent: (i) x is a weak market optimum of (u, ρ); (ii) x is a strong market optimum of (u, ρ); (iii) $x \in A$ is such that $\sum_{i \in N} x_i = \rho$, and there exists a price system $p >> 0$ such that, for all i: either $x_i = 0$; or $x_i >> 0$ and $\partial u_i(x_i) = \partial_{r_i} v_i(p, r_i)p$; (iv) there exists a price system $p >> 0$ such that $(p, (p.x_1, \ldots, p.x_n))$ is a price-wealth market equilibrium of (u, ρ); (v) x is a market price equilibrium of (u, ρ).

Lindahl equilibrium allocation therefore is ((0,1/4),(0,1/4),(0,1/4),(1,1/4)).

7.2 Theorem 1

Proof. (i) A distributive optimum x is by definition a local weak maximum of the product function $(w_1 \circ u, \ldots, w_n \circ u)$ in the set of attainable allocations A. Assumptions 1-(i)-(d) and 1-(ii)-(d) readily imply $x >> 0$ and $u(x) >> 0$. The first-order necessary conditions (f.o.c.) for this smooth optimization problem (e.g. Mas-Colell, 1985: D.1) then state that there exists $(\mu, p) \in \mathbb{R}^n_+ \times \mathbb{R}^l_+$ such that: (i) $(\mu, p) \neq 0$; (ii) $p.(\rho - \sum_{i \in N} x_i) = 0$; (iii) $\sum_{i \in N} \mu_i \partial_j w_i(u(x)) \partial u_j(x_j) - p = 0$ for all $j \in N$. We must have $\mu > 0$, for otherwise $p = 0$ by f.o.c. (iii), which contradicts f.o.c. (i). Since $\mu > 0$, (μ, p) can be replaced by $(\mu / \sum_{i \in N} \mu_i, p / \sum_{i \in N} \mu_i)$ in the f.o.c., that is, we can suppose from there on that $\mu \in S_n$. F.o.c. (iii) is equivalent to: $(\sum_{i \in N} \mu_i \partial_j w_i(u(x))) \partial u_j(x_j) = p$ for all j. Differentiable non-satiation of the Paretian preordering and strictly increasing private utilities then imply that $p >> 0$ and $\sum_{i \in N} \mu_i \partial_j w_i(u(x)) > 0$ for all j. The necessary first-order conditions reduce therefore to the following, equivalent proposition: $x >> 0$ is such that $\sum_{i \in N} x_i = \rho$, and there exists $(\mu, p) \in S_n \times \mathbb{R}^l_{++}$ such that, for all $j \in N$, $\sum_{i \in N} \mu_i \partial_j w_i(u(x)) > 0$ and $\partial u_j(x_j) = (1 / \sum_{i \in N} \mu_i \partial_j w_i(u(x)))p$. The latter system of conditions characterizes a market optimum of (w, u, ρ) under Assumption 1-(i), by application of standard results on the characterization of Pareto optima of differentiable economies (see the Proposition of the Appendix). This establishes the first part of Theorem 1.

(ii) Let (p, x) be a competitive market equilibrium with free disposal of (w, u, ω). The set $A(x) = \{z \in A : w_i(u(z)) \geq w_i(u(x)) \text{ for all } i \in N\}$ of attainable allocations unanimously weakly preferred to x is nonempty (it contains x), and compact (as a subset of compact set A, which is closed by continuity of $w_i \circ u$ for all i). Continuous function $\sum_{i \in N} \mu_i (w_i \circ u)$ therefore has at least one maximum in $A(x)$, for any given $\mu \in S_n$. Let ω' be such a maximum, that is: $\sum_{i \in N} \mu_i (w_i(u(\omega')) \geq \sum_{i \in N} \mu_i (w_i(u(z))$ for all $z \in A(x)$, for a given $\mu \in S_n$. We suppose moreover that $\mu >> 0$. We want to prove that there exists a price system p' such that $(\omega', (p', \omega'))$ is a distributive liberal social contract of (w, u, ω) relative to (p, x).

If $z \in A(x)$ is not a strong distributive optimum, that is, if there exists $z' \in A$ such that $w(u(z')) > w(u(z))$, then $z' \in A(x)$, and $\sum_{i \in N} \mu_i (w_i(u(z')) > \sum_{i \in N} \mu_i (w_i(u(z))$ (since $\mu >> 0$), so that z does not maximize $\sum_{i \in N} \mu_i (w_i \circ u)$ in $A(x)$. Therefore, ω' is a strong distributive optimum of (w, u, ρ), unanimously weakly preferred to x by construction. It suffices to establish, to finish with, that there exists a price system p' such that (p', ω') is a competitive market equilibrium with free disposal of (w, u, ω'). But this readily follows from the first-order conditions of the end of part (i) of this proof (recall that $P^*_w \subset P_w$), by application of standard results on the characterization of competitive equilibria of differentiable economies.

(iii) Theorem 1-(iii) is a simple consequence of Theorem 1-(i) and the standard properties of Walrasian exchange economies recalled in the Proposition of the Appendix. ■

7.3 Theorem 2

Proof. : The proof of Theorem 2 is a simple extension of an argument developed in the first part of the proof of Theorem 1, where we already established that (i) implies the following set of necessary first-order conditions (f.o.c.): $x >> 0$ is such that $\sum_{i \in N} x_i = \rho$, and there exists $(\mu, p) \in S_n \times \mathbb{R}^l_{++}$ such that, for all $j \in N$, $\sum_{i \in N} \mu_i \partial_j w_i(u(x)) > 0$ and $\partial u_j(x_j) = (1/\sum_{i \in N} \mu_i \partial_j w_i(u(x)))p$.

We now prove the following: If x verifies the f.o.c., then it maximizes $\sum_{i \in N} \mu_i(w_i \circ u)$ in A. Note that the f.o.c. imply the necessary first-order conditions for a local maximum of $\sum_{i \in N} \mu_i(w_i \circ u)$ in A (apply to the latter program the argument developed in the proof of Theorem 1 for the derivation of the f.o.c. for a weak distributive optimum). It will suffice, therefore, to establish that these necessary conditions for a local maximum of $\sum_{i \in N} \mu_i(w_i \circ u)$ in A are also sufficient conditions for a global maximum of the same program. But this readily follows from our assumptions and the Theorem 1 of Arrow and Enthoven, 1961 (notably their conditions (b) or (c), which are both verified under our assumptions).

We have established at this point that at (i)$\Rightarrow$(ii). Let us prove the converse to finish with.

If x is not a weak distributive optimum, that is, if $x \notin A$, or if $x \in A$ and there exists $x' \in A$ such that $w(u(x')) >> w(u(x))$, then, clearly, x is not a maximum of $\sum_{i \in N} \mu_i(w_i \circ u)$ in A, whatever $\mu \in S_n$. Therefore, (ii)$\Rightarrow$(i). ■

7.4 Theorem 3

Proof. : Suppose, first, that proposition (ii) of Theorem 3 is not verified, that is, suppose that the unique $(p, r) \in S_l \times S_n$ such that $x = f(p, r)$ is not a price-wealth distributive optimum. Definitions 3 and 6 then readily imply that x is not a weak distributive optimum. Therefore (i)$\Rightarrow$(ii).

We now prove the converse. Suppose that market optimum $x = f(p, r)$ is not a distributive optimum, and let x' denote some attainable allocation such that $w(u(x')) >> w(u(x))$. From part (ii) of the proof of Theorem 1, there exists a strong distributive optimum x'' such that $w(u(x'')) \geq w(u(x'))$. From Theorem 1-(i), x'' also is a market optimum. From the Proposition of the Appendix, there exists a price-wealth market equilibrium (p'', r'') such that $x'' = f(p'', r'')$. Therefore (p, r) is not a price-wealth distributive optimum, and the proof is completed. ■

References

Archibald, G.C. and D. Donaldson (1976), "Non-paternalism and the basic theorems of welfare economics". Canadian Journal of Economics 9: 492-507.

Arrow, K.J. and G. Debreu (1954), "Existence of an equilibrium for a competitive economy", Econometrica 22:265-290.

Arrow, K.J. and A.C. Enthoven (1961), "Quasi-concave programming", Econometrica 29:779-800.

Balasko, Y. (1988), Foundations of the Theory of General Equilibrium (Academic Press, London).

Becker, G.S. (1974), “A theory of social interactions”, Journal of Political Economy 82: 1063-1093.

Bergstrom, T.C. (1970), “A “Scandinavian consensus” solution for efficient income distribution among non-malevolent consumers”, Journal of Economic Theory 2: 383-98.

Debreu, G. (1952), “A social equilibrium existence theorem”, Proceedings of the National Academy of Sciences 38: 886-893. Reprinted in G. Debreu (1983), Mathematical Economics (Cambridge University Press, Cambridge) 50-58.

Debreu, G. (1954), “Valuation equilibrium and Pareto optimum”, Proceedings of the National Academy of Sciences 40: 588-92. Reprinted in G. Debreu (1983), Mathematical Economics (Cambridge University Press, Cambridge) 98-104.

Debreu, G. and H. Scarf (1963), “A limit theorem on the core of an economy”, International Economic Review 4: 235-46.

Edgeworth, F.Y. (1881), Mathematical Psychics (C. Keagan Paul and Co, London).

Foley, D.K. (1970), “Lindahl’s solution and the core of an economy with public goods”, Econometrica 38: 66-72.

Harsanyi, J.C. (1955), “Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility”, Journal of Political Economy 63: 309-321.

Hochman, H.M. and J.D. Rodgers (1969), “Pareto optimal redistribution”, American Economic Review 59:542-557. Reprinted in H.M. Hochman (2002), Economic Behaviour and Distributional Choice, (Edward Elgar, Cheltenham, United Kingdom and Northampton, Massachusetts) 5-20.

Kolm, S.-Ch. (1966), “The optimal production of social justice”, in H. Guitton, and J. Margolis eds: Proceedings of the International Economic Association on Public Economics, Biarritz; Economie Publique (CNRS, Paris, 1968) 109-177; Public Economics (Macmillan, London, 1969) 145-200.

Kolm, S.-Ch. (1985), Le Contrat Social Libéral (Presses Universitaires de France, Paris).

Kolm, S.-Ch. (1987a), “Public economics”, in J. Eatwell, M. Newgate and P. Newman eds, The New Palgrave (Macmillan, London).

Kolm, S.-Ch. (1987b), “The freedom and consensus normative theory of the state: The liberal social contract”, in P. Kolowski (ed.), Individual Liberty and Democratic Decision-Making: The Ethics, Economics and Politics of Democracy (J.C.B. Mohr, Tübingen) 97-127.

Kolm, S.-Ch. (1996), Modern Theories of Justice (MIT Press; Cambridge, Massachusetts and London, England).

Kolm, S.-Ch. (2004), Macrojustice: The Political Economy of Fairness (MIT Press; Cambridge, Massachusetts and London, England).

Kolm, S.-Ch. (2006), “Reciprocity: Its scope, rationales, and consequences”, Chap. 7 in S.-C. Kolm and J. Mercier Ythier eds., Handbook of the Economics of Giving, Altruism and Reciprocity (North-Holland, Amsterdam) 371-541.

Lemche, S.Q. (1986), "Remarks on non-paternalism and the second theorem of welfare economics", Canadian Journal of Economics 19:270-280.

Lévy-Garboua, L., Meidinger, C. and B. Rapoport (2006), "The formation of social preferences: Some lessons from psychology and biology", Chap. 7 in S.-C. Kolm and J. Mercier Ythier eds., Handbook of the Economics of Giving, Altruism and Reciprocity (North-Holland, Amsterdam) 545-613.

Mas-Colell, A. (1985), The Theory of General Equilibrium. A Differentiable Approach (Cambridge University Press, London).

Mercier Ythier, J. (1993), "Equilibre général de dons individuels", Revue Economique 44: 925-950.

Mercier Ythier, J. (1997), "First-order conditions for distributive efficiency". Discussion paper 98A39. Marseille: GREQAM, EHESS and Université d'Aix-Marseille II et III.

Mercier Ythier, J. (1998a), "The distribution of wealth in the liberal social contract", European Economic Review 42:329-347.

Mercier Ythier, J. (1998b), "Existence of an equilibrium of property rights", Mathematical Social Sciences 35:261-272.

Mercier Ythier, J. (2006), "The economic theory of gift-giving: Perfect substitutability of transfers and redistribution of wealth", Chap. 5 in S.-C. Kolm and J. Mercier Ythier eds., Handbook of the Economics of Giving, Altruism and Reciprocity (North-Holland, Amsterdam) 227-339.227-339.

Mercier Ythier, J. (2007), "The aggregation of individual distributive preferences through the distributive liberal social contract: Normative analysis", contribution to the conference in honor of Serge Kolm "Normative economics and social ethics", 18-19 May 2007.

Musgrave, R.A. (1970), Pareto optimal redistribution: Comment, American Economic Review 60: 991-93. Reprinted in H.M. Hochman (2002), Economic Behaviour and Distributional Choice, (Edward Elgar, Cheltenham, United Kingdom and Northampton, Massachusetts) 21-23.

Nozick, R. (1974). Anarchy, State and Utopia (New York, Basic Books).

Pareto, V. (1913), "Il massimo di utilità per una colletività ", Giornale degli Economisti 3: 337-341.

Pareto, V. (1916), Traité de Sociologie Générale (Droz, Genève).

Rader, T. (1980), "The second theorem of welfare economics when utilities are interdependent", Journal of Economic Theory 23: 420-424.

Rawls, J. (1971). A Theory of Justice (Cambridge, Harvard University Press).

Smith, A. (1759), The Theory of Moral Sentiments. Edited by D.D. Raphael and A.L. Macfie (Clarendon Press, Oxford,1976).

Warr, P.G. (1983), The private provision of a public good is independent of the distribution of income, Economic Letters 13: 207-11.

Wicksteed, P.H. (1910), The common sense of political economy, London: Macmillan.

Winter, S. G. (1969), A simple remark on the second optimality theorem of welfare economics, Journal of Economic Theory 1: 99-103.